PRACTICAL OUTPUT-FEEDBACK RISK-SENSITIVE CONTROL FOR STOCHASTIC NONLINEAR SYSTEMS WITH STABLE ZERO-DYNAMICS

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Abstract: This paper addresses the design problem of practical (or satisfaction) output-feedback controls for stochastic strict-feedback nonlinear systems in observer canonical form with stable zero-dynamics under long-term average tracking risk-sensitive cost criteria. The cost function adopted here is of quadratic-integral type usually encountered in practice, rather than of quartic-integral one used to avoid difficulty in control design and performance analysis of the closed-loop system. By coordinate diffeomorphism the zero-dynamics are separated from the entire system so that the transformed system has an appropriate form suitable for integrator backstepping design. For any given risk-sensitive parameter and desired cost value, by using the integrator backstepping methodology, an output-feedback control is constructively designed such that (a) the closed-loop system is bounded in probability and, (b) the long-term average risk-sensitive cost is upper bounded by the desired value. Furthermore, under some additional conditions, another output-feedback control is designed to ensure the closed-loop system be asymptotically stable in the large and admit a zero risk-sensitive cost. Among others, this paper does not require the uniform boundedness of the gain functions of the system noises.

Key words: Nonlinear stochastic systems, integrator backstepping methodology, risk-sensitive cost criterion, output-feedback, stable zero-dynamics.

1 Introduction

Research on the global stabilization control design for nonlinear systems has been accelerated over the recent two decades. After the celebrated characterization of the feedback linearizable systems (see [10]), a breakthrough is achieved with the introduction of the integrator backstepping design methodology (see [16]), which provides a general constructive tool for designing global stabilization controls for nonlinear systems in or feedback equivalent to strict-feedback form. Since the early 1990s, a series of research results on strict-feedback systems have been obtained by using this methodology together with other design tools such as nonlinear damping, tuning functions, and MT filters (see. e.g., [7], [11], [15], [14], [18], [19], [24], [26], and [29]).

The design of the controls for strict-feedback stochastic nonlinear systems has received intense investigation recently (see, e.g., [1], [5], [6], [8], [20], [23], [21], [22], [25] and [27]), where [6], [8] and [25] considered full state-feedback control design, and [1], [5], [20], [23], [21], [22] and [27] considered output-feedback control design. Under the assumption (A): "the disturbance vector field vanishes at the origin", [5], [6] and [8] studied the problem of designing a control to asymptotically stabilize the closed-loop systems in the large. While [1], [20], [23], [21], [22], [25] and [27] considered the control design to achieve the boundedness in probability of the closed-loop system without using the assumption (A). Specifically, [6] considered the disturbance attenuation problem; [27] considered the stabilization problem of systems with stable zero-dynamics; [25], [20], [1] and [23] considered the design of satisfaction control under a quadratic, a quartic regulation and quadratic tracking risk-sensitive cost criterion, respectively. [1] used the assumption (B): "the gain functions of stochastic noise are uniformly bounded", while [20], [23] and [25] did not; [21] and [22] considered the reduced-order observer-based stabilization control design of the single-input multi-output stochastic nonlinear systems.

This paper studies the problem of output feedback control design for a class of stochastic nonlinear systems in observer canonical form with stable zero-dynamics under a quadratic tracking risk-sensitive cost criterion. In general, the design of output-feedback control is more difficult and challenging than that of full state-feedback control. Since the early 1990s, a general framework to study output-feedback control problem has been developed. The key thought is to first introduce the so-called INFORMATION STATE, which is a generalization of observer or filter, and then, by a measure transformation, to change the output-feedback control design problem into a full state-feedback one of an augmented system (see, e.g., [2], [3], [9], [13], or [12]). However, generally speaking, the equality (or inequality) of the information state satisfied is infinite-dimensional, to which an explicit finite-dimensional solution exists only for linear or special nonlinear systems (see [2]). The method of this paper is different from the information state one, and can be used to deal with more general inherently nonlinear sys-

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tems. The objective of this paper is very practical: to search for a SATISFACTION control rather than an optimal one. This makes it possible to avoid the measure transformation. In order to get the explicit formula of the control, strict-feedback nonlinear systems are considered. The main results of this paper indicate that: for any given risk-sensitive parameter and desired tracking risk-sensitive cost value, a dynamic output-feedback control can always be constructively designed so that the closed-loop system is bounded in probability and the long-time average risk-sensitive cost is upper bounded by the desired value. Under certain conditions, we can design control so that closed-loop system is asymptotically stable in the large and the long-time average risk-sensitive cost equals zero. Unlike [1], where the assumption (B) is essential, this paper does not use this assumption. In addition, the value range of the characteristic parameter of the value function used for backstepping design is enlarged from $\frac{2}{3}$ (see [20]) to set $(\frac{1}{2}, 1)$. This provides control designers with a freedom in choosing value function.

The remainder of the paper is organized as follows. Section 2 describes the system model and formulates the control problem to be studied. Section 3 describes the constructive design procedure of the control by employing integrator backstepping approach. Section 4 addresses the main results of this paper. Under some additional conditions, Section 5 considers the control design of asymptotically tracking almost surely and zero tracking long-term average risk-sensitive cost. Section 6 gives some concluding remarks. The paper ends with an appendix, which introduces the definitions of stability notions: *asymptotically stable in the large* and *bounded in probability*(see [17]), a key theorem of sufficient conditions for the solvability of the control problem, and provides two technical lemmas which play an important role in the control design and performance analysis.

Throughout this paper, $x_{[i]}$ denotes $x_{[i]} = [x_1, \ldots, x_i]^\tau$; for any given ith continuously differentiable function $y_d(t)$, $y_d^{(i)}(t)$ denotes the ith derivative with respect to the time variable t, the first and second derivatives are denoted by $\dot y_d$ and $\ddot y_d$ respectively, and $y_d^{[i]}$ denotes the $(i+1)$-dimensional column vector consisting of $y_d, \dot y_d, \ldots, y_d^{(i)}$, i.e., $y_d^{[i]} = \left[y_d, \dot y_d, \ddot y_d, \ldots, y_d^{(i)}\right]^\tau$. Obviously, $x_{[1]} = x_1$, $x_{[n]} = x$, $y_d^{[0]} = y_d$. $0_{i\times j}$ denotes the $(i\times j)$-dimensional matrix with all zero elements and will be written as 0 for brevity when there is no confusion caused. We use I_i to denote the identity matrix of i-dimension.

2 Problem Formulation

2.1. System model

Consider the stochastic nonlinear systems as following (see [27]):

$$\begin{cases} dx_i = x_{i+1}\,dt + f_i(y)\,dt + h_i(y)\,dw, \\ \qquad i = 1, \ldots, \rho-1, \\ dx_j = [x_{j+1} + f_j(y) + b_k g(y) u]\,dt + h_j(y)\,dw, \\ \qquad j = \rho, \ldots, n; k = m - j + \rho;\ x_{n+1} = 0, \end{cases} \tag{2.1}$$

where $y = x_1$ is the measurable output; $x = [x_1, \ldots, x_n]^\tau \in I\!R^n$ is the state vector, and the initial condition $x(0)$ is fixed but unknown; $u \in I\!R$ is the control input; $w \in I\!R^s$ is a vector-valued standard Brownian motion defined on probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with Ω being a sample space, $\mathcal{F}$ being a filtration, and $\mathcal{P}$ being the probability measure; $\rho = n - m$ is the relative degree of the system and supposed nonzero in this paper.

The main results of this paper are based on the following assumptions:

A1 The nonlinear functions $f_i(\cdot)$ and $h_i(\cdot)\,(i = 1, \ldots, n)$ are smooth, i.e., $f_i(\cdot) \in \mathcal{C}^\infty$ and $h_i(\cdot) \in \mathcal{C}^\infty$; the nonlinear function $g(\cdot)$ is continuous, and, for any $y \in I\!R$, $g(y) \neq 0$.

A2 All the roots of the polynomial $b_m s^m + \cdots + b_1 s + b_0$, $b_m \neq 0$ have negative real parts.

A3 Desired system output y_d is deterministic, it and its derivatives $\dot y_d, \ldots, y_d^{(n)}$ are known, uniformly bounded.

2.2. Control objective

The goal of control design is to make the solution process of the system (2.1) be bounded in probability and the following quadratic tracking risk-sensitive cost criterion achieve a pre-defined long-term cost value:

$$J_\theta(u) = \limsup_{T\to\infty} \frac{1}{T}\cdot\frac{2}{\theta}\ln\left\{E\left[\exp\left(\frac{\theta}{2}\int_0^T (y - y_d)^2\,dt\right)\right]\right\}, \tag{2.2}$$

that is, for any given positive cost value R_l (maybe arbitrarily close to zero), the risk-sensitive cost $J_\theta(u)$ is not larger than R_l, where $\theta > 0$ is called the risk sensitivity parameter, $y - y_d$ is called the output tracking error.

In addition to the purposes of cost upper bound, we are also interested in achieving *boundedness in probability* and *asymptotical stability in the large* for the closed-loop system. These two stability notions, which were introduced in the classical book [17], have been widely used now and is restated in Appendix.

As in [20] and [25], with the long-term average risk-sensitive cost criterion $J_\theta(u)$, for a given desired cost value $R_l > 0$, a practical risk-sensitive output-feedback tracking control is designed as

$$\begin{cases} \dot\xi = \alpha(t, \xi, y), & \alpha(\cdot,\cdot,\cdot) \in \mathcal{C}^1\times\mathcal{C}^1\times\mathcal{C}^1, \\ u = \mu(t, \xi, y), & \mu(\cdot,\cdot,\cdot) \in \mathcal{C}^0\times\mathcal{C}^0\times\mathcal{C}^0 \end{cases} \tag{2.3}$$

so that there exists a nonnegative value function $V(t, x, \xi) \in \mathcal{C}^0\times\mathcal{C}^2\times\mathcal{C}^2$ and radially unbounded with respect to x and ξ, satisfying the following Hamilton-Jacobi-Bellman (HJB) inequality:

$$\frac{\partial V}{\partial t} + \left[\frac{\partial V}{\partial x} \quad \frac{\partial V}{\partial \xi}\right]\left[\begin{matrix} f + Bg\mu \\ \alpha \end{matrix}\right] + \frac{\theta}{4}\frac{\partial V}{\partial x}hh^\tau \cdot\left(\frac{\partial V}{\partial x}\right)^\tau + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V}{\partial x^2}hh^\tau\right) + (y - y_d)^2 \le R_l. \tag{2.4}$$

From (2.4), it is easy to see that the essential difference between the stochastic HJB and deterministic HJB is that the former has the Itô term $\frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V}{\partial x^2}hh^\tau\right)$. How to deal with this term is the key to the control design and performance analysis.

3 Control Design

3.1. Observer design

Design the following observer to rebuild the states of (2.1):

$$\begin{cases} \dot{\widehat{x}}_i = \widehat{x}_{i+1} + k_i(y - \widehat{x}_1) + f_i(y), \\ \qquad i = 1, \ldots, \rho - 1, \\ \dot{\widehat{x}}_j = \widehat{x}_{j+1} + k_j(y - \widehat{x}_1) + f_j(y) + b_k g(y)u, \\ \qquad j = \rho, \ldots, n; k = m - j + \rho;\ \widehat{x}_{n+1} = 0, \end{cases} \tag{3.1}$$

where $k_1, k_2, \ldots, k_n$ are design constants such that all the roots of polynomial $s^n + k_1 s^{n-1} + \cdots + k_n$ have negative real parts. The initial condition for observer (3.1) is set by certain value $\widehat{x}(0)$.

Let $\widehat{x} = [\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n]^\tau$. Both system output y and observer state vector $\widehat{x}$ are available for control design. Denote the state estimation error as $\widetilde{x} = x - \widehat{x}$. Then, we have

$$\begin{aligned} d\widetilde{x} &= \begin{bmatrix} -k_1 & \\ \vdots & I_{n-1} \\ -k_n & 0 \cdots 0 \end{bmatrix} \widetilde{x}\, dt + h(y)\, dw \\ &\triangleq A\widetilde{x}\, dt + h(y)\, dw. \end{aligned} \tag{3.2}$$

Thus, the systems with observer dynamics (3.1) in the loop can be rewritten into

$$\begin{cases} d\widetilde{x} = A\widetilde{x}\, dt + h(y)\, dw, \\ dy = (\widehat{x}_2 + \widetilde{x}_2)\, dt + f_1(y)\, dt + h_1(y)\, dw, \\ \dot{\widehat{x}}_i = \widehat{x}_{i+1} + k_i(y - \widehat{x}_1) + f_i(y), \\ \qquad i = 2, \ldots, \rho - 1, \\ \dot{\widehat{x}}_j = \widehat{x}_{j+1} + k_j(y - \widehat{x}_1) + f_j(y) + b_k g(y)u, \\ \qquad j = \rho, \ldots, n; k = m - j + \rho;\ \widehat{x}_{n+1} = 0, \end{cases} \tag{3.3}$$

To carry out an integrator backstepping design, some appropriate coordinate transformations should be taken to transform the system (3.3) into a lower-triangular form in the case where $\widetilde{x} \equiv 0$.

3.2. State diffeomorphisms

In this subsection, we introduce a coordinate diffeomorphism (see [28]) transforming the system (3.3) into an appropriate form which is amenable to the application of integrator backstepping methodology.

$$\begin{aligned} d\widetilde{x} &= A\widetilde{x}\, dt + h(y)\, dw, \\ d\zeta &= \left\{ \begin{bmatrix} -\frac{b_{m-1}}{b_m} & \\ \vdots & I_{m-1} \\ -\frac{b_1}{b_m} & \\ -\frac{b_0}{b_m} & 0 \cdots 0 \end{bmatrix} \zeta \right. \\ &\quad + \begin{bmatrix} d_{\rho,\rho+1} \\ \vdots \\ d_{\rho n} \end{bmatrix} y + \begin{bmatrix} -d_{\rho-1,\rho} \\ \vdots \\ -d_{\rho-1,n-1} \end{bmatrix} \widetilde{x}_2 \\ &\quad \left. + \begin{bmatrix} g_{\rho,\rho+1}(y, \widetilde{x}_1) \\ \vdots \\ g_{\rho n}(y, \widetilde{x}_1) \end{bmatrix} \right\} dt + \begin{bmatrix} \widehat{h}_{\rho+1}(y) \\ \vdots \\ \widehat{h}_n(y) \end{bmatrix} dw \\ &\triangleq [E\zeta + L_1\widetilde{x}_1\, dt + L_2\widetilde{x}_2 + \Omega(y)]\, dt + \Psi(y)\, dw, \end{aligned} \tag{3.4}$$

$$\begin{aligned} d\eta_1 &= [d_{\rho 1} y + (\eta_2 + \widetilde{x}_2) + g_{\rho 1}(y, \widetilde{x}_1)]\, dt \\ &\quad + \widehat{h}_1(y)\, dw, \\ d\eta_i &= [d_{\rho i} y + \eta_{i+1} + g_{\rho i}(y, \widetilde{x}_1) - d_{\rho-1,i-1}\widetilde{x}_2] dt \\ &\quad + \widehat{h}_i(y)\, dw, \quad i = 2, \ldots, \rho - 1, \\ d\eta_\rho &= \left[1, 0_{1\times(\rho-1)}\right] \zeta dt + d_{\rho\rho} y dt + g_{\rho\rho}(y, \widetilde{x}_1) dt \\ &\quad + b_m g(y) u dt - d_{\rho-1,\rho-1}\widetilde{x}_2 dt + \widehat{h}_\rho(y) dw, \end{aligned}$$

where $L_1 = \left[\overline{d}_{\rho,\rho+1}, \ldots, \overline{d}_{\rho n}\right]^\tau$ and

$$\Omega = \left[\overline{g}_{\rho,\rho+1}(y) + d_{\rho,\rho+1} y, \ldots, \overline{g}_{\rho,n}(y) + d_{\rho n} y\right]^\tau.$$

This structure allows the design of output feedback controller by using integrator backstepping methodology.

3.3. Control design procedure

Actually, we can regard both dynamics ξ and $\widetilde{x}$ as a new dynamics $\chi = [\zeta^\tau, \widetilde{x}^\tau]^\tau \in I\!R^{n+m}$, which satisfy:

$$\begin{aligned} d\chi &= \begin{bmatrix} E & L \\ 0_{n\times m} & A \end{bmatrix} \chi\, dt + \begin{bmatrix} \Omega(y) \\ 0_{n\times 1} \end{bmatrix} dt + \begin{bmatrix} \Psi(y) \\ h(y) \end{bmatrix} dw \\ &\triangleq W\chi\, dt + F(y)\, dt + \Phi(y)\, dw. \end{aligned} \tag{3.5}$$

Here $L = \left[L_1,\ L_2,\ 0_{m\times(n-2)}\right]$.

The dynamics χ can be partitioned as $\chi \triangleq [\chi_a^\tau,\ \chi_b^\tau]^\tau$, where $\chi_a = [\zeta^\tau,\ \widetilde{x}_1]^\tau \in I\!R^{m+1}$ is available for feedback design, while $\chi_b = [\widetilde{x}_2, \ldots, \widetilde{x}_n]^\tau \in I\!R^{n-1}$ is not. Specifically, χ_a and χ_b satisfy the following stochastic differential equations, respectively:

$$\begin{cases} d\chi_a = \left\{ \begin{bmatrix} E & L_1 \\ 0_{1\times m} & -k_1 \end{bmatrix} \chi_a + \begin{bmatrix} L_2 & 0_{m\times(m-2)} \\ 1 & 0_{1\times(m-2)} \end{bmatrix} \chi_b \right. \\ \qquad \left. + \begin{bmatrix} \Omega(y) \\ 0 \end{bmatrix} \right\} dt + \begin{bmatrix} \Psi(y) \\ h_1(y) \end{bmatrix} dw \\ \qquad \triangleq W_a \chi_a dt + L_a \chi_b dt + F_a(y) dt + \Phi_a(y) dw, \\ d\chi_b = \begin{bmatrix} 0 & \\ \vdots & I_{n-2} \\ 0 & 0 \cdots 0 \end{bmatrix} \chi_b dt + \begin{bmatrix} 0 \cdots 0 & -k_2 \\ \vdots \ddots \vdots & \vdots \\ 0 \cdots 0 & -k_n \end{bmatrix} \chi_a dt \\ \qquad + \begin{bmatrix} h_2(y) \\ \vdots \\ h_n(y) \end{bmatrix} dw \\ \qquad \triangleq W_b \chi_b dt + L_b \chi_a dt + \Phi_b(y) dw. \end{cases}$$

Thus, we have the following overall systems which is amenable for integrator backstepping design.

$$\begin{cases} d\chi = W\chi\, dt + F(y)\, dt + \Phi(y)\, dw, \\ d\eta_i = \eta_{i+1} dt + d_i \chi_b dt + g_i(y, \chi_a) dt + \widehat{h}_i dw, \\ \qquad i = 1, \ldots, \rho - 1, \\ d\eta_\rho = [b_m g(y) u + d_\rho \chi_b + g_\rho(y, \chi_a)]\, dt + \widehat{h}_\rho dw, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} d_1 &= [1, 0_{1\times(n-2)}], \\ d_i &= [-d_{\rho-1,i-1}, 0_{1\times(n-2)}],\ i = 2, \ldots, \rho, \\ g_i &= g_{\rho i}\left(y, [0_{1\times m}, 1]\chi_a\right) + d_{\rho i} y, \quad i = 1, \ldots, \rho - 1, \\ g_\rho &= g_{\rho\rho}\left(y, [0_{1\times m}, 1]\chi_a\right) + d_{\rho\rho} y + [1, 0_{1\times m}]\, \chi_a. \end{aligned}$$

Now, we turn to the development of a recursive construction procedure for the desired risk-sensitive controller.

First, introduce a set of transformed state variables:

$$z_i = \eta_i - \alpha_{i-1}(t,\, \chi_a,\, \eta_{[i-1]}),\ i = 1, \ldots, \rho. \qquad (3.7)$$

where $\alpha_1,\ \ldots,\ \alpha_{\rho-1}$ are smooth functions, which are known as virtual controllers in the backstepping procedure. For the sake of simplicity, let $\alpha_0(t) = y_d(t)$ and $u = \alpha_n(t,\, \chi_a,\, \eta)$.

In fact, (3.7) defines a smooth diffeomorphism. Clearly, there exist smooth functions $\vartheta_i(t,\, \chi_a,\, \eta_{[i]})$, $i = 1, \ldots, \rho$, such that

$$z_{[i]} = \vartheta_i(t,\, \chi_a,\, \eta_{[i]}), \quad i = 1,\, \ldots,\, \rho. \qquad (3.8)$$

From the transformation (3.7) and $y = \eta_1$, the system (3.6) can be expressed, in terms of the z coordinates, as

$$\begin{cases} d\chi &= W\chi\, dt + F(y)\, dt + \Phi(y)\, dw, \\ dz_i &= [z_{i+1} + \alpha_i + F_i(t,\, \chi_a,\, \eta_1)]\, dt \\ & \quad + S_i(t,\, \chi_a,\, \eta_{[i]})\chi_b\, dt + \Psi_i(t,\, \chi_a,\, \eta_{[i]})\, dw, \\ & \qquad i = 1,\, \ldots,\, \rho - 1, \\ dz_\rho &= [\chi_1 + b_m g(y) u + F_\rho(t,\, \chi_a,\, \eta_{[\rho]})]\, dt \\ & \quad + S_\rho(t,\, \chi_a,\, \eta_{[\rho]})\chi_b\, dt + \Psi_\rho(t,\, \chi_a,\, \eta_{[\rho]})\, dw, \end{cases} \qquad (3.9)$$

where, for $i = 1,\, 2,\, \ldots,\, \rho$,

$$F_i = g_i(y,\, \chi_a) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \eta_j}[\eta_{j+1} + g_j(y,\, \chi_a)]$$
$$- \frac{\partial \alpha_{i-1}}{\partial \chi_a}[W_a \chi_a + F_a(y)] - \frac{\partial \alpha_{i-1}}{\partial t}$$
$$- \frac{1}{2}\mathrm{Tr}\left[\frac{\partial^2 \alpha_{i-1}}{\partial [(\chi_a^\tau,\, \eta_{[i-1]}^\tau)^\tau]^2} \begin{pmatrix} \Phi_a \\ \widehat{h}_1 \\ \vdots \\ \widehat{h}_{i-1} \end{pmatrix} \begin{pmatrix} \Phi_a \\ \widehat{h}_1 \\ \vdots \\ \widehat{h}_{i-1} \end{pmatrix}^\tau \right],$$

$$S_i = d_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \eta_j} d_j - \frac{\partial \alpha_{i-1}}{\partial \chi_a} L_a,$$

$$\Psi_i = \widehat{h}_i(y) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \eta_j} \widehat{h}_j(y) - \frac{\partial \alpha_{i-1}}{\partial \chi_a} \Phi_a(y).$$

Note that $F_1(t, \chi_a, \eta_1) = F_1(\chi_a,\, \eta_1) = g_1(\eta_1,\, \chi_a)$, $S_1(t,\, \chi_a,\, \eta_1) = S_1 = d_1$ and $\Psi_1(t,\, \chi_a,\, \eta_1) = \Psi_1(\eta_1) = \widehat{h}_1(\eta_1)$.

Then, we can design controller by using backstepping approach. The detailed design procedure is omitted for the space limitation.

It should be pointed out that during the backstepping design, for χ, we design the value function (or LYAPUNOV function) as

$$V_0(\chi) = \phi(\xi) = \delta(c + \xi)^\gamma - \delta c^\gamma,\ \xi = \chi^\tau P \chi,$$

where $0 < \delta \leq 1$, $c > 0$, $\frac{1}{2} < \gamma < 1$, and γ, c and δ are design constants to be specified. Constant γ is called as the CHARACTERISTIC PARAMETER of value function V_0. Clearly, V_0 is positively definite and radially unbounded. This is the necessary feature of a LYAPUNOV function used to stability analysis.

4 Main Results

We summarize the main results of this paper into a theorem.

THEOREM 4.1 *Consider the system (2.1) and the tracking risk-sensitive cost criterion (2.2). Suppose the system (2.1) satisfies Assumptions A1–A3. Then, for any given risk-sensitive parameter $\theta > 0$ and desired cost value $R_l > 0$, there exists an output-feedback control such that the closed-loop system:*

1. *has a unique solution on $[0,\ \infty)$ almost surely,*
2. *admits a guaranteed cost value R_l for the risk-sensitive cost criterion (2.2),*
3. *is bounded in probability.*

PROOF. The proof is omitted for the space limitation.

5 Control Design of Asymptotically Tracking

If in addition to Assumptions A1–A3, we have $f_i(y_d) \equiv 0$ and $h_i(y_d) \equiv 0$ $(i = 1, \ldots, n)$, then there exists an output-feedback risk-sensitive control so that the closed-loop system is asymptotical stability in the large, and at the same time, admits a zero risk-sensitive cost, i.e., the tracking error converges to zero almost surely. A special case (with y_d being constant) was studied for control design with zero risk-sensitive cost in [1].

THEOREM 5.1 *Consider the system (2.1) and the tracking risk-sensitive cost (2.2). Suppose that system (2.1) satisfies Assumptions A1–A3. For any given risk sensitivity parameter $\theta > 0$ and desired cost value $R_l > 0$, if $f_i(y_d) \equiv 0$, $h_i(y_d) \equiv 0$ $(i = 1, \ldots, n)$, then there is an output-feedback control such that the resulting closed-loop system (a) has a unique solution on $[0, \infty)$ almost surely, (b) is asymptotically stable in the large, and (c) admits a zero risk-sensitive cost.*

6 Concluding Remark

In this paper, the practical output-feedback control design problem of stochastic nonlinear strict-feedback systems in observer canonical form with stable zero-dynamics under a long-term tracking risk sensitive cost criterion is investigated. A state observer is designed to guarantee an exponentially convergent state estimate when there is no disturbance. By coordinate transformation, we transforms the system with the state observer in the loop into a lower triangular structure. And then, for any given risk-sensitive parameter and desired cost value, by using integrator backstepping method, we can present constructively the output-feedback control design algorithm. The cost function adopted here is of quadratic form usually encountered in practice, rather than of quartic one used to avoid difficulty on controller design and performance analysis of the closed-loop systems. It is shown that under the control designed, (a) the closed-loop system is bounded in probability, (b) the long-term average risk-sensitive cost of the closed-loop systems is upper bounded by the desired value. Besides, the value range of the characteristic parameters of the value function is investigated. When system vector nonlinearity and stochastic disturbance vector field vanish at the desired output y_d, i.e., $f_i(y_d) \equiv 0$, $h_i(y_d) \equiv 0$ $(i = 1, \ldots, n)$, we can design a control based on this information such that the resulting closed-loop system is asymptotically stable in the large and admits a zero risk-sensitive cost.

Appendix Preliminary Results

In this appendix, we give the definitions of *bounded in probability* and *asymptotically stable in the large*, and a key theorem to present the sufficient conditions for these two stability notions.

For a general control-free stochastic nonlinear system:

$$dx = f(t,x)\,dt + h(t,x)\,dw, \tag{A.1}$$

where f and h are assumed to be continuous in t and locally Lipschitz in x, w is vector-valued Brown motion defined as in Section 2.

DEFINITION A.1 *The solution process $\{x(t),\ t \geq 0\}$ of the system (A.1) is said to be* bounded in probability, *if*

$$\lim_{\varepsilon\to\infty} \sup_{t\in[0,\,\infty)} P\{\|x(t)\| > \varepsilon\} = 0.$$

DEFINITION A.2 *Consider the system (A.1), with $f(t,0) \equiv 0$ and $h(t,0) \equiv 0$. The solution $x(t) = 0$ of the stochastic differential equation (A.1) is said to be* asymptotically stable in the large, *if for any $\varepsilon > 0$,*

$$\lim_{\|x(0)\|\to 0} P\left\{\sup_{t\geq 0}\|x(t)\| \geq \varepsilon\right\} = 0,$$

and for any initial condition $x(0)$,

$$P\left\{\lim_{t\to\infty} x(t) = 0\right\} = 1.$$

The following theorem can be directly derived from Theorems 5 and 6 of [25] and will be the base of the design and analysis of this paper.

THEOREM A.1 *Consider stochastic nonlinear system (A.1) and tracking risk-sensitive cost criterion (2.2). Suppose that Assumptions A1–A3 are satisfied. For any given risk-sensitive parameter $\theta > 0$ and desired risk-sensitive cost value $R_l \geq 0$, if there exist a nonnegative value function $V(t,x) \in \mathcal{C}^1 \times \mathcal{C}^2$, a function $\sigma(t,x) \in \mathcal{C}^1$, and a nonnegative function $l(t,x)$ such that*

$$W_1(x) \leq V(t,\,x) \leq W_2(x), \tag{A.2}$$

$$dV(t,\,x) \leq \sigma(t,\,x)\,dw - \frac{\theta}{4}\sigma(t,\,x)\sigma^\tau(t,\,x)\,dt - l(t,\,x)\,dt - (y-y_d)^2\,dt + R_l\,dt, \tag{A.3}$$

where $W_1(x)$ and $W_2(x)$ are positive definite and radially unbounded, then

1. *system (A.1) has a unique solution on $[0,\infty)$ almost surely;*
2. *the risk-sensitive cost $J_\theta(u)$ is upper bounded by R_l;*
3. *in addition, if there are constants $c_{l1} > 0$ and $c_{l2} \geq 0$, such that*

$$\mathcal{L}V(t,x) \leq -c_{l1}V(t,\,x) + c_{l2}, \tag{A.4}$$

then system (A.1) is bounded in probability, and when $c_{l2} = 0$, system (A.1) is asymptotically stable in the large and admits a zero risk-sensitive cost.

PROOF. The proof is omitted for the space limitation.

The following is key to the backstepping design procedure.

LEMMA A.1 *For any given positive definite matrix P and constant $\gamma \in (\frac{1}{2},\,1)$, set*

$$\Pi_\gamma(x,c) \triangleq \|x\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P) - (c + x^\tau Px)^{\gamma-1}\|x\|^2, \quad x \in I\!R^n,$$

and

$$\mathcal{M}_\gamma(c) = \sup_{x\in I\!R^n} \Pi_\gamma(x,c).$$

If c is positive and finite, then $\mathcal{M}_\gamma(c)$ is also finite and satisfies

$$\lim_{c\to 0^+} \mathcal{M}_\gamma(c) = 0, \qquad \lim_{c\to+\infty} \mathcal{M}_\gamma(c) = +\infty.$$

PROOF. The proof is omitted for the space limitation.

REMARK A.1 *Lemma A.1 shows that: term $\|x\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P)$ is $\mathcal{M}_\gamma(c)$-close to term $(c + x^\tau Px)^{\gamma-1}$ $\|x\|^2$, and $\mathcal{M}_\gamma(c)$ is arbitrarily close to zero as constant c is.*

LEMMA A.2 *For every given positive definite matrix P and constant $\gamma \in (\frac{1}{2},\,1)$, define function:*

$$\Delta_\gamma(x,c) = (c + x^\tau Px)^\gamma - c^\gamma - \lambda_{\max}^\gamma(P)\|x\|^{2\gamma}, \quad c \geq 0, \quad x \in I\!R^n.$$

Then, we have

$$\Delta_\gamma(x,c) \leq 0. \tag{A.5}$$

REMARK A.2 *From Lemma A.1 and Lemma A.2, we know that for any $\chi \in I\!R^{n+m}$, there exist the following inequalities:*

$$\|\chi_i\|^{2\gamma} \leq \|\chi\|^{2\gamma} \leq \lambda_{\max}^{1-\gamma}(P)\left[\mathcal{M}_\gamma(c) + \frac{\|\chi\|^2}{(c + \chi^\tau P\chi)^{1-\gamma}}\right], \tag{A.6}$$

$$(c + \chi^\tau P\chi)^\gamma - c^\gamma \leq \lambda_{\max}(P)\left[\mathcal{M}_\gamma(c) + \frac{\|\chi\|^2}{(c + \chi^\tau P\chi)^{1-\gamma}}\right].$$

LEMMA A.3 *For any given constants a_1 and a_2 satisfying $1 < a_1 < a_2$, set $f_{a_1,\,a_2}(x) = x^{a_1} - x^{\alpha_2}$. Then*

$$\sup_{x\in I\!R^+\cup\{0\}} f_{a_1,\,a_2}(x) = \frac{a_2 - a_1}{a_2}\left(\frac{a_1}{a_2}\right)^{\frac{a_1}{a_2-a_1}}. \tag{A.7}$$

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Switching Control of Singular and Impulsive Systems

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Abstract— **This paper discusses the problem of switching control for singular and impulsive systems which has not been studied up to now. A new criterion for exponential stability of the controlled impulsive and switching singular systems is established. Finally, an example is given to demonstrate the effectiveness of the theoretic results.**

Keywords Singular and impulsive systems, switching control, exponential stability

I. INTRODUCTION

Singular systems have been of interest in the literature since they have many important applications in, for example, circuit systems, robotics, aircraft modelling, social, biological, and multisector economic systems, dynamics of thermal nuclear reactors, control systems, singular perturbation systems, and so on. Progress in dealing with singular systems can be found in books [1], [2], [4], [6].

Many singular systems exhibit impulsive behaviors, which characterized by abrupt changes in the states at certain instants in the fields of information science, electronics, automatic control systems, computer networks, artificial intelligence, robotics, and telecommunications [3], [7], [9], [10], [12]. Many sudden and sharp changes occur instantaneously, in the form of impulses, which cannot be well described by using pure continuous or pure discrete models. Therefore, it is important and, in fact, necessary to study singular and impulsive systems. For singular systems with infinite impulses and jumps in the solutions of the systems often happen [5], [15], but, except for the work by Guan et al. [10], [11] and Liu et al. [14], [13], there are few papers dealing with singular and impulsive systems.

On the other hand, switched systems have received increasing interest in recent years [17], [18], [19], [20] due to the fact that switched systems have numerous applications in control of robotics, multi-media, manufacturing, power electronics, switched-capacitor networks, chaos generator, automated highway systems, air traffic management systems, etc. A switched system arises when a switched controller is used. The major reason why one chooses a switched controller rather than a continuous controller is that a switched controller can be applied to achieve better performances [16].

In this paper, a basic model of singular and impulsive system is introduced and the switching control for this model is studied. The paper is organized as follows. In Section 2, using the switched controller, the hybrid impulsive and switching singular model is presented. Then, in Section 3, the exponential stability for such impulsive and switching singular systems is investigated. An example and conclusions are given in Section 4 and 5, respectively.

II. PROBLEM FORMULATION

Let $R_+ = [0, +\infty)$, $J = [t_0, +\infty)$, $(t_0 \geq 0)$, and R^n denote the n-dimensional Euclidean space. For $x = (x_1, \ldots, x_n)^\top \in R^n$, the norm of x is $\|x\| := \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$. Correspondingly, for $A = (a_{ij})_{n\times n} \in R^{n\times n}$, $\|A\| := \lambda_{\max}^{\frac{1}{2}}\left(A^T A\right)$. The identity matrix of order n is denoted as I_n (or simply I if no confusion arises).

An linear singular impulsive control system with impulses at fixed times is described by

$$\begin{cases} E\dot{x} = Ax + Bu, & t \neq t_k \\ \triangle x = U_k(t, x), & t = t_k, \end{cases} \tag{2.1}$$

where $t \in J, x \in R^n$ is the state variable, $u \in R^m$ is the control input. The matrix $E \in R^{n\times n}$ may be singular, and it is assumed that $\text{rank}(E) = r \leq n$. A and B are known real constant matrices with appropriate dimensions. It is assumed that the pair (E, A) is regular, that is, $\det(sE - A) \neq 0$ for some complex number s.

A sequence $\{t_k, U_k(t_k, x(t_k))\}$ has the effect of suddenly changing the state of the system (2.1) at the points t_k, where

$$t_1 < t_2 < \cdots < t_k < \cdots, \qquad \lim_{k\to\infty} t_k = \infty, \tag{2.2}$$

and $t_1 > t_0$; that is

$$\triangle x|_{t_k} =: x(t_k^+) - x(t_k^-) = U_k(t_k, x(t_k)),$$

where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$. For simplicity, it is assumed that $x(t_k^-) = x(t_k)$. Furthermore, $U_k(t_k, x(t_k))$ can be chosen as $c_{2k}x(t_k)$ with c_{2k} being constants for $k = 1, 2, \cdots$.

Construct a switching controller u for system (2.1) as follow:

$$u(t) = \sum_{k=1}^{\infty} C_{1k} x(t)\, l_k(t), \tag{2.3}$$

where C_{1k} are $n \times n$ constant matrices and $l_k(t)$ is the ladder function, i.e.

$$l_k(t) = \begin{cases} 1, & t_{k-1} < t \le t_k, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

If one chooses the switching control gain matrices $\{C_{1k}\}$ as $C_1, C_2, \cdots, C_m$ that is, $C_{1k} \in \{C_1, C_2, \cdots, C_m\}$, then the closed-loop system of (2.1) under control (2.3) becomes

$$\begin{cases} E\dot{x} = (A + BC_{i_k})x, & t \in (t_{k-1}, t_k] \\ \triangle x = c_{2k}x(t_k), & t = t_k \\ x(t_0^+) = x_0, & k = 1, 2, \cdots \end{cases} \quad (2.5)$$

where $i_k \in \{1, 2, \cdots, m\}$.

System (2.5) is also called *a hybrid impulsive and switching singular system*. It can be rewritten as the following compact form:

$$\begin{cases} E\dot{x} = A_{i_k}x, & t \in (t_{k-1}, t_k] \\ \triangle x = c_k x(t_k), & t = t_k \\ x(t_0^+) = x_0, & k = 1, 2, \cdots \end{cases} \quad (2.6)$$

where $t \in J$, $x \in R^n$ is the state variable, $A_{i_k} = A + BC_{i_k}$ are $n \times n$ matrices, c_k are constants for $k = 1, 2, \cdots$, $i_k \in \{1, 2, \cdots, m\}$ is the switch index, the sequence $\{t_k\}$ satisfies (2.2).

It is obvious that system (2.6) has m different modes, that is,

$$E\dot{x} = A_i x, \qquad i = 1, 2, \cdots m, \quad (2.7)$$

switching in the interval J.

Definition 2.1: ([8]) The system (2.1) (by setting $u(t) = 0$) is said to be E-exponentially stable if there exist constants $a > 0, b > 0$ such that

$$\|Ex(t)\| \le \|Ex(t_0)\| a\, e^{-b(t-t_0)}, \qquad t > t_0. \quad (2.8)$$

III. Hybrid Impulsive and Switching Singular System

In this section, we discuss the asymptotic properties of the hybrid system (2.6) under arbitrary switch.

For the system (2.6), let

$$(1 + c_k)^2 \le \beta_k, \quad k = 1, 2, \cdots. \quad (3.1)$$

Theorem 3.1: (i) It is assumed that there exists a constant $0 < \alpha < \lambda$ such that

$$\ln \beta_k - 2\alpha(t_k - t_{k-1}) \le 0, \quad k = 1, 2, \cdots. \quad (3.2)$$

(ii) If there exists a constant invertible matrix P satisfying

$$E^\top P = P^\top E \ge 0, \quad (3.3)$$

$$A_i^\top P + P^\top A_i + 2\lambda E^\top P < 0, \qquad i = 1, 2, \cdots m, \quad (3.4)$$

then the trivial solution of system (2.6) is globally E-exponentially stable under arbitrary switch, where β_k is given by (3.1).

Proof. For system (2.6), construct a Lyapunov function in the form of

$$v(t) = x^\top(t) E^\top P x(t), \quad (3.5)$$

and let $v(t){:=}v(x(t))$, where P is a constant invertible matrix satisfying (3.3) and (3.4). The total derivative of $v(x(t))$ with respect to (2.6) is

$$\begin{aligned} \dot{v}(x(t))\Big|_{(2.6)} &= x^\top A_{i_k}^\top P x + x^\top P^\top A_{i_k} x \\ &= x^\top (A_{i_k}^\top P + P^\top A_{i_k}) x \\ &< -2\lambda x^\top E^\top P x = -2\lambda v(t), \quad t \in (t_{k-1}, t_k], \end{aligned}$$

which implies that

$$v(t) \le v(t_{k-1}^+) e^{-2\lambda(t-t_{k-1})}, \quad t \in (t_{k-1}, t_k], \quad k = 1, 2, \cdots. \quad (3.6)$$

On the other hand, it follows from (2.6) and (3.5) that

$$\begin{aligned} v(t_k^+) &= x^\top(t_k^+) E^\top P x(t_k^+) \\ &= (1 + c_k)^2 x^\top(t_k) E^\top P x(t_k) \\ &= (1 + c_k)^2 v(t_k) \\ &\le \beta_k v(t_k), \qquad k = 1, 2, \cdots, \end{aligned} \quad (3.7)$$

where β_k are defined by (3.1).

The following results come from (3.6) and (3.7). For $t \in (t_0, t_1]$,

$$v(t) \le v(t_0^+) e^{-2\lambda(t-t_0)},$$

which leads to

$$v(t_1) \le v(t_0^+) e^{-2\lambda(t_1-t_0)},$$

and

$$v(t_1^+) \le \beta_1 v(t_1) \le \beta_1 v(t_0^+) e^{-2\lambda(t_1-t_0)}.$$

Similarly, for $t \in (t_1, t_2]$,

$$v(t) \le v(t_1^+) e^{-2\lambda(t-t_1)} \le \beta_1 v(t_0^+) e^{-2\lambda(t-t_0)}.$$

In general, for $t \in (t_k, t_{k+1}]$, notice the assumption (3.2),

$$\begin{aligned} v(t) &\le v(t_0^+)\beta_1 \cdots \beta_k e^{-2\lambda(t-t_0)} \\ &= v(t_0^+)\beta_1 \cdots \beta_k e^{-2\alpha(t-t_0)} e^{-2(\lambda-\alpha)(t-t_0)} \\ &\le v(t_0^+)\beta_1 \cdots \beta_k e^{-2\alpha(t_k-t_0)} e^{-2(\lambda-\alpha)(t-t_0)} \\ &= v(t_0^+)\beta_1 e^{-2\alpha(t_1-t_0)} \cdots \beta_k e^{-2\alpha(t_k-t_{k-1})} \\ &\quad e^{-2(\lambda-\alpha)(t-t_0)} \\ &\le v(t_0^+) e^{-2(\lambda-\alpha)(t-t_0)}, \qquad t \in (t_k, t_{k+1}], \end{aligned}$$

namely,

$$v(t) \le v(t_0^+) e^{-2(\lambda-\alpha)(t-t_0)}, \qquad t > t_0.$$

Since $E^\top P = P^\top E \ge 0$ and P is invertible, there exists a positive definite symmetric matrix Q such that $E^\top P = E^\top Q E$. Thus, we obtain

$$\begin{aligned} \lambda_{\min}(Q)\|Ex(t)\|^2 &\le v(x(t)) \\ &\le v(t_0^+) e^{-2(\lambda-\alpha)(t-t_0)} \\ &\le \|Ex(t_0^+)\|^2 \lambda_{\max}(Q) e^{-2(\lambda-\alpha)(t-t_0)}, \\ &\qquad t > t_0, \end{aligned}$$

that is,

$$\|Ex(t)\| \le \|Ex(t_0^+)\| \sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} e^{-(\lambda-\alpha)(t-t_0)}, \qquad t > t_0,$$

implying that the system (2.6) is E-exponentially stable under arbitrary switch. This immediately completes the proof. $\diamondsuit$

IV. NUMERICAL EXAMPLE

In this section, we give an example to demonstrate the effectiveness of the proposed method.

Consider the linear singular and impulsive system as follows:

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 2 & 0 \\ 5 & 10 \end{bmatrix} x + \begin{bmatrix} 3 & -20 \\ 5 & 0 \end{bmatrix} u, & t \neq t_k \\ \triangle x = \begin{bmatrix} -1.2 & 0 \\ 0 & -0.6 \end{bmatrix} x(t_k), & t = t_k. \end{cases}$$

It is easy to see that $(1 + c_k)^2 \le \beta_k = 1$. Select $\alpha = 0.5$, $\lambda = 1$ and $t_k - t_{k-1} = 0.06$ such that $\ln \beta_k - 2\alpha(t_k - t_{k-1}) = -0.06 < 0$.

If we choose

$$C_1 = \begin{bmatrix} -1.2 & 0 \\ 0.8 & -0.5 \end{bmatrix}, \quad \text{and} \quad C_2 = \begin{bmatrix} 1 & 12 \\ 1 & 0 \end{bmatrix},$$

then substituting the matrices specified above into Eqs. (3.3), (3.4) and solving it, we have

$$P = \begin{bmatrix} 2.6192 & 0 \\ 0.5598 & -10.0913 \end{bmatrix},$$

which implies, from Theorem 3.1, that the solution of controlled system (2.6) is globally E-exponentially stable under arbitrary switching control. Simulation results are shown as follows.

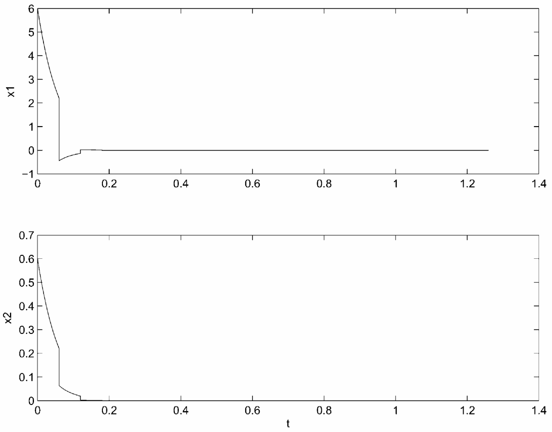

Fig.1. System states of controlled system

V. CONCLUSIONS

The problem of linear singular and impulsive systems via switching controller has been studied. Some new fundamental properties of singular systems with impulse and switch effects have been derived. Several sufficient conditions guaranteeing the stability for the corresponding closed-loop systems via switching controller are presented. A numerical example has been provided to verify the effectiveness of the proposed approach.

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非线性系统分岐解的动态输出反馈抑制

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摘要: 本文研究用动态输出反馈抑制分岐解的出现问题. 证明了能用动态输出反馈进行分岐抑制的充要条件是系统的零动态无平稳分岐解. 此外, 提出了保证闭环系统无平稳分岐解并渐近稳定的充分条件. 利用这些结果, 研究了轴流式压缩机的分岐控制问题, 建立了保证轴流式压缩机有预定工作点并且渐近稳定的动态补偿器设计方法.

关键词: 非线性系统, 分岐控制, 动态补偿器

Bifurcation Suppression of Nonlinear Systems via Dynamic Output Feedback

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Abstract: This paper deals with the problem of bifurcation suppression of nonlinear systems. It is proved that the bifurca-ted solution of a nonlinear system can be suppressed via dynamic output feedback if and only if its zero dynamics has nobifurcated solutions. In addition, a sufficient condition for guaranteeing bifurcation suppression and asymptotical stabilityof the closed loop system is presented. These results are applied to bifurcation control of axial compressor. A dynamiccompensator that guarantees bifurcation suppression and asymptotical stability is constructed for the axial compressor.

Key words: nonlinear system, bifurcation control, dynamic compensator

1 引言

最近十多年来非线性系统的分岐控制得到了广泛深入的研究. 文[1]对 2000 年以前的非线性系统的分岐控制问题的研究情况进行了系统的总结, 文中包含了丰富的文献. 平衡分岐解的反馈抑制或镇定是分岐控制中的二个重要问题. 反馈镇定的含义是通常的, 就是通过反馈使闭环系统的分岐平衡点是渐近稳定的. 所谓分岐解的反馈抑制, 就是通过反馈控制使闭环系统不出现分岐平衡点的. 分岐平衡点是渐近稳定的. 所谓分岐解的反馈抑制, 就是通过反馈控制使闭环系统不出现分岐平衡点的.

如果一个系统有分岐现象的话, 由于系统参数的变化, 系统设定的工作点(平衡点) 就会漂移. 这种漂移可能影响系统的性能, 甚至使系统不能正常工作. 文[2]讨论了状态反馈分岐解抑制问题. 但该文中的控制空间和状态空间是相同的. 在这种情况下, 控变量可以任意地改变系统的向量场. 我们通常研究的大多数控制系统的控制变量并不具有这种性质.

本文中, 我们研究用动态输出反馈抑制分岐解的问题. 概略地说, 我们的主要结果是: 分岐解能用动态输出反馈抑制的充要条件是系统的输出为常值的动态系统没有平稳分岐解. 当系统的输出常值动态没有平稳分岐解时, 我们给出了动态补偿器构造性设计方法. 作为一种应用, 我们研究了轴流式压缩机的平稳分岐解的压制问题.

从上面可以看到，当系统的所有输出常值的动态系统都会发生平稳分岐解时，闭环系统出现平稳分岐解是必然的．在此情形下，分岐解应是稳定的．因此本文的第二个问题是研究分岐解的镇定问题．最近十多年来，许多作者研究了分岐镇定问题．但是他们用的方法，主要是状态反馈，例如文献[3]，[4]，[5]，[6]和[7]等．另外还有用静态输出反馈的，例如文献[8]等．但我们沒有見到用动态输出反馈进行分岐镇定的文献．我们知道，当不是所有变量能测量的情况下，状态反馈不是可行的方法．而静态输出反馈控制，其能镇定的系统是不多的．本文用动态输出反馈进行分岐镇定，主要结果是：动零动态分岐渐近稳定的系统能用动态输出反馈镇定．

本文的安排如下：第二节是问题描述．第三节是本的主要结果．提出了能用动态输出反馈分岐抑制的充要条件，以及能稳定分岐抑制的充分条件．第四节研究了轴流式压缩机的稳定分岐抑制问题．提出了保证轴流式压缩机稳定分岐抑制的动态补偿器的构造方法．第五节是结束语．

2 问题的描述和预备知识

考虑非线性系统

$$\begin{aligned}\dot{x} &= f(x)+g(x,\mu)u,\\ y &= h(x),\end{aligned}\qquad (2.1)$$

其中 $x\in R^n$ 是状态变量，$u\in R$ 是控制变量，$y\in R$ 是输出变量，$\mu\in R$ 是参变量；$f,g:U\times(-\delta,\delta)\to R^n$，$h:U\to R$；$f,g$ 和 h 都是光滑映射，U 是 $x=0$ 的一个开邻域，$\delta>0$．

为叙述方便，我们将 $u=0$ 时的系统(2.1)记为

$$\dot{x} = f(x,\mu).\qquad (2.2)$$

定义 2.1. 设 $x=0$ 是系统(2.2)的一个平稳分岐点．如果存在 $\delta_1>0$ 和动态补偿器

$$\begin{aligned}\dot{\varsigma} &= \Phi(\varsigma,y),\\ u &= \alpha(\varsigma,y),\end{aligned}\qquad (2.3)$$

使得 $(x,\varsigma)=(0,0)$ 是闭环系统

$$\begin{aligned}\dot{x} &= f(x,\mu)+g(x,\mu)\alpha(\varsigma,y),\\ \dot{\varsigma} &= \Phi(\varsigma,y)\end{aligned}\qquad (2.4)$$

在 $B((0,0),\varepsilon)\times(-\delta_1,\delta_1)$ 上的唯一平衡点，则称系统(2.1)能用动态输出反馈分岐抑制．

$B((x,\varsigma),\varepsilon)$ 表示以为 (x,ς) 中心，以 $\varepsilon>0$ 为半径的开球．

有关平稳分岐点的概念可见文[2-8]．注意，在动态补偿器(2.3)中，我们并没有要求 $\alpha(0,h(0))=0$．

被分岐抑制的闭环系统只保证平衡点是唯一的．平衡点唯一但不稳定的话，系统的设定性能仍然无法保证．因此我们引入稳定分岐抑制的概念．

定义 2.2. 设 $x=0$ 是非线性系统(2.2)分岐点．称(2.1)为能用动态输出反馈稳定分岐抑制，如果它能分岐抑制并且闭环系统是渐近稳定的．

在本文中，我们假定系统(2.1)具有相对阶 r．这时系统(2.1)能用局部坐标变换化为

$$\begin{aligned}\dot{z} &= f_0(z,\xi,\mu)+g_0(z,\xi,\mu)u,\\ \dot{\xi}_1 &= \xi_2,\\ &\vdots\\ \dot{\xi}_{r-1} &= \xi_r,\\ \dot{\xi}_r &= f_1(z,\xi,\mu)+g_1(z,\xi,\mu)u,\\ y &= \xi_1\end{aligned}\qquad (2.5)$$

其中 $z\in R^{n-r}$，$\xi=(\xi_1,\dots,\xi_r)^T\in R^r$，$f_0,g_0,f_1$ 和 g_1 都是光滑函数，$g_1(0,0,\mu)\neq 0$，$\mu\in(-\delta_1,\delta_1)$．

显然系统(2.5)的零动态为

$$\begin{aligned}\dot{z} &= f_0(z,0,\mu)\\ &\quad -g_0(z,0,\mu)g_1^{-1}(z,0,\mu)f_1(z,0,\mu)\end{aligned}.\qquad (2.6)$$

3 分岐抑制

本节考虑分岐抑制问题．

定理 3.1. 设 $x=0^n$ 是的分岐点．系统(2.1)能用动态输出反馈分岐抑制的充分必要条件是：$z=0$ 是其零动态的唯一平衡点，即，$\forall\mu\in(-\delta_1,\delta_1)$，

$$f_0(z,0,\mu)-g_0(z,0,\mu)g_1^{-1}(z,0,\mu)f_1(z,0,\mu)=0$$

在 $z=0$ 的某个邻域内，意味看着 $z=0$．

下面我们来考虑稳定分岐抑制问题．设

$$\begin{aligned}&\bar{f}_0(z,\mu)\\ &=f_0(z,0,\mu)-g_0(z,0,\mu)g_1(z,0,\mu)f_1(z,0,\mu)\end{aligned}.$$

记

$$A=\frac{\partial}{\partial z}\bar{f}_0(z,\mu)\Big|_{z=0,\mu=0}.$$

在本文中我们始终假设 A 有一个特征值为零，其余特征值都具有负实部．此假设在一般分岐控

制的研究中是成立，参见文献[5]和[8]等.

定理 3.2. 设$(z,\xi)=(0,0)$是系统(2.5)的分岐点. 如果$\forall\mu\in(-\delta_1,\delta_1)$， $z=0$是系统(2.6)的唯一平衡点,而且渐近稳定， 则系统(2.5)能用动态输出反馈分岐稳定抑制.

4 轴流式压缩机的分岐稳定抑制

许多作者研究过轴流式压缩机的分岐控制，例如，见文献[5]和[8]．轴流式压缩机的数学模型可以表示成(参见文[5])：

$$\begin{aligned}&\dot{\psi}=\phi+2-\sqrt{\psi+\psi_0}\left(\frac{2}{\sqrt{\psi_0}}+\mu+u\right),\\&\dot{\phi}=-\psi-\frac{3}{2}\phi^2-\phi^3-3A^2-3\phi A^2,\\&\dot{A}=\sigma A(-2\phi-\phi^2-A^2),\end{aligned}\qquad(4.1)$$

其中$\sigma>0$ 是常数，$\psi_0=\dfrac{11}{3}$， $\mu\in(-\delta_1,\delta_1)$ 是分岐参数， u是控制变量．关于 ϕ，ψ 和A的物理意义见文[5]．

容易看到当μ在$(-\delta_1,\delta_1)$上变化时,系统(4.1)在$u=0$时产生了平稳分岐解．许多作者用状态反馈或静态输出反馈对(4.1)的分岐镇定进行了研究(参见文[5]和[8])．在他们的研究中，平稳分岐解不会消失.下面我们研究用动态输出反馈进行分岐稳定抑制的问题.

我们用$y=\phi$作为系统的输出．设

$$\begin{aligned}&\xi_1=\phi,\\&\xi_2=-\psi-\frac{3}{2}\xi_1^2-\xi_1^3-3A^2-3\xi_1A^2.\end{aligned}\qquad(4.2)$$

则系统(4.1)被变换成

$$\begin{aligned}&\dot{A}=\sigma A(-2\xi_1-\xi_1^2-A^2),\\&\dot{\xi}_1=\xi_2,\\&\dot{\xi}_2=f_1(A,\xi_1,\xi_2,\mu)+g_1(A,\xi_1,\xi_2,\mu)u,\end{aligned}\qquad(4.3)$$

其中

$$\begin{aligned}&f_1(A,\xi_1,\xi_2,\mu)\\&=-\xi_1-2-3\xi_1\xi_2-3\xi_1^2\xi_2-6\sigma A^2(-2\xi_1-\xi_1^2)\\&\quad-3\xi_2A^2-6\sigma\xi_1A^2(2\xi_1-\xi_1^2-A^2)\\&\quad+\sqrt{-\xi_2-\frac{3}{2}\xi_1^2-\xi_1^3-3A^2-3\xi_1A^2+\psi_0}\\&\quad\times\left(\frac{2}{\sqrt{\psi_0}}+\mu\right),\end{aligned}$$

$$\begin{aligned}&g_1(A,\xi_1,\xi_2,\mu)\\&=\sqrt{-\xi_2-\frac{3}{2}\xi_1^2-\xi_1^3-3A^2-3\xi_1A^2+\psi_0}.\end{aligned}$$

显然，系统(4.3)的零动态是

$$\dot{A}=-\sigma A^3,\qquad(4.4)$$

其零解$A=0$是不随参数μ变化的唯一渐近稳定平衡点．根据定理 3.2，系统能用动态输出反馈分岐稳定抑制.

下面我们来构造动态补偿器．事实上，对于本例，我们只须构造下面这样简单的控制器

$$\begin{aligned}&\dot{\lambda}=-\alpha y,\\&u=\lambda,\end{aligned}\qquad(4.5)$$

其中$\alpha>0$是待确定的常数.

我们先来证明， $\forall\mu\in(-\delta_1,\delta_1)$， 闭环系统(4.3)和(4.5)只有唯一的平衡点.为此只须证明, $\forall\mu\in(-\delta_1,\delta_1)$，方程组

$$\begin{aligned}&\sigma A(-\xi_1-\xi_1^2-A^2)=0,\\&\xi_2=0,\\&f_1(A,\xi_1,\xi_2,\mu)+g_1(A,\xi_1,\xi_2,\mu)\lambda=0,\\&-\alpha y=0,\end{aligned}\qquad(4.6)$$

有唯一解.

事实上，从方程组(4.6)的第一，第二和第四个方程可以得出

$$\xi_1=\xi_2=A=0.$$

再将它们代入方程组(4.6)的第三个方程，可得

$$-2+\sqrt{\psi_0}\left(\frac{2}{\sqrt{\psi_0}}+\mu\right)+\sqrt{\psi_0}\lambda=0.$$

解上述方程得$\lambda=-\mu$．因此，$\forall\mu\in(-\delta_1,\delta_1)$，闭环系统(4.3)和(4.5)有唯一平衡点

$$\begin{pmatrix}A\\\xi_1\\\xi_2\\\lambda\end{pmatrix}=\begin{pmatrix}0\\0\\0\\-\mu\end{pmatrix}.$$

下面我们证明此平衡点是渐近稳定的．设

$$\xi_3=f_1(A,\xi_1,\xi_2,\mu)+g_1(A,\xi_1,\xi_2,\mu)\lambda.$$

则闭环系统(4.3)和(4.5)被变换成

$$\begin{aligned}&\dot{A}=\sigma A(-\xi_1-\xi_1^2-A^2),\\&\dot{\xi}_1=\xi_2,\\&\dot{\xi}_2=\xi_3,\\&\dot{\xi}_3=-\alpha\sqrt{\psi_0}\xi_1-\xi_2-\frac{1}{2}\sqrt{\psi_0}\xi_3\\&\qquad+f_2(A,\xi_1,\xi_2,\xi_3,\mu),\end{aligned}\qquad(4.7)$$

其中 $f_2(A,\xi_1,\xi_2,\xi_3,\mu)$具有性质:

$$f_2(A,\xi_1,\xi_2,\xi_3,\mu)=O(\|(A,\xi_1,\xi_2,\xi_3)\|^2),$$
$$f_2(A,0,0,0,\mu)=O(|A|^4). \qquad (4.8)$$

我们取 $\alpha>0$ 使得多项式

$$p(s)=s^3+\frac{1}{2}\sqrt{\psi_0}s^2+s+\alpha\sqrt{\psi_0}$$

是 Hurwitz 多项式．注意，那样的必定存在．根据中心流形理论，(4.8)具在中心流形(参见[9]或[10])：

$$\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=O(|A^4|), \qquad (4.9)$$

而且其简化方程为

$$\dot{A}=-\sigma A^3+O(|A|^4). \qquad (4.10)$$

显然(4.10)的零解是渐近稳定的．由此可见闭环系统(4.3)和(4.5)的平衡点渐近稳定.

另外,如果采用形如

$$\dot{\lambda}=-\alpha(y-y_0),$$
$$u=\lambda$$

的动态补偿器,其中 $y_0>0$ 是常数，那么闭环系统平衡点中 ξ_1 分量将是 y_0．注意，这时，闭环系统的零动态变为

$$\dot{A}=-\sigma(2y_0+y_0^2)A-\sigma A^3.$$

其零解 $A=0$ 仍然是渐近稳定的．也就是说，我们可以适当调整原开环系统平衡点的位置.

5 结束语

本文建立了能用动态输出反馈分岐抑制的充要条件并提出了动态补偿器的构造方法，提出了能分岐抑制并同时保证闭环系统渐近稳定的一个充分条件．这些结被用于轴流式压缩机的分岐控制．本文的结果和方法也可应用于其他实际系统的分岐控制.

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遗传算法在变间距全息光栅设计中的应用

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摘 要：本文将遗传算法应用于变间距全息光栅设计。将变间距全息光栅记录参数的选择问题转化成多变量函数的寻优，利用遗传算法，结合局部搜索，寻找合适的记录参数。计算结果表明该方法具有良好的优化效果。
关键词：变间距全息光栅；遗传算法；局部搜索

Varied-Line-Spacing Holographic Grating design By Applying Genetic Algorithm

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Abstract: This article applies genetic algorithm in the design of varied-line-spacing holographic grating. Transforms the problem of choosing recording parameters of varied-line-spacing holographic grating to the optimization of multi-variable function, combines genetic algorithm with local search method to find the proper recording parameters. Comupting result shows that this method is effective in optimization.
Key words: Varied-Line-Spacing Holographic Grating, Genetic Algorithm, Local Search

1 引 言

以衍射光栅为主体的光谱仪器，在光谱学、天文学、环境检测、工业控制等领域发挥着巨大作用。早期的衍射光栅由机械刻划，难以控制光栅条纹的形状和间距，不利于校正光学系统的像差。全息光栅采用激光干涉技术制作，与机械光栅相比，光栅加工效率高，杂散光少，且无鬼线(Ghosts)[1]。更重要的是，在加工全息光栅的过程中，可以通过调整光路布置，控制条纹的形状和间距变化规律，制作出具有良好消像差能力的变间距全息光栅[2]。

早期的变间距全息光栅使用两束球面波形成干涉条纹，不足以消除高阶像差。Namioka 与 Koike 发展了变间距全息光栅技术，使用非球面波形成干涉条纹，增加了光路设计的自由度，有利于消除高阶像差[3]。

变间距全息光栅加工参数的选择，可以转化为多变量有约束优化问题。但其目标函数形式复杂，具有众多局部极值，使用解析方法和经典优化方法难以有效寻优。本文根 据变间距全息光栅加工的特点，推导了优化问题的目标函数，采用遗传算法，结合局部搜索进行记录参数的寻优，取得了良好的效果。

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2 变间距全息光栅的加工

变间距全息光栅的加工采用激光干涉技术。在待加工的光栅基底上均匀涂布光刻胶，利用两束相干激光产生干涉条纹，曝光后，干涉条纹被记录在光刻胶上，这一过程称为记录(Recording)。显影除去不需要的光刻胶，在光栅基底上形成浮雕光刻胶光栅。使用离子束刻蚀，将光栅条纹蚀刻到光栅基底上，并镀上反射膜，形成可使用的光栅。

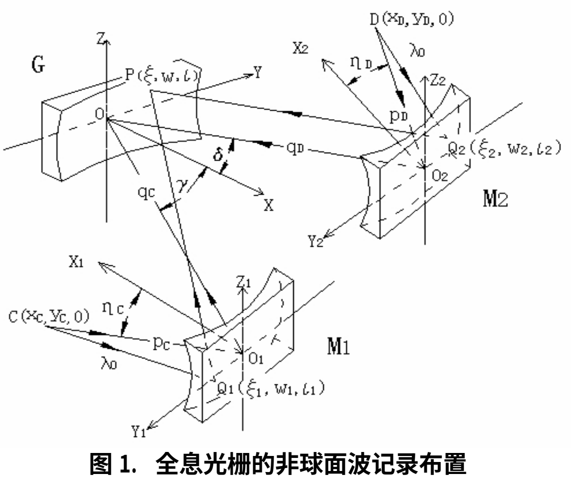

图 1. 全息光栅的非球面波记录布置

非球面波记录变间距全息光栅的光路布置如图 1 所示。其中O是待记录的椭球光栅基底G的中心，O_1与O_2代表两个椭球面镜M_1与M_2的中心，C和D是两个相干光源[3]。影响干涉条纹形状和间距的因素包括光栅面型，椭球镜面型，记录光源波长，以及八个可调的长度与角度参数p_C、p_D、q_C、q_D、γ、δ、η_C、η_D。这八个参数称为记录参数(Recording Parameters)。

考虑到加工工艺与光路调试的难度，通常采用平面镜代替椭球面镜M_1，球面镜代替椭球面镜M_2，分别生成球面波与非球面波。变间距光栅基底一般为平面。此时有六个记录参数：p_C+q_C、p_D、q_D、γ、δ、η_D。

3 记录参数优化问题

对于平面变间距光栅上的一点$P(0,w,l)$，从O点所在的第零条条纹出发，记P点条纹数为j，$j\in R$，对j泰勒展开到四阶：

$$j=\frac{1}{\lambda_0}\left[j_{10}w+\frac{1}{2}(j_{20}w^2+j_{02}l^2+j_{30}w^3+j_{12}wl^2)+\frac{1}{8}(j_{40}w^4+2j_{22}w^2l^2+j_{04}l^4)\right] \tag{1}$$

其中λ_0为光源波长，j_{10}、j_{20}、j_{02}、j_{30}、j_{12}、j_{40}、j_{22}、j_{04}是记录参数p_C+q_C、p_D、q_D、γ、δ、η_D的函数[3]。仅考虑光栅Y轴上的情况：

$$j=\frac{1}{\lambda_0}\left[(j_{10}w+\frac{1}{2}j_{20}w^2+\frac{1}{2}j_{30}w^3+\frac{1}{8}j_{40}w^4)\right] \tag{2}$$

一般用线密度表示光栅条纹的变化规律。由(2)式得到的条纹线密度为：

$$\frac{dj}{dw}=\frac{j_{10}}{\lambda_0}+\frac{j_{20}}{\lambda_0}w+\frac{3j_{30}}{2\lambda_0}w^2+\frac{j_{40}}{2\lambda_0}w^3 \tag{3}$$

要求加工的平面变间距光栅在$(0,w,0)$点条纹线密度为：

$$\begin{aligned}n&=n_0(1+b_2w+b_3w^2+b_4w^3)\\&=n_0+n_0b_2w+n_0b_3w^2+n_0b_4w^3\end{aligned} \tag{4}$$

需要选择一组记录参数，使光栅具有线密度参数n_0、b_2、b_3、b_4。通常联立求解方程组：

$$\begin{cases}n_0=\dfrac{j_{10}(p_C+q_C,p_D,q_D,\gamma,\delta,\eta_D)}{\lambda_0}\\ n_0b_2=\dfrac{j_{20}(p_C+q_C,p_D,q_D,\gamma,\delta,\eta_D)}{\lambda_0}\\ n_0b_3=\dfrac{3j_{30}(p_C+q_C,p_D,q_D,\gamma,\delta,\eta_D)}{2\lambda_0}\\ n_0b_4=\dfrac{j_{40}(p_C+q_C,p_D,q_D,\gamma,\delta,\eta_D)}{2\lambda_0}\end{cases} \tag{5}$$

寻找记录参数。但上述方程组复杂难解，且求得的解往往不满足工艺条件约束，实际计算中常需反复试凑[4]。

将问题转化为极小化某一目标函数的形式。考虑使条纹密度误差的平方在Y轴上的积分达到最小。$(0,w,0)$点条纹密度误差：

$$\begin{aligned}n_r-n_e&=\left(\frac{j_{10}}{\lambda_0}-n_0\right)+\left(\frac{j_{20}}{\lambda_0}-n_0b_2\right)w\\&\quad+\left(\frac{3j_{30}}{2\lambda_0}-n_0b_3\right)w^2+\left(\frac{j_{40}}{2\lambda_0}-n_0b_4\right)w^3\\&=r_1+r_2w+r_3w^2+r_4w^3\end{aligned} \tag{6}$$

n_r为实际的光栅线密度，n_e为期望的光栅线密度，其中：

$$\begin{aligned}&r_1=\frac{j_{10}}{\lambda_0}-n_0,\quad r_2=\frac{j_{20}}{\lambda_0}-n_0b_2\\&r_3=\frac{3j_{30}}{2\lambda_0}-n_0b_3,\quad r_4=\frac{j_{40}}{2\lambda_0}-n_0b_4\end{aligned} \tag{7}$$

光栅Y轴坐标范围$w_{\min}\le w\le w_{\max}$，一般情况下取$w_{\max}=-w_{\min}=w_0$，$w_0$为光栅长度的一半。目标：

$$\begin{aligned}Min\quad J_1&=\int_{w_{\min}}^{w_{\max}}(n_r-n_e)^2dw\\&=\int_{-w_0}^{+w_0}(r_1+r_2w+r_3w^2+r_4w^3)^2dw\\&=2w_0\left[r_1^{\ 2}+\frac{1}{3}w_0^{\ 2}(2r_1r_3+r_2^{\ 2})\right.\\&\quad\left.+\frac{1}{5}w_0^{\ 4}(r_3^{\ 2}+2r_2r_4)+\frac{1}{7}w_0^{\ 6}r_4^{\ 2}\right]\end{aligned} \tag{8}$$

因$w_0>0$，故消去$2w_0$，则取优化问题的目标函数：

$$\begin{aligned}Min\quad J_2&=r_1^{\ 2}+\frac{1}{3}w_0^{\ 2}(2r_1r_3+r_2^{\ 2})\\&\quad+\frac{1}{5}w_0^{\ 4}(r_3^{\ 2}+2r_2r_4)+\frac{1}{7}w_0^{\ 6}r_4^{\ 2}\end{aligned} \tag{9}$$

优化问题的约束表现为，记录参数必须满足全息工作台几何尺寸的限制。

$$\begin{aligned}s.t.\quad&(p_C+q_C)_{\min}\le p_C+q_C\le(p_C+q_C)_{\max}\\&(p_D)_{\min}\le p_D\le(p_D)_{\max}\\&(q_D)_{\min}\le q_D\le(q_D)_{\max}\\&(\gamma)_{\min}\le\gamma\le(\gamma)_{\max}\\&(\delta)_{\min}\le\delta\le(\delta)_{\max}\\&(\eta_D)_{\min}\le\eta_D\le(\eta_D)_{\max}\end{aligned} \tag{10}$$

4 基于遗传算法的参数优化

遗传算法广泛应用于多极值目标函数的优化。其优点在于拓展搜索空间的能力，缺点在于，对新开发的搜索空间进行更深入的局部搜索时效率太低。而单纯形法的局部搜索效率较高，但是过于依赖初值，不适合多极值目标函数的优化。考虑到在记录参数优化问题中，实际可用的光栅记录参数，并不需要使目标函数达到全局极值，性能较好的局部极值就可以满足要求。因此本文将遗传算法拓展搜索空间的能力与单纯形法的局部搜索能力相结合，首先采用遗传算法优化目标函数，再将其结果

作为起始点进行单纯形搜索。

算法流程如下[5][6]：

1) 在变量约束范围内，随机生成 $Nind$ 个个体，每个个体包含一组参数，采用实值编码方式，精度为小数点后 15 位，$Nind$ 取 1000。将循环计数器 $counter$ 置 0。

2) 计算各个体的目标函数值，按从大到小进行排序。设某个体的序号为 x_i，则它的适应度函数定义为：

$$F(x_i) = 2 - SP + 2(SP-1)\frac{x_i - 1}{Nind - 1} \tag{11}$$

其中 SP 为选择压力。取 SP=2。

3) 采用随机均匀采样（Stochastic Universal Sampling），选取 $Nind \times GGAP$ 个参与交叉与变异操作的个体，$GGAP$ 称为代沟。设 $N = Nind \times GGAP$，$Sum = \sum_{i=1}^{Nind} F(x_i)$，生成[0,1]之间均匀分布的随机数 ptr，构成等差数列 $\left\{ ptr \times \frac{Sum}{N}, ptr \times \frac{2Sum}{N}, \Lambda, ptr \times Sum \right\}$，对于该数列中每个数值，若其处于 $\left[\sum_{i=1}^{n-1} F(x_i), \sum_{i=1}^{n} F(x_i) \right]$ 范围内，则将第 x_i 个数选择进入下一步交叉和变异操作，允许重复选择。

4) 对于被选择的 N 个个体，交叉生成 N 个新个体。设两个父代的分别为 P_1，P_2，均为包含一组参数的向量，那么生成的两个新个体：

$$O_1 = \alpha P_1 + (1-\alpha) P_2$$
$$O_2 = \alpha P_2 + (1-\alpha) P_1 \tag{12}$$

其中 α 为[-0.25,1.25]之间均匀分布的随机数。

5) 对生成的 N 个新个体进行变异。设个体中参数数目为 $Nvar$，任一参数 Var_j 的取值范围是 $[Min_j, Max_j]$，则该参数变异的结果：

$$Var_j' = Var_j + MutMx_j \times \left(\frac{Max_j - Min_j}{2}\right) \times \sum_{i=0}^{m-1} \beta_i 2^{-i} \tag{13}$$

其中 $MutMx$ 以 $1/Nvar$ 的概率为 +1，以 $1/Nvar$ 的概率为 −1，$1\text{-}2/Nvar$ 的概率为 0。β_i 为 1 的概率为 $1/m$，为 0 的概率为 $1\text{-}1/m$。m 取 20。

若变异结果超出参数取值范围，则将其取为最近的边界点的值。

6) 对于 $Nind$ 个父代与 N 个子代，选择其中适应值较大的 $Nind$ 个个体，作为新一代个体。同时将循环计数器 $counter$ 加 1，若 $counter < MaxGen$，则返回 2，否则至 7。取 $MaxGen$ 为 100。

7) 选择遗传算法得到的最好个体，作为初始值进行单纯形搜索，并返回结果。

5 优化设计结果

要求的平面 VLS 光栅的线密度为：

$$n = n_0(1 + b_2 w + b_3 w^2 + b_4 w^3)$$
$$n_0 = 1400 (groove/mm)$$
$$b_2 = 1.1256 \times 10^{-3} (1/mm)$$
$$b_3 = 7.8845 \times 10^{-6} (1/mm^2)$$
$$b_4 = 0.0000 \times 10^{-9} (1/mm^3)$$

记录范围 $50 \times 10 (mm^2)$，即 $w_0 = 25mm$。记录光源为波长 $457.9nm$ 的氩激光，即 $\lambda_0 = 4.579 \times 10^{-4} mm$，使用一个球面波与一个非球面波的方式记录。辅助镜为半径 $1000mm$ 的球面镜。角度限定在 $-90°$ 到 $+90°$ 之间，距离限定在 $100mm$ 到 $2000mm$ 之间。为得到更多可行解，重复运行 50 次，从而为实际加工提供更多选择方案。

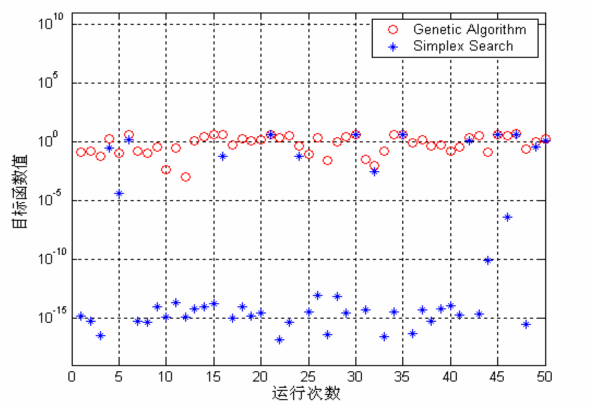

图 2. 各次运行时遗传算法与局部搜索的目标函数值

表 1. 角度精度 $10^{-15} rad$，距离精度 $10^{-15} mm$ 时的光栅参数期望值与设计值

	期望值	设计值
n_0 $(groove/mm)$	1400.0000	1400.0000
b_2 $(1/mm)$	1.1256×10^{-3}	1.1256×10^{-3}
b_3 $(1/mm^2)$	7.8845×10^{-6}	7.8845×10^{-6}
b_4 $(1/mm^3)$	0.0000×10^{-9}	0.0000×10^{-9}

表 2. 角度精度 $0.001rad$，距离精度 $1mm$ 时的光栅参数期望值与设计值

	期望值	设计值
n_0 $(groove/mm)$	1400.0000	1399.6869
b_2 $(1/mm)$	1.1256×10^{-3}	1.1226×10^{-3}
b_3 $(1/mm^2)$	7.8845×10^{-6}	7.8699×10^{-6}
b_4 $(1/mm^3)$	0.0000×10^{-9}	0.4037×10^{-9}

比较每次运行时，遗传算法最好结果的目标函数值，以及局部搜索结果的目标函数值，如图 2 所示。可以看出，局部搜索对遗传算法的优化结果有着非常大的改进。同时局部搜索的结果并没有出现两组相同的参数值，这说明目标函数形式复杂，具有众多局部极值。

选择一组记录参数，代入计算设计值，并与期望值相比较，列于表 1。可以发现优化结果与期望

值完全符合。考虑到工艺中的容许误差，计算角度精度 0.001*rad*，距离精度 1*mm* 时的结果，列于表 2。优化结果可以满足实际加工的容差要求。

6 结论

变间距全息光栅的设计，至今没有成熟高效的方法，实践中常需要反复试凑。本文将变间距全息光栅设计中的记录参数选择问题，转化成多变量有约束的优化问题，采用遗传算法与局部搜索相结合的搜索策略进行寻优。计算表明，优化结果完全满足设计指标的要求。引入加工参数的容许误差后，优化结果仍然具有良好的性能。

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