Diffusion Processes and Partial Differential Equations

KAZUAKI TAIRA

扩散过程和偏微分方程

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Preface

This book is devoted to the functional analytic approach to the problem of construction of Markov processes in probability theory. It is well known that, by virtue of the Hille–Yosida theory of semigroups, the problem of construction of Markov processes can be reduced to the study of boundary value problems for degenerate elliptic (integro-)differential operators of second order. Several recent developments in the theory of partial differential equations have made possible further progresss in the study of boundary value problems and hence of the problem of construction of Markov processes. The presentation of these new results is the main purpose of this book. Unlike many other books on Markov processes, this book focuses on the relationship between Markov processes and elliptic boundary value problems, with emphasis on the study of maximum principles. Our approach is distinguished by the extensive use of the theory of partial differential equations.

This book grew out of lecture notes for graduate courses given by the author at Sophia University, Hokkaido University, Tôhoku University, Tokyo Metropolitan University and University of Tsukuba. It is addressed to graduate students and mathematicians with an interest in probability, functional analysis and partial differential equations. The contents of the book are divided into five principal parts.

The first part (Chapters 1-4) provides the elements of the Lebesgue theory of measure and integration, manifold theory, functional analysis and distribution theory which are used throughout the book. The material in these preparatory chapters is given for completeness, to minimize the necessity of consulting too many outside references. This makes the book fairly self-contained.

In the second part (Chapters 5-6), the basic definitions and results about Sobolev spaces are summarized, and the calculus of pseudo-differential operators—a modern theory of potentials—is developed. The theory of pseudo-differential operators forms a most convenient tool in the study of elliptic boundary value problems in Chapter 8.

Our subject proper starts with the third part (Chapter 7), where various maximum principles for degenerate elliptic differential operators of second order are studied. In particular, the underlying analytical mechanism of propagation of maximums is revealed here. This plays an important role in the interpretation and study of Markov processes in terms of partial differential equations in Chapter 10.

The fourth part (Chapter 8) is devoted to general boundary value problems for second-order elliptic differential operators. The basic questions of existence, uniqueness and regularity of solutions of general boundary value problems with spectral parameter are studied in the framework of Sobolev spaces, using the calculus of pseudo-differential operators. Our approach is not far removed from the classical potenial approach. A fundamental existence and uniqueness theorem is proved here. The importance of such a theorem is visible in constructing Markov processes in Chapter 10.

The fifth and final part (Chapters 9-10) is devoted to the functional analytic approach to the problem of construction of Markov processes. General existence theorems for Markov processes in terms of boundary value problems are proved in Chapter 9, and then the construction of Markov processes is carried out in Chapter 10, by solving general boundary value problems with spectral parameter.

Bibliographical references are discussed primarily in Notes at the end of the chapters. These notes are intended to supplement the text and place it in better perspective.

To make the material in Chapters 7-10 accessible to a broad spectrum of readers, I have added *Introduction and Summary.* In this introductory chapter, I have included ten elementary (but important) examples of diffusion processes, and further I have attempted to state our problems and results in such a fashion that a broad spectrum of readers could understand, and also to describe how these problems can be solved, using the mathematics I present in Chapters 1-6.

I hope that this book will lead to a better insight into the study of three interrelated subjects of analysis: Markov processes, semigroups and elliptic boundary value problems, and further that the reader will appreciate intimate connections between partial differential equations and Markov processes.

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This work was begun at Ecole Normale Supérieure d'Ulm and Université de Paris-Sud (1976/78) with the financial support of the French Government while I was on leave from Tokyo Institute of Technology, and a major part of the work was done at University of Tsukuba (1978/79) with the aid of a grant-in-aid for scientific research and at the Institute for Advanced Study (1980/81) with the financial support of the National Science Foundation while I was on leave from University of Tsukuba. I take this opportunity to express my gratitude to all these institutions.

I would like to extend my warmest thanks to Professor Richard Bellman who originally suggested that my work be published in book form. Thanks are also due to the editorial and production staff of Academic Press Boston and Tokyo for their unfailing helpfulness and cooperation during the production of the book.

Last, but not least, I owe a great debt of gratitude to my wife, Naomi, who not only typed a part of the manuscript but also gave me moral support during the preparation of this book.

Notation and Conventions

The following notation is used for sets of numbers:

$\mathbf{N}$	the positive integers,
$\mathbf{Z}$	the integers,
$\mathbf{R}$	the real numbers,
$\mathbf{C}$	the complex numbers,
$[a, b]$	the closed interval $\{x \in \mathbf{R}; a \leq x \leq b\}$,
$[a, b)$	the semiclosed interval $\{x \in \mathbf{R}; a \leq x < b\}$,
(a, b)	the open interval $\{x \in \mathbf{R}; a < x < b\}$.

The notation for set-theoretic concepts is standard, and is described on pages 37 and 38. Other symbols introduced in the text are listed on pages 431 through 442.

We shall use without explanation the following:

$\equiv$	defined as (or "be identically equal to" as usual)
$\Rightarrow$	implication sign
$\Leftrightarrow$	two-sided implication sign (if and only if)

- ■ end of a proof
- ▼ proof of a lemma is done, but the proof of the theorem or proposition goes on.

Numbers in square brackets, e.g. [1], refer to the bibliography.

Definitions, results, remarks and examples are numbered within sections. For example, in Section $n.m$ (where n denotes the chapter), the theorems are indexed by Theorem $n.m.k$. Formulas and conditions are also numbered within sections, and those in Section $n.m$ are indexed by (k). When in another section we refer to such a formula (k) or condition (k), we designate it by ($n.m.k$). The really important ones are called theorems, and the slightly less important ones propositions.

Contents

Introduction and Summary

I. Markov Processes and Semigroups

I.1 Brownian Motion

In 1828, the English botanist R. Brown observed that pollen grains suspended in water move chaotically, incessantly changing their direction of motion. The physical explanation of this phenomenon is that a single grain suffers innumerable collisions with the randomly moving molecules of the surrounding water.

A mathematical theory for Brownian motion was put forward by A. Einstein in 1905. Let $p(t, x, y)$ be the probability density function that a one-dimensional Brownian particle starting at position x will be found at position y at time t. Einstein derived the following formula from statistical mechanical considerations:

$$p(t, x, y) = \frac{1}{\sqrt{2\pi Dt}} \exp\left[-\frac{(y-x)^2}{2Dt}\right].$$

Here D is a positive constant determined by the radius of the particle, the interaction of the particle with surrounding molecules, temperature and the Boltzmann constant. This gives an accurate method of measuring Avogadro's number by observing particles undergoing Brownian motion. Einstein's theory was experimentally tested by J. Perrin between 1906 and 1909.

Brownian motion was put on a firm mathematical foundation for the first time by N. Wiener in 1923. Let Ω be the space of continuous functions $\omega: [0, \infty) \to \mathbf{R}$ with coordinates $x_t(\omega) = \omega(t)$ and let $\mathscr{F}$ be the smallest σ-algebra in Ω which contains all sets of the form $\{\omega \in \Omega; a \leq x_t(\omega) < b\}$, $t \geq 0$, $a < b$. Wiener constructed probability measures P_x, $x \in \mathbf{R}$, on $\mathscr{F}$ for which the following formula holds:

$$
\begin{aligned}
P_x\{\omega \in \Omega;\ & a_1 \leq x_{t_1}(\omega) < b_1, a_2 \leq x_{t_2}(\omega) < b_2, \ldots, a_n \leq x_{t_n}(\omega) < b_n\} \\
&= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} p(t_1, x, y_1) p(t_2 - t_1, y_1, y_2) \cdots \\
&\quad \times p(t_n - t_{n-1}, y_{n-1}, y_n)\, dy_1\, dy_2 \cdots dy_n, \\
&\qquad 0 < t_1 < t_2 < \cdots < t_n < \infty. \qquad (1)
\end{aligned}
$$

This formula expresses the "starting afresh" property of Brownian motion that if a Brownian particle reaches a position, then it behaves subsequently as though that position had been its initial position. The measure P_x is called the *Wiener measure* starting at x.

P. Lévy found another construction of Brownian motion, and gave a profound description of qualitative properties of the individual Brownian path in his book *Processus stochastiques et mouvement brownien* (1948).

I.2 Markov Processes

Markov processes are an abstraction of the idea of Brownian motion. Let K be a locally compact, separable metric space and $\mathscr{B}$ the σ-algebra of all Borel sets in K, that is, the smallest σ-algebra containing all open sets in K. (The reader may content himself with thinking of $\mathbf{R}$ while reading about K.) Let $(\Omega, \mathscr{F}, P)$ be a probability space. A function X defined on Ω taking values in K is called a *random variable* if it satisfies

$$\{X \in E\} = X^{-1}(E) \in \mathscr{F} \qquad \text{for all} \quad E \in \mathscr{B}.$$

We express this by saying that X is $\mathscr{F}/\mathscr{B}$-measurable. A family $\{x_t\}_{t \geq 0}$ of random variables is called a *stochastic process*, and may be thought of as the motion in time of a physical particle. The space K is called the *state space* and Ω the *sample space*. For a fixed $\omega \in \Omega$, the function $x_t(\omega)$, $t \geq 0$, defines in the state space K a *trajectory* or *path* of the process corresponding to the sample point ω.