Comparison between Trajectory Tracking Control and Path Following Control: A Linear System Example

Saif A. Al-Hiddabi

Department of Mechanical and Industrial Engineering
Sultan Qaboos University
P.O.Box 33, Al-Khod 123
Muscat, Sultanate of Oman
Email: hiddabi@squ.edu.om

Abstract: In this paper we study the problem of trajectory tracking control and path following control for linear control systems. First we formulate the problem as an output trajectory tracking control problem. Next we formulate a related path following problem. A controller is developed for this case, and it is compared with a controller that solves the related output trajectory tracking problem. The advantages of the path following controller are found to be smaller control values are typically required thereby avoiding control saturation effects. The simulation example that have been provided clearly confirm this conclusion.

Keywords: Path Control, Trajectory Tracking Control, Maneuver Regulation Control.

1 Introduction

In a trajectory tracking problem, the desired outputs, parameterized by time, are provided by a command generator. The trajectory tracking controller processes the desired outputs and forces the system outputs to follow the desired outputs as closely as possible. In the presence of tracking errors, the trajectory tracking controller attempts to make the outputs "catch up" with the time-parameterized desired outputs; this may lead to closed loop performance difficulties and to large control signals.

One approach to eliminate such problems is to use a path following controller instead of a trajectory tracking controller. The objective of path following controller is to track a specified geometric path in the output space with specified "velocity" along the path. The path tracking controller eliminates the aggressiveness of the trajectory tracking controller by forcing convergence to the desired path in a smooth way.

Path following, maneuver regulation or path tracking controllers have been studied for robotic systems [9, 15] and for aerospace vehicles [12, 14]. A general approach has been developed by Hauser and Hindman [11, 13] for feedback linearizable nonlinear control system.

The present paper compares the trajectory tracking controller to the path following controller. The path tracking controller regulates the position errors transverse to the desired path but it does not regulate the position error along the desired path. Based on our experience with the planar vertical take off and landing (PVTOL) aircraft model treated in [7] and the simplified longitudinal aircraft model treated in [5], this method improves closed loop properties and reduces the size of control inputs. The path tracking approach is based on designing a tracking controller that maintain a desired speed along a desired path with closed loop stability. This design approach is different from the trajectory tracking approach treated in [6] where vehicle speed and position are regulated along the desired path.

As suggested in [3, 4, 8], first a trajectory tracking controller, consisting of static state feedback, is designed to guarantee uniform asymptotic trajectory tracking. The constant feedback gains are determined based on LQR optimization. A path tracking controller is then obtained from the tracking controller by introducing a suitable state projection that is related to the LQR feedback gains. Properties of the closed loop, including local asymptotic convergence of the transverse errors, are presented.

2 System definition

Given a linear time invariant system,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^p$, and $y \in R^q$ and (A, B) is assumed to be stabilizable.

A prescribed reference output is denoted by $y_R(t)$. We assume $y_R(t)$ is arbitrary but sufficiently smooth.

3 Trajectory Tracking Control approach

3.1 A tracking Controller

In the trajectory tracking problem, control action is required to achieve the following objectives:

- Asymptotic stability, which means the closed loop system has an asymptotically stable equilibrium at $x = 0$ when $y_R(t) = 0$.
- Asymptotic output regulation, which means that for any initial state, the output error $e(t) = y(t) - y_R(t)$, converges to zero as $t \to \infty$.

- Exact output regulation, which means that there exists an initial state for a given $y_R(t)$ such that the output error $e(t) = y(t) - y_R(t) = 0$,for all $t \geq 0$.

Assume that a state trajectory $\alpha(t) \in R^n$ and a control action $\mu(t) \in R^p$ satisfy (1); that is

$$\dot{\alpha} = A\alpha + B\mu, \quad y_R = C\alpha$$

A tracking control law is given by

$$u(t) \;=\; \mu(t) + K(x(t) - \alpha(t)) \qquad (2)$$

where K is a control gain chosen as that $A_c = (A + BK)$ is Hurwitz. This control law achieves the closed loop objectives indicated previously.

3.2 Comments on determining $\alpha(t)$ and $\mu(t)$

Some comments are given on how to obtain the desired state trajectory α and control μ for a given output trajectory $y_R(t)$:

3.2.1 Inversion based approach for sufficiently smooth $y_R(t)$

This approach depends on the relative degree r and stability of the zero dynamics [1] of the system; for simplicity we consider a SISO system.

1. $r = n$. In this case, α and μ can be expressed in term of y_R and its derivatives up to $y_R^{(n)}$

$$\alpha = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} y_R \\ \dot{y}_R \\ \vdots \\ y_R^{n-1} \end{bmatrix}, \quad \mu = \frac{y_R^n - CA^n\alpha}{CA^{(n-1)}B}$$

The proof of the above result is omitted for space limitation.

2. $r < n$ and the system is minimum phase, i.e. the zero dynamics is asymptotically stable. If y_R and its derivatives up to $y_R^{(r)}$ are smooth and bounded, the state trajectory α depends on y_R and its derivatives up to $y_R^{(r)}$ and is bounded as well. For simplicity, the system is expressed in the normal form: [1]

$$\begin{array}{rcl} \dot{\zeta}_1 & = & \zeta_2 \\ & \vdots & \\ \dot{\zeta}_{r-1} & = & \zeta_r \\ \dot{\zeta}_r & = & F_1\zeta + F_2\eta + b\mu, \quad b \neq 0 \\ \dot{\eta} & = & R_1\zeta + R_2\eta \end{array} \qquad (3)$$

and

$$y = \zeta_1$$

where $\alpha = [\zeta^T; \eta^T]^T \in R^n, \quad \zeta = [\zeta_1, \ldots, \zeta_r] \in R^r, \quad \eta \in R^{n-r}$ and R_2 is Hurwitz.

It is easy to observe that ζ can be determined by $\zeta_i = y_R^{(i-1)}, \quad i = 1, 2, \ldots, r$. Hence, η is determined by the last equation in the normal system (3) since ζ is known. Finally, μ is given by:

$$\mu = \frac{y_R^{(r)} - F_1\zeta - F_2\eta}{b} = \frac{y_R^{(r)} - F\alpha}{b}$$

where $F = [F_1 F_2]$. Obviously, ζ and η are bounded; so α and μ.

3. $r < n$ and the system is hyperbolic and non-minimum phase, i.e. the zerodynamics has right half plane poles but no poles on the imaginary axis. In this case, the above inversion based approach can not be used directly but the noncausal stable inversion approach introduced in [10] can be invoked to find α and μ by solving a certain integral equation.

4 Path Following Control Approach

4.1 Path Following Controller

Suppose $\alpha(t)$ and $\mu(t)$ are available in the tracking control. A path following control law introduced in [11] is determined by replacing the trajectory time t with the parameter $\theta \quad (\theta \in R)$ that corresponds to the point on the state path $\alpha(\theta)|\theta \in R$ closest to the current state x with respect to the norm defined by $P > 0$, where $PA_c + A_c^TP = -Q, \quad Q > 0$. For each state x, θ is determined by solving the following optimization problem:

$$\theta(x) = \arg\min \|x - \alpha(\theta)\|_P^2, \quad \theta \in R \qquad (4)$$

Conditions which guarantee that the above optimization problem has a unique solution are:

- α is at least C^2;
- $\alpha'(\theta)$ is nonzero;
- $\alpha''(\theta)$ is bounded;
- No self-intersection of the curve $\alpha(\theta)|\theta \in R$ in R^n.
- The state x is sufficiently close to the curve $\alpha(\theta)|\theta \in R$.

Under these assumptions, the maneuver regulation control law is

$$u(x) = \mu(\theta(x)) + K(x - \alpha(\theta(x))) \qquad (5)$$

The closed loop system becomes

$$\dot{x} = Ax + B\mu(\theta(x)) + K(x - \alpha(\theta(x))) \qquad (6)$$

4.2 Closed loop properties of path following control

The stability of the closed loop transverse dynamics under the path following controller can be shown as follows: [11]

Using the maneuver regulation control law, the error $e = x - \alpha(\theta(x))$ satisfies

$$
\begin{aligned}
\dot{e} &= \dot{x} - \dot{\alpha}(\theta) \\
&= Ax + B(\mu(\theta) + K(x - \alpha(\theta)) - A\alpha(\theta) \\
&- B\mu(\theta) + \alpha'(\theta)(1 - \dot{\theta}) \\
&= A_c e + \alpha'(\theta)(1 - \dot{\theta})
\end{aligned} \tag{7}
$$

where $\alpha' := \frac{d\alpha(\theta)}{d\theta}$. Note that here we use the identity $\dot{\alpha}(\theta) = \alpha'\dot{\theta}$.

Consider an error-based Lyapunov function

$$V(e) = \| x - \alpha(\theta(x)) \|_P^2 = e^T P e \tag{8}$$

Hence, we have

$$
\begin{aligned}
\dot{e} &= \dot{e}^T P e + e^T P \dot{r} \\
&= e^T(A_c^T P + P A_c)e + 2e^T P \alpha'(\theta)(1 - \dot{\theta}) \qquad (9) \\
&= e^T(A_c^T P + P A_c)e = -e^T Q e = - \| e \|_Q^2
\end{aligned}
$$

where $Q > 0$. Here we make use of $e^T P \alpha'(\theta) = 0$ which is a necessary condition for satisfaction of condition (4).

As a result, it follows that for any $x(0)$, $V(e) \to 0$ as $t \to \infty$; that is, $(x - \alpha(\theta(x))) \to 0$ as $t \to \infty$. This means that the transverse dynamics of the closed loop are asymptotically stable.

4.3 Properties of θ

Condition (4) implies $\theta(x)$ satisfies

$$\dot{\theta} = \frac{\dot{x}^T P \alpha'(\theta)}{\| \alpha'(\theta) \|_P^2 - (x - \alpha(\theta))^T P \alpha''(\theta)} \tag{10}$$

where $\theta(0)$ solves $h'(\theta) = (x(0) - \alpha(\theta))^T P \alpha'(\theta) = 0$.

Thus, $\dot{x} = \dot{e} + \dot{\alpha}(\theta) = A_c e + \alpha'(\theta)\dot{\theta}$ and condition (10) can be written as

$$\dot{\theta} = \frac{\| \alpha'(\theta) \|_P^2 + e^T A_c^T P \alpha'(\theta)}{\| \alpha'(\theta) \|_P^2 - e^T P \alpha''(\theta)} \tag{11}$$

4.4 LQR Path Following Controller

Consider the maneuver regulation controller

$$
\begin{aligned}
u(x) &= \mu(\theta(x)) + K(x - \alpha(\theta(x))) \\
\theta(x) &= \arg\min \|x - \alpha(\theta)\|_P^2, \quad \theta \in R
\end{aligned}
$$

where we choose the gain K according to LQR theory: $K = -R^{-1}B^T P$ and P is the positive definite solution to the Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0, \quad Q \geq 0, \quad R > 0. \tag{12}$$

We show briefly that the closed loop using the above LQR maneuver regulation controller has the properties identified previously.

As before, we define the error $e = x - \alpha(\theta(x))$ and choose an error based Lyapunove function as $V(e) = e^T P e$. Following the same development and combing the Riccati equation, we obtain

$$\dot{V}(e) = - \| e \|_{\overline{Q}}^2 \tag{13}$$

where $\overline{Q} = Q + K^T R K > 0$. Hence, the closed loop transverse dynamics is asymptotically stable and θ has the same properties indicated previously.

5 Example

We illustrate the above analysis using a dynamic example of a vehicle moving on a two dimensional plane. The input/output model of such system is given by

$$
\begin{aligned}
\ddot{x} &= u_x \\
\ddot{y} &= u_y
\end{aligned}
$$

where x and y are the inertial position of the vehicle.

5.1 Trajectory Tracking Control

Define tracking error as $e_{xT1} = x - x_r(t)$, $e_{xT2} = \dot{x} - \dot{x}_r(t)$, $e_{yT1} = y - y_r(t)$ and $e_{yT2} = \dot{y} - \dot{y}_r(t)$, where $x_r(t), y_r(t)$ are time parameterized reference trajectory for the vehicle center of mass.

The system dynamics in error coordinates is given by

$$
\begin{aligned}
\dot{e}_{xT1} &= e_{xT2} \\
\dot{e}_{xT2} &= u_x \\
\dot{e}_{yT1} &= e_{yT2} \\
\dot{e}_{yT2} &= u_y
\end{aligned}
$$

Consider the following state feedback control laws

$$
\begin{aligned}
u_x &= \ddot{x}_r(t) - K_{dx}(\dot{x} - \dot{x}_r(t)) - K_{px}(x - x_r(t)) \qquad (14) \\
u_y &= \ddot{y}_r(t) - K_{dy}(\dot{y} - \dot{y}_r(t)) - K_{py}(y - y_r(t)) \qquad (15)
\end{aligned}
$$

The resulting closed loop system using the above control law is given by

$$
\begin{aligned}
\dot{e}_{xT1} &= e_{xT2} \\
\dot{e}_{xT2} &= -K_{dx}e_{xT2} - K_{px}e_{xT1} \\
\dot{e}_{yT1} &= e_{yT2} \\
\dot{e}_{yT2} &= -K_{dy}e_{yT2} - K_{py}e_{yT1}
\end{aligned}
$$

It is straightforward to show that for any positive K_{dx}, K_{px}, K_{dy} and K_{py}, the above closed loop system is asymptotically stable and all errors converge to zero as time goes to infinity. We use the LQR theory to determine the feedback gains by setting A, B, Q and R in (12) as

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, Q = 4I_{4\times4}, R = 0.06I_{2\times2}.$$

Solving (12) for P, we obtain

$$P = \begin{pmatrix} 4.4631 & 0.4899 & 0 & 0 \\ 0.4899 & 0.5466 & 0 & 0 \\ 0 & 0 & 4.4631 & 0.4899 \\ 0 & 0 & 0.4899 & 0.5466 \end{pmatrix}$$

and the gains K_{dx}, K_{px}, K_{dy} and K_{py} can be computed using $K = -R^{-1}B^T P$ to obtain $K_{dx} = K_{dy} = 9.1102$ and $K_{px} = K_{py} = 8.1650$

5.2 Path Following Control

Define transverse error as $e_{xP1} = x - x_r(\theta)$, $e_{xP2} = \dot{x} - x_r'(\theta)$, $e_{yP1} = y - y_r'(\theta)$, $e_{yP2} = \dot{y} - y_r'(\theta)$, where $x_r(\theta), y_r(\theta)$ are θ parameterized reference trajectory for the vehicle center of mass, $x_r'(\theta) = \frac{dx}{d\theta}$ and θ is the solution of (4).

The system dynamics in error coordinates is given by

$$\begin{aligned} \dot{e}_{xP1} &= e_{xP2} + x_r'(\theta)(1-\dot{\theta}) \\ \dot{e}_{xP2} &= u_x + x_r''(\theta)\dot{\theta} \\ \dot{e}_{yP1} &= e_{yP2} + y_r'(\theta)(1-\dot{\theta}) \\ \dot{e}_{yP2} &= u_y + y_r''(\theta)\dot{\theta} \end{aligned}$$

A possible choice of u_x and u_y is as follows

$$\begin{aligned} u_x &= x_r''(\theta) - K_{dx}(\dot{x} - x_r'(\theta)) - K_{px}(x - x_r(\theta)) \quad (16) \\ u_y &= y_r''(\theta) - K_{dy}(\dot{y} - y_r'(\theta)) - K_{py}(y - y_r(\theta)) \quad (17) \end{aligned}$$

The closed loop system can be written as

$$\begin{aligned} \dot{e}_{xP1} &= e_{xP2} + x_r'(\theta)(1-\dot{\theta}) \\ \dot{e}_{xP2} &= -K_{dx}e_{xP2} - K_{px}e_{xP1} + x_r''(\theta)(1-\dot{\theta}) \\ \dot{e}_{yP1} &= e_{yP2} + y_r'(\theta)(1-\dot{\theta}) \\ \dot{e}_{yP2} &= -K_{dy}e_{yP2} - K_{py}e_{yP1} + y_r''(\theta)(1-\dot{\theta}) \end{aligned}$$

If we consider using the same positive feedback gains used in the trajectory tracking control (14,15), than asymptotic stability of the path following closed loop system can proven using a Lyapunove function similar to that given by (8). Note that the path following closed loop system is nonlinear, and thus a Lyapunove function similar to that given by (8) is the only way to proof global asymptotic stability of the closed loop system.

To show clearly the comparison between the two design approaches, we consider a circular path for the vehicle center of mass. The time parametrized trajectory is given by $x_r(t) = 20\cos(t)$ and $y_r(t) = 20\sin(t)$. The initial states are selected so that vehicle is initialized on the path but not on a point correspond to $t = 0$. Figure 1, shows the desired path for the vehicle, the path that results from the trajectory tracking controller and the path that results from the path following controller assuming identical initial condition and control gains. It is clear from Figure 1 that the trajectory tracking controller causes the vehicle to track the time parameterized path but with large transient errors. On the other hand, the path following controller causes the vehicle to remain in the desired path in a smooth way with much smaller transient errors than for the trajectory tracking controller.

In Figure 2, we show the magnitude of the control signals required to execute the desired motion for the vehicle. It is clear that both the x control and the y control for the trajectory tracking controller are large and may cause control saturation. On the other hand the control signals required by the path following control design are much smaller in magnitude than for the trajectory tracking controller.

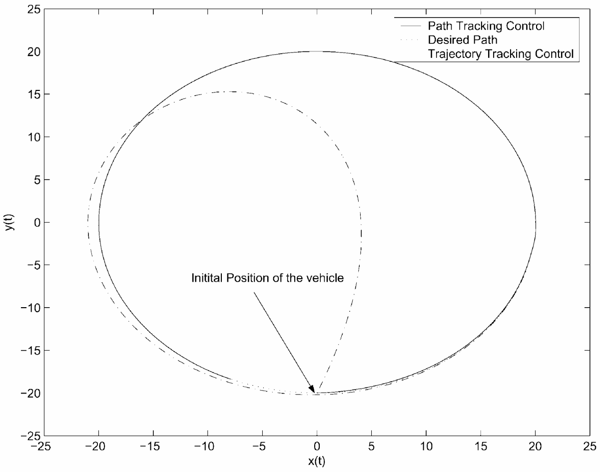

Figure 1: Vehicle position using trajectory tracking control and path following control

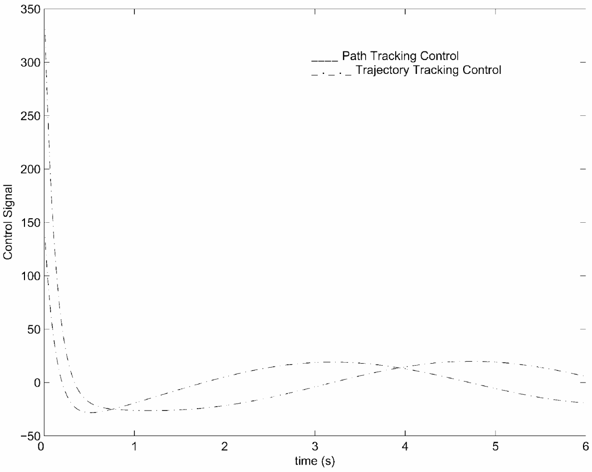

Figure 2: Control signals when using trajectory tracking control and path following control

6 Conclusions

We have constructed a trajectory tracking controller and a path following controller to achieve desired closed loop properties for a linear system. The main contribution of this paper is use of LQR theory for comparing the trajectory tracking control design approach and the path following control design approach. We demonstrate these approaches on a linear dynamical system. We concluded that the advantages of the path following controller are found to be smaller control values are typically required thereby avoiding control saturation effects.

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基于特征结构配置的最优结构主动控制及其在土木工程中的应用

王国胜，段广仁

(**哈尔滨工业大学 控制理论与制导技术研究中心，哈尔滨** 150001)

E-mail: gswang@126.com

摘要：考虑了具有最优控制的结构主动控制问题，目的是在系统满足闭环特性的前提下，设计状态反馈控制器使得系统性能泛函极小化。利用特征结构配置方法提供的自由度，给出了性能泛函的显示参数化表示，从而该问题转化为带有约束条件的优化问题，参与优化的变量仅为一组参量。并给出了求解该优化问题的算法，该算法直接基于结构系统矩阵，故其简单性为工程应用提供方便。地震作用下对三层剪切结构建筑模型进行仿真分析，结果表明本文所提具有结构主动控制方法的有效性。

关键词：结构系统；主动控制；特征结构配置；地震控制；优化.

Optimal structural active control based on eigenstructure assignment and its applications in civil engineering

Wang Guo-sheng，Duan Guang-ren

(Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin 150001)

Abstract: The design of structural active control with minimum control effort is investigated. The aim is to design a state feedback controller, such that the closed-loop system has desired eigenvalues and a system performance function is optimized. By utilizing design degrees of freedom offered by parametric eigenstructure assignment, a parametric expression for the system performance index is presented. Thus the optimization problem is changed into a minimization problem with some constraints and the optimized variables are a group of parameters. An algorithm is proposed for this minimization and utilizes the original system data, and thus it is simple to use in applications. A three-story shearing model under earthquake excitation is analyzed by using the proposed algorithm and the simulation results show the effect of this algorithm.

Key words: Structural system; active control; eigenstructure assignment; earthquake control; optimization.

1 引 言

自从 1972 **年美国** Yao[1]**结合现代控制理论，提出了土木工程结构振动控制的概念，开创了结构振动的主动控制研究的历程，结构振动控制从理论到应用都取得了很大进展。结构振动控制方法按照控制系统有无能源输入分为主动控制、被动控制、半主动控制和混合控制等，其中主动控制是一种积极的抗振手段，具有效果好，适用范围广等优点，成为国内外相关领域研究的前沿课题**[2,3]**。近** 30 **年来应用和发展起来的、适用于土木结构的主动控制算法主要包括二次型最优控制、独立模态最优控制、极点配置和滑动模态控制等。极点配置或特征结构配置作为土木工程结构的主动控制算法之一，虽然很早被提出，但在土木结构领域中的应用却很少，可查到的文献仅有**[4]**。**

本文则把特征结构配置参数化方法[5, 6]**和最优控制问题相结合，引入土木结构领域中，考虑了具有最优控制力的结构主动控制问题，其目的是设计状态反馈控制器，使得闭环系统具有希望的极点外，还使得系统性能泛函极小化。利用特征结构配置方法提供的自由度，给出了性能指标参数化表达式。把优化问题最终转化为含有约束条件的极小化问题，参与优化的变量为特征结构配置方法提供的自由参量。给出了解决该优化问题的方法，该方法直接基于结构系统矩阵，不涉及系统增广或变换，其简单性为工程应用提供了方便。最后，应用该算法设计了地震作用下三层剪切结构建筑模型的状态反馈控制器，仿真结果表明了本文所提方法的有效性。**

2 结构系统状态空间模型

考虑在水平地震地面运动加速度 $\ddot{x}_g(t)$ **作用下** n **自由度的层间剪切型结构模型，其运动方程为**

$$M\ddot{X}(t)+C\dot{X}(t)+KX(t)=Hu(t)-MI_{n\times1}\ddot{x}_g(t) \quad (1)$$

式中 $X(t)$ **，** $\dot{X}(t)$ **和** $\ddot{X}(t)$ **分别为各楼层相对于地面的位移、速度、加速度向量；**$u(t)$ **为** r **维控制力向量；**$\ddot{x}_g$ **为地震地面运动加速度；**M **，** C **和** K **分别为结构系统的** $n\times n$ **阶质量矩阵、阻尼矩阵和刚度**

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矩阵；H 为控制力作用位置矩阵；$I_{n\times1}$ 为 n 行元素均为1的列向量。要求 M 和 H 均为满秩，且矩阵对 $(-M^{-1}C,\ M^{-1}H)$ 可控，即

$$\mathrm{rank}[-M^{-1}C-sI_{n\times n}\quad M^{-1}H]=n\text{，}\ \forall s\in\mathbb{C} \tag{2}$$

系统(1)的等价状态方程为

$$\dot{Z}(t)=AZ(t)+Bu(t)-E\ddot{x}_g(t) \tag{3}$$

式中

$$A=\begin{bmatrix}0 & I_n\\ -M^{-1}K & -M^{-1}C\end{bmatrix},\ B=\begin{bmatrix}0\\ M^{-1}H\end{bmatrix},$$

$$E=\begin{bmatrix}0\\ I_n\end{bmatrix},\ Z=\begin{bmatrix}X\\ \dot{X}\end{bmatrix}$$

选取状态反馈控制律

$$u=F_0X(t)+F_1\dot{X}(t)=FZ(t)\text{，}\ F=[F_0\quad F_1] \tag{4}$$

式中 F 为 $r\times2n$ 的反馈增益矩阵，反馈控制作用是状态变量(速度和位移)的线性组合，此时闭环系统为

$$\dot{Z}(t)=A_cZ(t)-E\ddot{x}_g(t)\text{，}\ A_c=A+BF \tag{5}$$

3　特征结构配置控制算法

因非亏损矩阵较亏损矩阵对系统参数扰动具有良好的鲁棒性，故本文仅考虑闭环系统矩阵 A_c 的特征值为互异且自共轭情形，记特征值为 $s_i\in\mathbb{C}$ ，$i=1,2,\cdots,2n$，其对应的特征向量分别为 v_i ，$i=1,2,\cdots,2n$ ，则有

$$s_iv_i=A_cv_i,\ i=1,2,\cdots,2n \tag{6}$$

记

$$\Lambda=\mathrm{diag}(s_1,s_2,\cdots,s_{2n}),\ V=[v_1\ v_2\ \cdots\ v_{2n}] \tag{7}$$

则方程(6)等价于

$$A_cV=V\Lambda\ \text{或}\ AV+BFV=V\Lambda \tag{8}$$

因矩阵对 $(-M^{-1}C,M^{-1}H)$ 可控，故对矩阵 $[-M^{-1}C-sI\quad M^{-1}H]$ 进行初等变换,可得 $n\times n$ 阶单模阵 $P(s)$ 和 $(n+r)\times(n+r)$ 阶单模阵 $Q(s)$ 满足下式：

$$P(s)[-M^{-1}C-sI\quad M^{-1}H]Q(s)=[0\quad I]\text{，}\ \forall s\in\mathbb{C} \tag{9}$$

对 $Q(s)$ 进行如下分块：

$$Q(s)=\begin{bmatrix}Q_{11}(s) & Q_{12}(s)\\ Q_{21}(s) & Q_{22}(s)\end{bmatrix} \tag{10}$$

其中 $Q_{11}(s)$ 为 $n\times r$ 阶矩阵。从而有下述定理，它给出了方程(8)中增益阵 F 的参数化表达式，其证明过程详见文[5，6]。

定理 1　给定二阶动力学系统(1), 那么

1) **若矩阵对 $(-M^{-1}C,M^{-1}H)$ 可控，则矩阵对 $(A,\quad B)$ 仍可控的充要条件是存在 $n\times n$ 阶单模阵 $H(s)$ 和 $(n+r)\times(n+r)$ 阶单模阵**

$$L(s)=\begin{bmatrix}L_{11}(s) & L_{12}(s)\\ L_{21}(s) & L_{22}(s)\end{bmatrix} \tag{11}$$

其中 $L_{11}(s)$ 为 $n\times r$ 阶矩阵，满足下式

$$H(s)[-Q_{12}(s)P(s)M^{-1}K+sI\quad -Q_{11}(s)]L(s)=[0\quad I_n]\text{，} \tag{12}$$

2) **若上述条件成立，那么满足(8)的状态反馈增益阵 F 可如下给出：**

$$F=WV^{-1} \tag{13}$$

式中

$$V=[v_1\ v_2\ \cdots\ v_{2n}]\text{，}\ v_i=\begin{bmatrix}L_{11}(s_i)\\ s_iL_{11}(s_i)\end{bmatrix}g_i \tag{14}$$

$$W=[w_1\ w_2\ \cdots\ w_{2n}]\text{，}$$

$$w_i=[Q_{21}(s_i)L_{21}(s_i)+Q_{22}(s_i)P(s_i)M^{-1}KL_{11}(s_i)]g_i \tag{15}$$

其中 g_i ，$i=1,2,\cdots,2n$ 是一组 $r\times1$ 阶参量，需满足

$$\det\begin{bmatrix}L_{11}(s_1)g_1 & \cdots & L_{11}(s_{2n})g_{2n}\\ s_1L_{11}(s_1)g_1 & \cdots & s_{2n}L_{11}(s_{2n})g_{2n}\end{bmatrix}\neq0 \tag{16}$$

和

$$s_i=\overline{s_j}\Leftrightarrow\ g_i=\overline{g_j}\text{，}\ i,j=1,2,\cdots,2n \tag{17}$$

综合上述特征结构配置参数化结果，其优越性可以归纳为如下几点：

1) **该方法给出了满足方程(8)的所有状态反馈增益阵和闭环特征向量矩阵的参数化表示，其含有的参量可进一步用来满足系统设计中其它性能指标，如鲁棒性等；**
2) **该方法计算过程中只涉及层间剪切型结构模型(1)中矩阵 M ，C ，K 和 H ，并不涉及增广系统(3)中矩阵 A 和 B ，故便于工程应用。**

4　最优结构主动控制设计

本文考虑的具有最优控制力的结构主动控制设计问题可如下描述：给定层间剪切建筑结构系统(1)以及一组自共轭且互异的复数 $s_i\in\mathbb{C}$ ，$i=1,2,\cdots,2n$ ，确定形如(4)的状态反馈控制律 $u=FZ(t)$ ，对于任意正定对称矩阵 R ，满足下述条件:

1) **闭环系统矩阵 A_c 的特征值为 $s_i\in\mathbb{C}$ ，$i=1,2,\cdots,2n$ ；**
2) $\min tr(P(F))$；

其中正定矩阵 P 是下述 Lyapunov 方程的解：

$$A_c^TP+PA_c=-F^TRF\ . \tag{18}$$

不难发现，上述优化问题等价于极小化下述二次型性能指标函数

$$I=\int_0^\infty u^T(t)Ru(t)dt\ . \tag{19}$$

为求解问题ESA，我们首先给出如下结论：给定系统(1)以及一组共轭互异的复数 s_i ，$i=1,2,\cdots,2n$ ，若矩阵对 $(-M^{-1}C,M^{-1}H)$ 和 $(A,\quad B)$ 均可控，那么对于任一正定对称矩阵 $R\in\mathbf{R}^{r\times r}$ ，方程(18)中矩阵 P 的解为

$$P=-\frac{1}{2}V^{-T}\left[\frac{g_i^T M^T(s_i)RM(s_j)g_j}{s_i+s_j}\right]_{2n\times 2n}V^{-1} \quad (20)$$

式中

$$M(s_i)=Q_{21}(s_i)L_{21}(s_i)+Q_{22}(s_i)P(s_i)M^{-1}KL_{11}(s_i) \quad (21)$$

矩阵V **由**(14)**决定；**$g_i\in \mathbf{C}^r$， $i=1,2,\Lambda\ ,2n$,**为一组满足**(16)**和**(17)**的自由参量。**

若记$V=V(g_i)$ **，由**(20)**易知**

$$\mathrm{tr}(P(F))=\mathrm{tr}(P(g_i,i=1,2,\Lambda\ ,2n)) \quad (22)$$

式中 $P(g_i,i=1,2,\Lambda\ ,2n)$ **由**(20)**给出，从而问题** ESA **转化为**

$$\min\ \mathrm{tr}(P(g_i,s_i,i=1,2,\Lambda\ ,2n)),\text{s. t. } (16)\text{和}(17) \quad (23)$$

综上分析，问题 ESA **的求解过程可归纳为如下步骤，我们称之为特征结构配置方法(以下简称算法** ESA)**。**

1) **计算满足**(9)**的单模阵** $P(s)$ **和** $Q(s)$ **，如**(10)**对** $Q(s)$ **进行分块；**
2) **计算满足**(12)**式的单模阵** $H(s)$ **和** $L(s)$ **，如**(11)**对** $L(s)$ **进行分块；**
3) **设定** g_i **，** $i=1,2,\Lambda\ ,2n$ **的参量表示，根据**(14)**和**(15)**分别计算矩阵** V **和** W **的参量表达式；**
4) **求解优化问题**(23)**，确定满足**(16)**和**(17)**的一组参量** g_i **，** $i=1,2,\Lambda\ ,2n$ **，将其代回上步计算矩阵** V **和** W **；**
5) **根据**(13)**计算状态反馈增益阵** F **。**

5 数值仿真分析

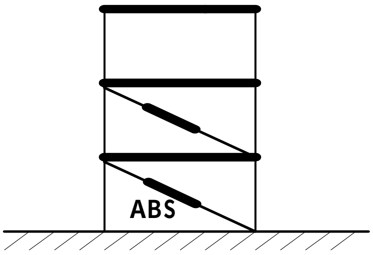

图1 三层剪切型建筑结构模型

考虑如图 1 **所示三层剪切型结构模型**[2,3]**，该模型的结构参数取自三层** Benchmark **模型，但与标准** Benchmark **不同的是采用在底层和中间层设置两个主动拉索控制装置，结构系统矩阵为**

$$M=\begin{bmatrix}981 & & \\ & 981 & \\ & & 981\end{bmatrix}(\mathrm{kg}),$$

$$C=\begin{bmatrix}382.7 & -57.3 & 61.6\\ -57.3 & 456.7 & -2.63\\ 61.6 & -2.63 & 437.3\end{bmatrix}(\mathrm{N\cdot s/m}),$$

$$K=\begin{bmatrix}2.741 & -1.641 & 0.369\\ -1.641 & 3.021 & -1.624\\ 0.369 & -1.624 & 1.333\end{bmatrix}\times 10^{6}(\mathrm{N/m}),$$

$$H=\begin{bmatrix}1 & -1\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

假设待配置的特征值为 $s_{1,2}=-3\pm 14i$ **，** $s_{3,4}=-6\pm 43i$ **，** $s_{5,6}=-9\pm 72i$ **，根据算法** ESA **有如下结果。**

1) **由奇异值分解易算得满足**(9)**的单模阵** $P(s_i)$ **和** $Q(s_i)$ **，** $i=1,2,\Lambda\ ,6$ **，并如**(10)**对** $Q(s_i)$ **进行分块**;
2) **由奇异值分解易求得满足**(12)**的单模阵** $H(s_i)$ **和** $L(s_i)$ **，** $i=1,2,\Lambda\ ,6$ **，并如**(11)**对** $L(s_i)$ **进行分块**;
3) **设定** $g_i=\begin{bmatrix}a_i\\ b_i\end{bmatrix}$ **，** $i=1,2,\Lambda\ ,6$ **，由**(14)**和**(15)**算得**

$$v_1=10^{-4}\begin{bmatrix}(0.0055+0.0098i)a_1+(0.0112i-0.0023)b_1\\ (0.0032+0.0210i)a_1+(0.0036+0.0246i)b_1\\ (0.0010+0.0266i)a_1+(0.0042+0.0315i)b_1\\ (0.0481i-0.1538)a_1-(0.14960-0.0661i)b_1\\ -(0.0180i+0.3050)a_1-(0.0186i-0.3566)b_1\\ -(0.3754+0.0658i)a_1-(0.0352i+0.4539)b_1\end{bmatrix}$$

$$v_2=\overline{v_1}$$

$$v_3=10^{-4}\begin{bmatrix}(0.0008+0.0125i)a_3-(0.0028i+0.0072)b_3\\ (-0.0020+0.0053i)a_3+(0.0016-0.0022i)b_3\\ -(0.0080i+0.0008)a_3-(0.0027-0.0051i)b_3\\ -(0.5438i+0.0400)a_3+(0.3285-0.0789i)b_3\\ -(0.2153+0.1189i)a_3+(0.0867+0.0843i)b_3\\ (0.3493+0.0149i)a_3-(0.2036+0.1471i)b_3\end{bmatrix}$$

$$v_4=\overline{v_3}$$

$$v_5=10^{-4}\begin{bmatrix}-(0.0016-0.0033i)a_5-(0.0003+0.0063i)b_5\\ -(0.0013+0.0030i)a_5+(0.0006+0.0076i)b_5\\ (0.0014i+0.0009)a_5-(0.0014+0.0035i)b_5\\ -(0.2206+0.1424i)a_5+(0.4484+0.0799i)b_5\\ (0.2279-0.0645i)a_5-(0.0227i+0.5560)b_5\\ -(0.1058-0.0496i)a_5+(0.2663-0.0741i)b_5\end{bmatrix}$$

$$v_6=\overline{v_5}$$

$$w_1=\begin{bmatrix}(1-9.8036\times 10^{-10}i)a_1-10^{-9}(3.2297-3.1088i)b_1\\ 10^{-9}(3.1087i-3.2324)a_1+(1-2.7765\times 10^{-9}i)b_1\end{bmatrix}$$

$$w_2=\overline{w_1}$$

$$w_3=\begin{bmatrix}(1+1.9636\times 10^{-10}i)a_3-10^{-11}(52.529i-8.4043)b_3\\ 10^{-11}(8.3967-52.529i)a_3+(1+1.2100\times 10^{-9}i)b_3\end{bmatrix}$$

$$w_4=\overline{w_3}$$

$$w_5=\begin{bmatrix}(1+1.9636\times10^{-10}i)a_5+10^{-11}(8.4043-52.529i)b_5\\10^{-11}(8.3967+52.529i)a_5+(1+1.2100\times10^{-9}i)b_5\end{bmatrix}$$

$$w_6=\overline{w_5}$$

4) 由 matlab 优化工具箱中函数 fmincon 算得优化问题(23)的优化参数为

$$g_1=\overline{g_2}=\begin{bmatrix}7.8380+8.1414i\\9.9984+9.9283i\end{bmatrix},$$

$$g_3=\overline{g_4}=\begin{bmatrix}3.1988+6.9593i\\-2.4319-4.6354i\end{bmatrix},$$

$$g_5=\overline{g_6}=\begin{bmatrix}-3.8124-2.1648i\\7.5629+4.0498i\end{bmatrix}$$

将其代入第三步，易得矩阵 V 和 W 。

5) 根据(13)算得状态反馈增益阵为

$$F=10^5\begin{bmatrix}-0.8626&0.9553&-0.5490&-0.1353&-0.0442&-0.0198\\0.4901&-1.4326&0.7733&0.0891&-0.1159&-0.0786\end{bmatrix}$$

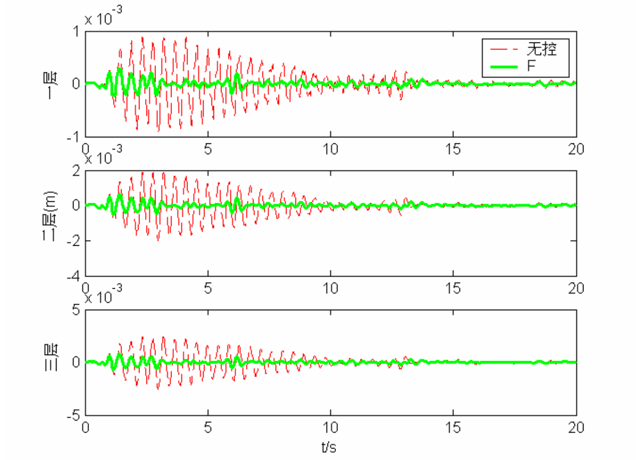

图 2 Elcentro 波作用下无控和 F 控制的结构响应比较

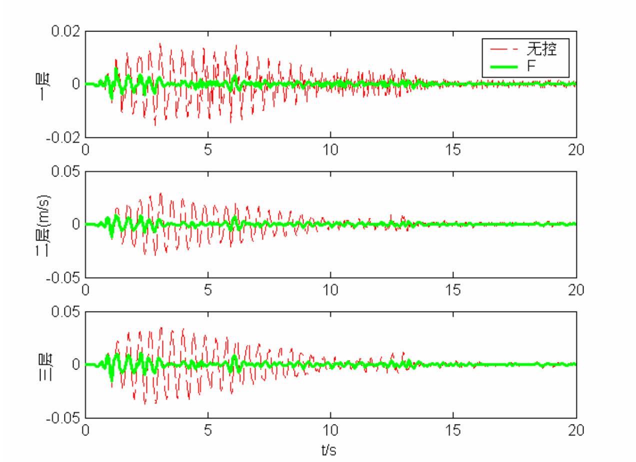

图 3 Elcentro 波作用下无控和 F 控制的各层速度比较

为进一步验证算法有效性，选取输入地震波为 El Centro(S00E)波。图 2、3 和 4 给出了无控和 F 控制结构系统的各层位移、速度和加速度反应曲线，图 5 给出了相应的控制力时程曲线。仿真结果表明，El Centro 地震输入下本文所提算法对结构的位移、速度和加速度响应均能起到良好的控制作用。

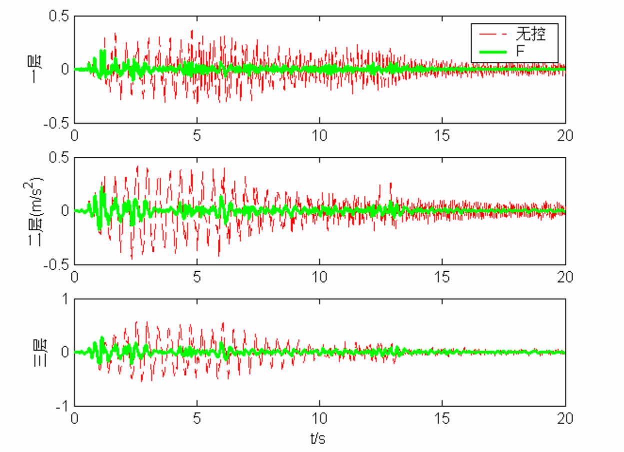

图 4 Elcentro 波作用下无控和 F 控制的各层加速度比较

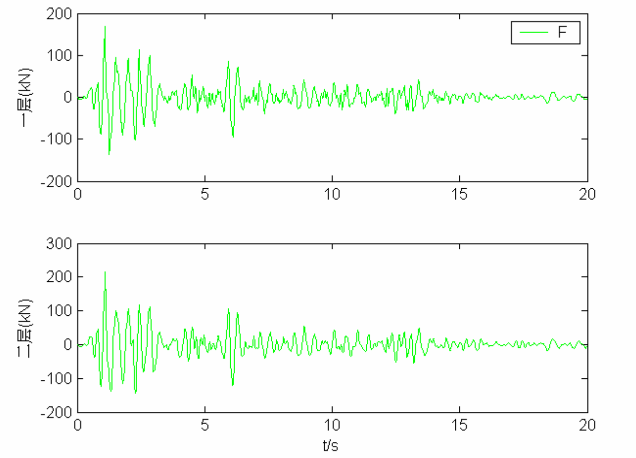

图 5 Elcentro 波作用下 F 控制的各层控制力时程

6 结 论

本文将现代控制理论中的特征结构配置方法引入土木结构中，考虑了结构系统的最优控制问题，利用特征结构配置的一种参数化方法，将该问题转化为含有约束条件的优化问题，并给出了一种简单有效的算法。最后，将利用该算法设计的控制器应用于地震作用下的三层剪切结构建筑模型并进行了仿真分析，仿真结果表明，El centro 地震输入下本文所提算法对结构的位移、速度和加速度响应均能起到良好的控制作用。

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探讨用最优控制方法解力学问题

叶庆凯　刘才山
（北京大学 力学与工程科学系 北京 100871）

摘要：力学原理有微分类原理和变分类原理两大类。微分类原理的数学表现形式是微分方程，而变分类原理是一种极值性原理。将力学中的变分类原理与变分法中的直接方法结合起来，就有可能产生一种求解力学问题的新方法。这种方法完全与微分方程无关。结合平面接触力学问题讨论了这种新方法的实现方案并比较了它与传统的有限单元方法的优缺点。

关键词： 接触，优化

The Discussion about Using Optimal Control Method to Solve Mechanical Problems

Ye Qing-Kai　Liu Cai-Shan
(*Department of Mechanics and Engineering Science, Peking University, Beijing 100871, China*)

Abstract: The mechanical principals have two kinds of classification in classical mechanics: differential principals and variation principals. The mathematic expressions of differential principals are differential equations, and the variation principals can be expressed in the form of searching for the extremum of one mathematic function. Combining the variation principals in mechanics with the direct methods of variation theory in mathematic, a new method that has no any relationship with the differential equations of mechanical system may be induced. To illuminate the scheme of this new method implement in mechanical problems, the plane contact problem is discussed in this paper. Finally, the advantages and disadvantages of this new method compared with the traditional finite element method mostly applied to solve these complicated mechanical problems are presented.

Keywords: Contact Problem, Optimization, Variation Method

1 引 言

在自然界和工程技术中，物质机械运动的表现形态是多种多样的。但从力学学科来看，这么多的错综复杂的机械运动只不过是由为数不多的几个“基本规律”所制约。也可以这样说，客观实际中表现出来的如此复杂的机械运动只是为数不多的几个“基本规律”的逻辑结果。当我们把力学学科逻辑推理的出发点缩小到最小程度之后，这个出发点就是我们所理解的“公理体系”，或简称为“原理”。

存在着多种多样的力学原理。它们主要可以区分为两大类：微分类原理和变分类原理。微分类原理的数学表现形式是微分方程（常微分方程或偏微分方程），例如牛顿三定理。物体的真实运动是这些微分方程的解。因而为获得物体的实际运动，我们需要在给定的初始条件（对于常微分方程）或边界条件（对于偏微分方程）下求出这些微分方程的解。变分类原理是一种极值性原理，例如最小位能原理，最小作用量原理等。按照这种原理，力学系统的实际运动（包括静态平衡）比起与它邻近的某种可能的运动来说，总是使某种函数或某种泛函取极值。

很早以前，就有用变分原理来解力学问题的研究，并取得了许多著名的成果。例如：圆环上两点间的最短距离[1]；超音速流动时的最小阻力形状[2] 共轴放置的两个环支撑的肥皂膜的形状[2]等。但是，在那个时期，人们对微分方程性质的

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了解要比对变分问题的了解深刻得多，而且只在微分方程形式下才能得到精确的分析解，因而虽然用的是变分原理，但在解具体的力学问题时，还是先通过变分法中的 Euler 方程将变分问题转换成微分方程形式再求解的。这样，事实上，在那个时期，不论从变分类原理出发，还是从微分类原理出发，结果都是归结为求解微分方程（常微分方程或偏微分方程)。

但是，实际情况下，处理微分方程的边值问题一般是困难的，能够得到精确分析解的情况并不多。可以说，把变分问题归结为微分方程的边值问题，往往反而把问题复杂化了。

二十世纪 60 年代以后,随着数字计算机的普及，一般矢量空间中极值问题求解方法的深入探讨,非线性领域中快速且可靠的计算方法的提出,使得变分问题有可能按其原有形式进行处理比转换为微分方程后再进行处理更方便一些。由此发展了完全不依赖于 Euler 方程而直接以极小问题为讨论对象的变分法中的直接方法。

将力学中的变分类原理与变分法中的直接方法结合起来，就有可能产生一种求解力学问题的新方法。这种方法完全与微分方程无关。由于变分法中的直接方法最早是在最优控制领域中发展起来的，这种方法称为用最优控制方法解力学问题。曾用这种方法求解平板中心有小孔时的应力集中问题[3]，谐波减速器中柔轮的应力问题[4]，喷管中的跨音速流动问题[5]。得到了比较理想的结果。

本文以上述几个问题为依托来探讨用最优控制方法解力学问题的基本步骤并将它与力学问题的传统解法（有限单元法等）进行比较。

2 力学中的变分原理

对于一个由 N 个质点组成的质点系而言，分析力学的普遍原理是

$$\sum_{i=1}^{3N}(F_i - m_i \ddot{x}_i)\delta_i = 0$$

其中 F_i 是作用在质点系上的给定力，包括非理想的约束力；$-m_i \ddot{x}_i$ 是质点系的惯性力；δ_i 是质点系约束组 π 的微变线性空间 $\varepsilon(\pi)$ 中的任一元素，亦即是任一可能的微位移。如果考虑的是平衡问题,而且假定作用在质点系上的诸力均是有势的，则上述原理转化为最小位能原理：在静止的平衡力学系统的所有容许位移中，真实的位移将使位能的变分为零。将此原理应用到弹性体这一具体情况，就有：对一个有势力作用下的保守，完整的力学系统的平衡问题而言，真实的位移将使位能的变分为零，即

$$\delta V = 0$$

若平衡是稳定的，则位能 V 应取最小值。

一般地说，一个弹性体中的总的位能由三部分组成。一部分是贮存于整个弹性体中的形变能 V_1；第二部分是作用在弹性体上的体积力产生的位能 V_2；第三部分是作用在弹性体表面的表面力产生的位能 V_3。

$$V = V_1 + V_2 + V_3$$

当体积力与表面力给定时，它是位移 u 及其梯度 $\partial u/\partial x$ 的泛函。可见，弹性体的平衡问题乃是索伯列夫空间中的极值问题，它有可能用最优控制中的计算方法来求解。为了套用最优控制中的直接方法,希望将指标泛函 V 表达为位移 u 的函数。为此我们需要将 $\partial u/\partial x$ 表示成 u 的函数。这可以将弹性体所占据的体积划分为若干个小单元，并在这些小单元中定义位移 u 所服从的变化规律来达到。对于我们所处理的平面问题，通常可取三角形单元，并假定在这些单元内部位移 u 服从线性变化规律。

当然，与有限单元方法不同，这里将弹性体所占据的体积划分为若干个小单元只是为了将 $\partial u/\partial x$ 表示成 u 的函数，而不是为了求解偏微分方程。

3 最优控制中的共轭梯度方法

从上面的讨论可知，一个平面力学问题可以提成为一个索伯列夫空间中的带有约束的极值问题。可以用最优控制中的共轭梯度方法来求解。其具体步骤如下：

(1) 在索伯列夫空间中给定一组满足约束的位移 u^0，v^0。并置 $i = 0$；$\varepsilon > 0$。

(2) 计算 $V(u^0, v^0)$ 及其梯度 $\nabla_u V(u^0, v^0)$ 和 $\nabla_v V(u^0, v^0)$。

(3) 若 $[\,|\nabla_u V(u^0, v^0)|^2 + |\nabla_v V(u^0, v^0)|^2\,]^{1/2} < \varepsilon$，停止；否则，令

$$h_0^{'} = g_0^{'} = -\nabla_u V(u^0, v^0),$$

$$h_0^{''} = g_0^{''} = -\nabla_v V(u^0, v^0)。$$

(4) 进行单维搜索，即计算 $\lambda^* > 0$ 使得

$$V(u^i + \lambda^* h_i^{'}, v^i + \lambda^* h_i^{''}) = \min_{\lambda}\{V(u^i + \lambda h_i^{'}, v^i + \lambda h_i^{''})\}$$

令 $u^{i+1} = u^i + \lambda^* h_i^{'}$，$v^{i+1} = v^i + \lambda^* h_i^{''}$

并计算梯度 $\nabla_u V(u^{i+1}, v^{i+1})$ 和 $\nabla_v V(u^{i+1}, v^{i+1})$。

(5) 若 $[\,|\nabla_u V(u^{i+1}, v^{i+1})|^2 + |\nabla_v V(u^{i+1}, v^{i+1})|^2\,]^{1/2} < \varepsilon$，仃止；否则，令

$g_{i+1}^{'} = -\nabla_u V(u^{i+1}, v^{i+1})$，　$g_{i+1}^{''} = -\nabla_v V(u^{i+1}, v^{i+1})$

$$h_{i+1}^{'} = g_{i+1}^{'} + \gamma h_i^{'}, \quad h_{i+1}^{''} = g_{i+1}^{''} + \gamma h_i^{''}$$

其中:

$$\gamma = \frac{(g_{i+1}^{'}, g_{i+1}^{'} - g_i^{'}) + (g_{i+1}^{''}, g_{i+1}^{''} - g_i^{''})}{\|g_i^{'}\| + \|g_i^{''}\|}$$

(6) 置 $i = i+1$，进行(4)。

在上述迭代过程中，每次求得位移 u^i，v^i 后，为保证其满足指定的约束，应在其上作用约束算子；每次求得搜索方向 $h_i^{'}$，$h_i^{''}$ 后，为保证搜索过程中不破坏约束，应在其上作用投影算子。

为实现上述计算，需要给出的是在给定了位移 u^i，v^i 后计算势能及其梯度的方法，以及约束算子（亦即如何强制位移 u^i，v^i 满足约束）和投影算子（亦即如何改变共轭梯度方向以保证搜索过程中不破坏约束）的形式。

4 新方法的优缺点分析

(1) 要能使用上述方法，所解的力学问题必须能表示为某个泛函的极值问题。

(2) 传统的有限单元方法是一种微分方法，而新方法是一种积分方法。微分方法给出每一个局部应满足的方程；而积分方法给出力学系统整体上应满足的方程。因而用积分方法求出的结果有可能在某些局部有较大的误差，而其整体性质却是完全可信的。

(3) 微分方法归结为求解微分方程（常微分方程或偏微分方程），因而必须事先给定边界条件。积分方法归结为求解有约束的极值问题，因而必须事先给定约束条件，但不必事先给定边界条件，这特别适用于无法事先给定边界条件的接触问题和跨音速问题。

(4) 求解微分方程在采用有限单元方法离散化后归结为求解线性代数方程组，因而需要求矩阵的逆。在矩阵维数庞大且为稀疏的时候，为使其逆仍是稀疏矩阵，必须对有限单元方法中的节点编号进行精心设计，这成为使用有限单元方法时的一个极具技巧的问题。求解有约束的极值问题时，在离散化后归结为反复进行矩阵相乘。即使矩阵维数庞大且为稀疏，由于我们只须储存矩阵中的非零元素就可以了，因而对离散化时节点的编号没有任何要求。

5 结 论

力学原理有微分类原理和变分类原理两大类。微分类原理的数学表现形式是微分方程，而变分类原理是一种极值性原理。将力学中的变分类原理与变分法中的直接方法结合起来，就有可能产生一种求解力学问题的新方法。这种方法完全与微分方程无关，而归结为求解有约束的极值问题，因而必须事先给定约束条件，但不必事先给定边界条件。这特别适用于无法事先给定边界条件的接触问题和跨音速问题。

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Suboptimal LQR Problem:Controller Uncertainty and Static Output Feedback Controller

Xu Xiaojie
School of Electrical Engineering,Wuhan University,Wuhan,430072,China
E-mail:xiaojiex@public.wh.hb.cn

Abstract: This paper considers the connection between controller uncertainty and static output feedback controller for suboptimal LQR problem.Necessary and sufficient conditions for the existence of the suboptimal LQR controller with controller uncertainty are given. Based on this , this paper shows how to use the result of suboptimal LQR problem with controller uncertainty to solve the static ourput feedback suboptimal LQR problem, and gives necessary and sufficient conditions for the existence of the static output feedback LQR controller

Key words: Optimal control, controller uncertainty, static output feedback.

1 INTRODUCTION

This paper considers the connection between suboptimal LQR problem with controller uncertainty and static output feedback (SOF) LQR problem. The SOF problem is practically important and theoretically appealing (Bernstein 1992, Blondel et al. 1995 and Syrmos et al. 1997). Recently, several approaches have been made for the SOF problems (Geromel et al. 1996, Moheimani et al. 1996, Geromel et al. 1998, Cao et al. 1998, Cao et al. 1999, Crusius et al. 1999, Benton et al. 1999, Fridman et al. 2000, Kim et al. 2000,Leibfritz 2001,Garcia et al. 2003, Leibfritz et al. 2003).

Xu (2003) presented analytical necessary and sufficient conditions for the existence of static output feedback controller.The result of Xu (2003) leads to a central solution of the admissible SOF controllers.

It should be noted that Kucera et al. (1995) presented a necessary and sufficient condition for static output feedback stabilizable.In Kucera et al. (1995), using the algorithm of Geromel and Peres (1985), an admissible SOF controller is yielded.However,the convergence for the algorithm of Geromel and Peres (1985) remained to be proved.

Also, we note that Keel et al. (1997) showed that, relatively small uncertainties/perturbations in controller parameters could even destabilize the consider closed-loop system. In order to design controller to be able to tolerate some lever of uncertainty in its parameters, Yang et al. (1999) ,Kim et al (1999) and Wu et al. (2002) investigated the controller uncertainty problems.

The main purpose of this paper is to establish the connection between the controller uncertainty and static output feedback controller for suboptimal LQR problem. Necessary and sufficient conditions for the existence of suboptimal LQR controller with controller uncertainty are given. Based on this and the result of Kucera et al. (1995) and the algorithm of Geromel and Peres (1985), we give a solution of static output feedback suboptimal LQR controller. Necessary and sufficient conditions for the existence of suboptimal LQR controller are given. Also, we show that the condition of the convergence for the algorithm of Geromel and Peres (1985).

The organization of this paper is as follows:in Section 2, we define the problems, and give some preliminary results. Section 3 contains the mian results.

2 DEFINITION AND PRELIMINARIES

Consider the controllable-observable linear time-invariant system

$$\dot{x} = Ax + Bu \tag{2.1}$$

$$y = Cx \tag{2.2}$$

where, $x \in R^n$ is the state, $u \in R^m$ is the control input, $y \in R^r$ is the measured output. A, B, C are known matrices of appropriate dimensions.Assume that r is less than n .

Now, let us define the state feedback suboptimal LQR problem with controller uncertainty and the static output feedback suboptimal LQR problem .

(1) The state feedback suboptimal LQR problem with controller uncertainty:

Consider the controllable-observable system (2.1)-(2.2) under the influence of state feedback of the form

$$u(t) = \hat{K}x(t)\ ,\ \hat{K} = (K + \Delta K) \tag{2.3}$$

with the cost function

$$J(\hat{K}) = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))dt$$

where, $Q > 0$ and $R > 0$;a controller $\hat{K}$ is said to define a suboptimal LQR controller with controller uncertainty if there exists a symmetric positive definite matrix X such that

$$\begin{aligned}&(A + B(K + \Delta K))^T X + X(A + B(K + \Delta K)) + Q\\&\quad + (K + \Delta K)^T R(K + \Delta K) < 0\end{aligned} \tag{2.4}$$

for any admissible uncertainty ΔK .The controller uncertainty considered here is assumed to be of following structure:

$$\Delta K = D_k \Delta E_k \tag{2.5}$$

where, D_K and E_K are known matrices of appropriate dimensions, Δ is an uncertain matrix satisfying

$$\Delta^T\Delta \le I \qquad (2.6)$$

with the elements of Δ being Lebesgue measurable.

(2)The SOF suboptimal LQR control problem:

Consider the controllable-observable system (2.1)-(2.2) under the influence of state feedback of the form

$$u(t)=K_{SOF}\,y(t)$$

with the cost function

$$J(K_{SOF})=\int_0^\infty (x^T(t)Qx(t)+u^T(t)Ru(t))dt$$

where, $Q>0$ and $R>0$; a controller K_{SOF} is said to define a SOF suboptimal LQR controller if there exists a symmetric positive definite matrix X such that

$$(A+BK_{SOF}C)^T X+X(A+BK_{SOF}C)+Q$$
$$+(K_{SOF}C)^T R(K_{SOF}C)<0 \qquad (2.7)$$

The following theorem is from Kucera et al.(1995).

Theorem 2.1 There exists a static output feedback suboptimal LQR controller iff for a sufficiently small scalar $\delta>0$, there exist real matrices K_{SOF} and M such that

$$R^{\frac{1}{2}}(K_{SOF}C+R^{-1}B^TX)=H \qquad (2.8)$$

where, X is the real symmetric positive definite solution of

$$A^TX+XA-XBR^{-1}B^TX+Q+H^TH+\delta I=0 \qquad (2.9)$$

It can be assumed without loss generality that $C=[C_r \quad C_{n-r}]$ such that C_r is invertible,and $K=[K_r \quad K_{n-r}]$,where, $C_r\in R^{r\times r}$, $C_{n-r}\in R^{r\times(n-r)}$, $K_r\in R^{m\times r}$, $K_{n-r}\in R^{m\times(n-r)}$.

Define the set of all admissible SOF controllers as

$$\Omega_L:=\{K_{SOF};A+BK_{SOF}C \text{ asym. stable}\}$$

and the set of all corresponding admissible state feedback controllers as

$$\Omega_K:=\{K:A+BK \text{ asym. stable}\}$$

Xu (2003) has given a central solution of the admissible SOF controllers as follows:

Theorem 2.2 Suppose that there exists a SF controller K that belongs to the set Ω_K, then there exists a SOF controller K_{SOF} that belongs to the set Ω_L iff the set $\Omega_0:=\{K:K_{n-r}=K_rC_r^{-1}C_{n-r}\}$ is not empty.

Moreover, this admissible SOF controller K_{SOF} is given by

$$K_{SOF}=KC(CC^T)^{-1}$$

Xu [26] has shown that there exists a stable region near the central solution of admissible SOF controllers if one assign the closed-loop poles to have some level stability margin using the state feedback controller. This result is demonstrated by the following example in Xu [26].

Example 2.1: Consider the system (2.1)-(2.2) , its parameter matrices are as follows:

$$A=\begin{bmatrix}1.8 & 0 & 1\\ 1.5 & 2 & 0\\ -2 & -1 & 1.5\end{bmatrix},B=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix},$$

$$C=\begin{bmatrix}1.3 & 2 & -0.7183\\ 0 & 1.2 & 0.3892\end{bmatrix}$$

The above system is controllable-observable and the eigenvalues of open-loop system are as follows:

$$p_1=2.0058+1.4519i, p_2=2.0058-1.4519i,$$
$$p_3= \ 1.2885;$$

thus the considered open-loop system is not stable.

(i) Let the desired closed-loop poles be as follows:

$p_1=-3.9023+1.3800i$, $p_2=-3.9023-1.3800i$,

$p_3=-6.2324$.

Then using MATLAB, we can find a state feedback controller as follows:

$$K=\begin{bmatrix}-11.4459 & -11.8646 & 8.7010\\ -1.6463 & -6.9693 & -0.9218\end{bmatrix}$$

thus, $K_{n-r}\neq K_rC_r^{-1}C_{n-r}$. In this time,we compute K_{SOF} using the formula: $K_{SOF}=KC^T(CC^T)^{-1}$, then K_{SOF} is as follows: $K_{SOF}=\begin{bmatrix}-8.9924 & 5.1630\\ -1.1228 & -3.9845\end{bmatrix}$ and the eigenvalues of $A+BK_{SOF}C$ are as follows:

$p_1=-4.0046+0.6271i$, $p_2=-4.0046-0.6271i$,

$p_3=-6.1521$.

Clearly, $\Delta K=K_{SOF}C-K_{SF}$ is admissible, the static output feedback controller K_{SOF} that stabilizes $A+BK_{SOF}C$ is given by $K_{SOF}=\begin{bmatrix}-8.9924 & 5.1630\\ -1.1228 & -3.9845\end{bmatrix}$.

(ii) Let the desired closed-loop poles be as follows:

$p_1=-0.1058+1.4520i$, $p_2=-0.1058-1.4520i$,

$p_3=-0.2885$.

Then we can determine a state feedback controller as follows:

$$K_{SF}=\begin{bmatrix}-4.1018 & -2.4851 & 0.5718\\ -0.0987 & -1.0943 & -0.6040\end{bmatrix}$$

Thus, $K_{n-r}\neq K_rC_r^{-1}C_{n-r}$,at this time, we still compute K_{SOF} using the formula: $K_{SOF}=K_{SF}C^T(CC^T)^{-1}$, then K_{SOF} is as follows: $K_{SOF}=\begin{bmatrix}-2.0813 & 1.0392\\ 0.0532 & -1.0437\end{bmatrix}$ and the eigenvalues of $A+BK_{SOF}C$ are as follows:

$p_1=0.6505+2.8905i$, $p_2=0.6505-2.8905i$,

$p_3=-0.2972$.

Obviously, $\Delta K=K_{SOF}C-K_{SF}$ is not admissible, and $A+BK_{SOF}C$ is not stable.

Remark 2.1 From Theorem 2.2 it is evident that if the set $\Omega_0:=\{K:K_{n-r}=K_rC_r^{-1}C_{n-r}\}$ is not empty for $K=-R^{-1}B^TX$,where X is the real symmetric positive-definite solution of Riccati equation (2.9); then an admissible SOF controller given by $K_{SOF}=KC(CC^T)^{-1}$ leads to a central solution of the SOF LQR controllers.

3 MAIN RESULT

In this section, first, we show that the result displayed in Theorem 2.1 by Kucera et al. (1995) ignored some condition.

For a sufficiently small scalar $\delta > 0$, The formula (2.4) can be rearranged as follows:

$$A^T X + XA - XBR^{-1}B^T X + Q$$

$$+(K + \Delta K - R^{-1}B^T X)^T R(K + \Delta K - R^{-1}B^T X) + \delta I = 0 \quad (3.1)$$

It is clear that if $K + \Delta K = K_{SOF} C$,then (2.9) equals (3.1).Together with Theorem 2.2, if the set Ω_0 is empty, then $\Delta K = K_{SOF} C - K \neq 0$ .From the two definitions displayed in Section 2, it is necessary that ΔK is admissible for the existence of the suboptimal LQR controller with controller uncertainty and the static output feedback suboptimal LQR controller. This fact has shown that the result displayed in Theorem 2.1 by Kucera et al. (1995) ignored the condition that ΔK is admissible.

Based on the above, we correct the result displayed in Theorem 2.1 by Kucera et al. (1995) as follows:

Theorem 3.1 There exists a static output feedback suboptimal LQR controller iff for a sufficiently small scalar $\delta > 0$, there exist real matrices K_{SOF} and M such that the following two conditions hold.

(i) $\Delta K = K_{SOF} C + R^{-1}B^T X \quad (3.2)$

where, X is the real symmetric positive definite solution of

$$A^T X + XA - XBR^{-1}B^T X + Q + \Delta K^T R \Delta K + \delta I = 0 \quad (3.3)$$

(ii) ΔK is admissible.

Moreover, this SOF suboptimal LQR controller is given by

$$K_{SOF} = -R^{-1}B^T XC^T (CC^T)^{-1}$$

Next,let us consider the state feedback suboptimal LQR control problem with uncertainty. The solution of state feedback suboptimal LQR control problem with controller uncertainty involves the Riccati equation

$$A^T X + XA - XBR^{-1}B^T X + \rho^2 E_k^T E_k + Q + \delta I = 0 \quad (3.4)$$

where, ρ is a positive scalar; $U = \rho^2 I - D_k^T RD_k > 0$.

Now, we can state the necessary and sufficient conditions for the existence of suboptimal LQR controller with controller uncertainty as follows:

Theorem 3.2 There exists a state feedback suboptimal LQR controller with controller uncertainty iff for a sufficiently small scalar $\delta > 0$, there exists a symmetric positive definite solution X that satisfies the Riccati equation (3.4).

Moreover,if this condition is met,the state feedback suboptimal LQR with controller uncertainty is given by

$$K = -R^{-1}B^T X \quad (3.5)$$

Proof:Sufficiency:Suppose that there exists a symmetric positive definite solution X that satisfies the Riccati equation (3.4), let $\overline{K} = -R^{-1}B^T X + D_k \Delta E_k$ and $A_c = A + B\overline{K}$,it follows from the square completion that

$$A_c^T X + XA_c + Q + \overline{K}^T R\overline{K}$$

$$= A^T X + XA + Q - XBR^{-1}B^T X + \Delta\overline{M} \quad (3.6)$$

where, $\Delta\overline{M} = E_k^T \Delta^T D_k^T RD_k \Delta E_k$.

Note that $U = \rho^2 I - D_k^T RD_k > 0$ and

$$\Delta\overline{M} = E_k^T \Delta^T D_k^T RD_k \Delta E_k$$

$$= -E_k^T \Delta^T (\rho^2 I - D_k^T RD_k)\Delta E_k + \rho^2 E_k^T \Delta^T \Delta E_k$$

$$\leq \rho^2 E_k^T E_k \quad (3.7)$$

By considering (3.6) and (3.7),we have

$$A_c^T X + XA_c + Q + \overline{K}^T R\overline{K}$$

$$\leq A^T X + XA - XBR^{-1}B^T X + \rho^2 E_k^T E_k + Q$$

and it follows from (3.4) that

$$A_c^T X + XA_c + Q + \overline{K}^T R\overline{K} < 0$$

Necessity: Suppose that there exists a state feedback suboptimal LQR controller with controller uncertainty, i.e., there exists a symmetric positive definite matrix X such that

$$(A + B(K + D_k \Delta E_k))^T X + X(A + B(K + D_k \Delta E_k))$$

$$+Q + (K + D_k \Delta E_k)^T R(K + D_k \Delta E_k) < 0 \quad (3.8)$$

or

$$(A + BK)^T X + X(A + BK) + Q + K^T RK + \Delta M < 0$$

$$\Delta M = E_k^T \Delta^T D_k^T (RK + B^T X) + (K^T R + XB)D_k \Delta E_k$$

$$+E_k^T \Delta^T D_k^T RD_k \Delta E_k$$

Note $U = \rho^2 I - D_k^T RD_k > 0$,it follows from the square completion for the above formula that

$$\Delta M = \rho^2 E_k^T \Delta^T \Delta E_k + (K^T R + XB)D_k (\rho^2 I - D_k^T RD_k)^{-1} D_k^T$$

$$\times(RK + B^T X) - ((K^T R + XB)D_k (\rho^2 I - D_k^T RD_k)^{-1} + E_k^T \Delta^T)$$

$$\times(\rho^2 I - D_k^T RD_k)((\rho^2 I - D_k^T RD_k)^{-1}(RK + B^T X) + \Delta E_k)$$

$$\leq \rho^2 E_k^T E_k + (XB + K^T R)D_k (\rho^2 I - D_k^T RD_k)^{-1} D_k^T (B^T X + RK)$$

and

$$(A + BK)^T X + X(A + BK) + Q + K^T RK + \Delta M$$

$$\leq (A + BK)^T X + X(A + BK) + Q + K^T RK + \rho^2 E_k^T E_k$$

$$+ (XB + K^T R)D_k (\rho^2 I - D_k^T RD_k)^{-1} D_k^T (B^T X + RK)$$

Thus, we conclude that there exists a symmetric positive definite matrix X such that (3.8) holds if there exists a symmetric positive definite matrix X such that

$$(A + BK)^T X + X(A + BK) + Q + K^T RK + \rho^2 E_k^T E_k$$

$$+ (XB + K^T R)D_k (\rho^2 I - D_k^T RD_k)^{-1} D_k^T (B^T X + RK) + \delta I = 0$$

or

$$A^T X + XA - XBR^{-1}B^T X + \rho^2 E_k^T E_k + Q + \delta I = 0$$

$$K = -R^{-1}B^T X$$

where, ρ is a positive scalar; $U = \rho^2 I - D_k^T RD_k > 0$.

Next,we establish a connection between the controller uncertainty and static output feedback controller for suboptimal LQR problem , and show how to use the result of suboptimal LQR problem with controller uncertainty to solve static output feedback suboptimal LQR problem.

Theorem 3.3 Suppose that there exists a factorisation of $\Delta K_c = K_{SOF} C + R^{-1}B^T X = K(C^T (CC^T)^{-1} C - I) = D_k \Delta_1 E_k$, where, $K = -R^{-1}B^T X$ and X is a symmetric positive definite solution to (3.2) and (3.3). Then there exists a SOF suboptimal LQR controller iff there exists a symmetric positive definite solution to the Riccati equation (3.4) for any admissible controller uncertainty ΔK that satisfies (2.5) and (2.6).

Moreover, this SOF suboptimal LQR controller is given by

$$K_{SOF} = -R^{-1}B^T XC^T (CC^T)^{-1}$$

It is clear that Theorem 3.3 does not say how to determine the real matrices D_k and E_k. However, In Kucera et al. (1995), the algorithm of Geromel and Peres (1985) is used to iterate the Riccati equation (3.3) up until a controller uncertainty $\Delta K_c = R^{-\frac{1}{2}}H$ is found that satisfies the constraint (3.2).This algorithm can be used to determine the real matrices D_k and E_k.

Step 1:Fix the two weighting matrices Q and R, a sufficiently small positive scalar δ, set $i = 0$, and $\Delta K_i = 0$.

Step 2:Solve the equation

$$A^T X_i + X_i A - X_i BR^{-1}B^T X_i + Q + \Delta K_i^T R\Delta K_i + \delta I = 0$$

for X_i symmetric positive definite.

Step 3:Set $\Delta K_{i+1} = -R^{-1}B^T X_i (C^T (CC^T)^{-1} C - I)$ increase i by 1 and go to Step 2.

If $\Delta K_0, \Delta K_1, \Delta K_2, \Lambda$ converges ,say to ΔK_c ,and ΔK_c is admissible, then the sequence X_0, X_1, X_2, Λ converges, say to X, the both (3.2) and (3.3) are satisfied for

$$K_{SOF} = -R^{-1}B^T XC^T (CC^T)^{-1}$$

and ΔK_c is given by

$$\Delta K_c = -R^{-1}B^T X(C^T (CC^T)^{-1} C - I).$$

Moreover, we factorize ΔK as follows:

$$\Delta K_c = -R^{-1}B^T X(C^T (CC^T)^{-1} C - I) = D_k \Delta_1 E_k$$

where, Δ_1 is constant matrix that satisfies

$$\Delta_1^T \Delta_1 \le I$$

Since ΔK_c has the same structure as ΔK that satisfies (2.5) and (2.6), the real matrices D_k and E_k are obtained using the above algorithm.

Proof of Theorem 3.3: Sufficiency:suppose that there exists a symmetric positive definite solution to the Riccati equation (3.4) for any admissible $\Delta K = D_k \Delta E_k$. By Theorem 3.1, $A + B(K + \Delta K) = A + BK_{SOF} C$ is asym. stable for $K + \Delta K = K_{SOF} C$ with $K = -R^{-1}B^T X$ and $K_{SOF} = -R^{-1}B^T XC^T (CC^T)^{-1}$.

Using the same argument as in the sufficient part of the proof of Theorem 3.2, we obtain

$$A_c^T X + XA_c + Q + \overline{K}^T R\overline{K} < 0$$

or

$$(A + BK_{SOF} C)^T X + X(A + BK_{SOF}) + Q$$
$$+(K_{SOF} C)^T R(K_{SOF} C) < 0$$

Necessity: suppose that there exists a SOF suboptimal LQR controller , then there exists a symmetric positive definite matrix X such that

$$(A + BK_{SOF} C)^T X + X(A + BK_{SOF} C) + Q$$
$$+(K_{SOF} C)^T R(K_{SOF} C) < 0$$

Note that the above formula equals the following one

$$(A + B(K + D_k \Delta E_k))^T X + X(A + B(K + D_k \Delta E_k))$$
$$+Q + (K + D_k \Delta E_k)^T R(K + D_k \Delta E_k) < 0$$

if $\Delta K = K_{SOF} C - K = D_k \Delta E_k$ is admissible.

Using the same argument as in the sufficient part of the proof of Theorem 3.2, we conclude that there exists a symmetric positive definite matrix X such that

$$A^T X + XA - XBR^{-1}B^T X + \rho^2 E_K^T E_K + Q + \delta I = 0$$

and the state feedback suboptimal LQR with controller uncertainty is

$$K = -R^{-1}B^T X$$

If $\Delta K = K(C^T (CC^T)^{-1} C - I) = D_k \Delta E_k$ is admissible,then one solution to the SOF suboptimal LQR controller is

$$K_{SOF} = KC^T (CC^T)^{-1}.$$

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