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美国大学生数学建模竞赛

**优秀论文集**

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# Modeling Forum

## Results of the 1997 Mathematical Contest in Modeling

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### Introduction

A total of 409 teams of undergraduates, from 226 schools, spent the second weekend in February working on applied mathematics problems. They were part of the twelfth Mathematical Contest in Modeling (MCM). On Friday morning, the MCM faculty advisor opened a packet and presented each team of three students with a choice of one of two problems. After a weekend of hard work, typed solution papers were mailed to COMAP on Monday. Nine of the top papers appear in this issue of *The UMAP Journal*.

Results and winning papers from the first twelve contests were published in special issues of *Mathematical Modeling* (1985–1987) and *The UMAP Journal* (1985–1996). The 1994 volume of *Tools for Teaching*, commemorating the tenth anniversary of the contest, contains all of the 20 problems used in the first ten years of the contest and a winning paper for each. Limited quantities of that volume and of the special MCM issues of the *Journal* for the last few years are available from COMAP.

### Problem A: The Velociraptor Problem

The velociraptor, *Velociraptor mongoliensis*, was a predatory dinosaur that lived during the late Cretaceous period, approximately 75 million years ago. Paleontologists think that it was a very tenacious hunter and may have hunted

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in pairs or larger packs. Unfortunately, there is no way to observe its hunting behavior in the wild, as can be done with modern mammalian predators. A group of paleontologists has approached your team and asked for help in modeling the hunting behavior of the velociraptor. They hope to compare your results with field data reported by biologists studying the behaviors of lions, tigers, and similar predatory animals.

The average adult velociraptor was 3 m long with a hip height of 0.5 m and an approximate mass of 45 kg. It is estimated that the animal could run extremely fast, at speeds of 60 km/hr, for about 15 sec. After the initial burst of speed, the animal needed to stop and recover from a buildup of lactic acid in its muscles.

Suppose that velociraptor preyed on *Thescelosaurus neglectus*, a herbivorous biped approximately the same size as the velociraptor. A biomechanical analysis of a fossilized thescelosaurus indicates that it could run at a speed of about 50 km/hr for long periods of time.

### Part 1

Assuming the velociraptor is a solitary hunter, design a mathematical model that describes a hunting strategy for a single velociraptor stalking and chasing a single thescelosaurus as well as the evasive strategy of the prey. Assume that the thescelosaurus can always detect the velociraptor when it comes within 15 m, but may detect the predator at even greater ranges (up to 50 m) depending upon the habitat and weather conditions. Additionally, due to its physical structure and strength, the velociraptor has a limited turning radius when running at full speed. This radius is estimated to be three times the animal's hip height. On the other hand, the thescelosaurus is extremely agile and has a turning radius of 0.5 m.

### Part 2

Assuming more realistically that the velociraptor hunted in pairs, design a new model that describes a hunting strategy for two velociraptors stalking and chasing a single thescelosaurus as well as the evasive strategy of the prey. Use the other assumptions and limitations given in Part 1.

## Problem B: Mix Well For Fruitful Discussions

Small group meetings for the discussion of important issues, particularly long-range planning, are gaining popularity. It is believed that large groups discourage productive discussion and that a dominant personality will usually control and direct the discussion. Thus, in corporate board meetings, the board will meet in small groups to discuss issues before meeting as a whole. These smaller groups still run the risk of control by a dominant personality. In an

attempt to reduce this danger, it is common to schedule several sessions with a different mix of people in each group.

A meeting of An Tostal Corporation will be attended by 29 board members of which nine are in-house members (i.e., corporate employees). The meeting is to be an all-day affair with three sessions scheduled for the morning and four for the afternoon. Each session will take 45 minutes, beginning on the hour from 9:00 A.M. to 4:00 P.M., with lunch scheduled at noon. Each morning session will consist of six discussion groups with each discussion group led by one of the corporation's six senior officers. None of these officers is a board member. Thus, each senior officer will lead three different discussion groups. The senior officers will not be involved in the afternoon sessions, and each of these sessions will consist of only four different discussion groups.

The president of the corporation wants a list of board-member assignments to discussion groups for each of the seven sessions. The assignments should achieve as much of a mix of the members as possible. The ideal assignment would have each board member with each other board member in a discussion group the same number of times while minimizing common membership of groups for the different sessions. The assignments should also satisfy the following criteria:

1. For the morning sessions, no board member should be in the same senior officer's discussion group twice.
2. No discussion group should contain a disproportionate number of in-house members.

Give a list of assignments for members 1–9 and 10–29 and officers 1–6. Indicate how well the criteria in the previous paragraphs are met. Since it is possible that some board members will cancel at the last minute or that some not scheduled will show up, an algorithm that the secretary could use to adjust the assignments with an hour's notice would be appreciated. It would be ideal if the algorithm could also be used to make assignments for future meetings involving different levels of participation for each type of attendee.

## The Results

The solution papers were coded at COMAP headquarters so that names and affiliations of the authors would be unknown to the judges. Each paper was then read preliminarily by two "triage" judges at Southern Connecticut State University (Problem A) or at Carroll College, Montana (Problem B). At the triage stage, the summary and overall organization are the basis for judging a paper. If the judges' scores diverged for a paper, the judges conferred; if they still did not agree on a score, a third judge evaluated the paper.

Final judging took place at Harvey Mudd College, Claremont, California. The judges classified the papers as follows:

	Outstanding	Meritorious	Honorable Mention	Successful Participation	Total
Velociraptor	5	37	58	134	234
Discussion Groups	<u>4</u>	<u>25</u>	<u>43</u>	<u>103</u>	<u>175</u>
	9	62	101	237	409

The nine papers that the judges designated as Outstanding appear in this special issue of *The UMAP Journal*, together with commentaries. We list those teams and the Meritorious teams (and advisors) below; the list of all participating schools, advisors, and results is in the **Appendix**.

## Outstanding Teams

### Institution and Advisor

### Team Members

#### Velociraptor Papers

“Pursuit–Evasion Games in the Late Cretaceous”

Calvin College  
Grand Rapids, MI  
Gary W. Talsma

Edward L. Hamilton  
Shawn A. Menninga  
David Tong

“The Geometry and the Game Theory of  
Chases”

Harvard University  
Cambridge, MA  
Howard Georgi

Charlene S. Ahn  
Edward Boas  
Benjamin Rahn

“Gone Huntin’: Modeling Optimal  
Predator and Prey Strategies”

Pomona College  
Claremont, CA  
Richard Elderkin

Hei (Celia) Chan  
Robert A. Moody  
David Young

“Lunch on the Run”

University of Alaska Fairbanks  
Fairbanks, AK  
John P. Lambert

Gordon Bower  
Orion Lawler  
James Long

“A Three-Phase Model for  
Predator–Prey Analysis”

Washington University  
St. Louis, MO  
Hiro Mukai

Lance Finney  
Jade Vinson  
Derek Zaba

## Discussion Groups Papers

“An Assignment Model for  
Fruitful Discussions”

East China Univ. of Science and Technology  
Shanghai, China  
Lu Xiwen

Han Cao  
Hui Yang  
Zheng Shi

“Using Simulated Annealing to Solve  
the Discussion Groups Problem”

Macalester College  
St. Paul, MN  
Karla V. Ballman

David Castro  
John Renze  
Nicholas Weininger

“Meetings, Bloody Meetings!”

Rose-Hulman Institute of Technology  
Terre Haute, IN  
Aaron D. Klebanoff

Joshua M. Horstman  
Jamie Kawabata  
James C. Moore, IV

“A Greedy Algorithm for Solving  
Meeting Mixing Problems”

University of Toronto  
Toronto, Ontario, Canada  
Nicholas A. Derzko

Adrian Corduneanu  
Cyrus C. Hsia  
Ryan O'Donnell

## Meritorious Teams

### Velociraptor Papers (37 teams)

Beijing Univ. of Aeronautics and Astronautics, Beijing, China (Li Weiguo)  
Brandon University, Brandon, Manitoba, Canada (Doug Pickering)  
California Polytechnic State Univ., San Luis Obispo, CA (Thomas O'Neil) (two teams)  
Dalian University of Technology, Dalian, Liaoning, China (He Ming-Feng)  
Duke University, Durham, NC (David P. Kraines)  
East China Univ. of Science and Technology, Shanghai, China (Shao Nianci)  
Experimental High School of Beijing, China (Zhang Jilin)  
Goucher College, Baltimore, MD (Megan Deeney)  
Harvey Mudd College, Claremont, CA (David L. Bosley)  
Hebei Institute of Technology, Tangshan, Hebei, China (Wan Xinghuo)  
Hope College, Holland, MI (Ronald Van Iwaarden)  
Macalester College, St. Paul, MN, (Daniel A. Schwalbe)  
N.C. School of Science and Mathematics, Durham, NC (Dot Doyle)  
Nankai University, Tianjin, China (Ruan Jishou)  
National Univ. of Defence Technology, Chang Sha, Hunan, China (Cheng LiZhi)  
North Carolina State University, Raleigh, NC (Robert T. Ramsay)  
Rose-Hulman Inst. of Technology, Terre Haute, IN (George Berzsenyi)  
Seattle Pacific University, Seattle, WA (Steven D. Johnson)

Southeast University, Nanjing, Jiangsu, China (Xu Liang)  
 Swarthmore College, Swarthmore, PA (Stephen B. Maurer)  
 Trinity University, San Antonio, TX (Diane G. Saphire)  
 Tsinghua University, Beijing, China (Wang Siqun)  
 United States Air Force Academy, USAF Academy, CO (Scott G. Frickenstein)  
 Univ. of Science and Technology of China, Hefei, Anhui, China, (Yu Feng)  
 Univ. of Science and Technology of China, Hefei, Anhui, China (Yu Tianyue)  
 University of Colorado, Boulder, CO (Anne M. Dougherty)  
 University of Puget Sound, Tacoma, WA (Robert A. Beezer)  
 University of Saskatchewan, Saskatoon, Canada (Raj Srinivasan)  
 Wake Forest University, Winston-Salem, NC (Stephen B. Robinson)  
 Wake Forest University, Winston-Salem, NC (Edward Allen)  
 Washington University, St. Louis, MO (Hiro Mukai)  
 Western Washington University, Bellingham, WA (Igor Averbakh)  
 Western Washington University, Bellingham, WA (Saim Ural)  
 Wuhan Univ. of Hydraulics and Engineering, Wuhan, Hubei, China (Peng Zhuzeng)  
 Youngstown State University, Youngstown, OH (J. Douglas Faires)  
 Zhejiang University, Hangzhou, China (Fang Daoyuan)

### **Discussion Groups Papers** (25 teams)

Colorado College, Colorado Springs, CO (Deborah P. Levinson)  
 David Lipscomb University, Nashville, TN (Gary C. Hall)  
 Eastern Mennonite University, Harrisonburg, VA (John L. Horst)  
 Eastern Oregon State College, LaGrande, OR (Holly S. Zullo)  
 Eastern Oregon State College, LaGrande, OR (Mark R. Parker)  
 Gettysburg College, Gettysburg, PA (James P. Fink)  
 Graceland College, Lamoni, IA (Ronald K. Smith)  
 Grinnell College, Grinnell, IA (Thomas L. Moore)  
 Harvey Mudd College, Claremont, CA (David L. Bosley)  
 Hebei Institute of Technology, Tangshan, Hebei, China (Liu Baoxiang)  
 Hiram College, Hiram, OH (Larry Becker)  
 Ithaca College, Ithaca, NY (James E. Conklin)  
 Kenyon College, Gambier, OH (Brian D. Jones)  
 National Univ. of Defence Technology, Chang Sha, Hunan, China (Wu MengDa)  
 Peking University, Beijing, China (Huang Sheng)  
 South China Univ. of Technology, Guangzhou, China (Hao Zhifeng)  
 Southeast University, Nanjing, Jiangsu, China (Zhu Dao-yuan)  
 Trinity College Dublin, Dublin, Ireland (Timothy G. Murphy)  
 Univ. of Northern Colorado, Greeley, CO (William W. Bosch)  
 University College Cork, Cork, Ireland (Martin Stynes)  
 University College Cork, Cork, Ireland (Gareth Thomas)  
 University of Colorado, Boulder, CO (Bengt Fornberg)  
 University of Richmond, Richmond, VA (James Davis)  
 University of Southern Queensland, Toowoomba, Queensland, Australia  
 (Christopher J. Harman)  
 Xidian University, Xian, Shaanxi, China (Mao Yong-cai)

## Awards and Contributions

Each participating MCM advisor and team member received a certificate signed by the Contest Director and the appropriate Head Judge.

INFORMS, the Institute for Operations Research and the Management Sciences, awarded to each member of two Outstanding teams a cash award and a three-year membership. The teams were from Calvin College (Velociraptor Problem) and Rose-Hulman Institute of Technology (Discussion Groups Problem). Moreover, INFORMS gave free one-year memberships to all members of Meritorious and Honorable Mention teams.

The Society for Industrial and Applied Mathematics (SIAM) designated one Outstanding team from each problem as a SIAM Winner. Each team member received a cash prize. The teams were from Washington University (Velociraptor Problem) and from University of Toronto (Discussion Groups Problem). Both teams presented their results at a special Minisymposium at the SIAM Annual Meeting at Stanford University in July.

The Mathematical Association of America (MAA) designated one Outstanding team from each problem as an MAA Winner. The teams were from Harvard University (Velociraptor Problem) and Macalester College (Discussion Groups Problem). The Macalester team gave a presentation at a special session of MAA Mathfest in Atlanta, GA, in August.

## Judging

### *Director*

Frank R. Giordano, COMAP, Lexington, MA

### *Associate Directors*

Chris Arney, Dept. of Mathematical Sciences, U.S. Military Academy,  
West Point, NY

Robert L. Borrelli, Mathematics Dept., Harvey Mudd College,  
Claremont, CA

William Fox, Dept. of Mathematical Sciences, U.S. Military Academy,  
West Point, NY

### **Velociraptor Problem**

#### *Head Judge*

Marvin S. Keener, Mathematics Dept., Oklahoma State University,  
Stillwater, OK

#### *Associate Judges*

James Case, Baltimore, Maryland

Alessandra Chiareli, Computational Science Center, 3M, St. Paul, MN

Courtney Coleman, Mathematics Dept., Harvey Mudd College,  
Claremont, CA  
Patrick Driscoll, Dept. of Mathematical Sciences, U.S. Military Academy,  
West Point, NY  
Ben A. Fusaro, Dept. of Mathematical Sciences,  
Florida State University, Tallahassee, FL  
Mario Juncosa, RAND Corporation, Santa Monica, CA  
California State University Los Angeles, Los Angeles, CA  
Mark Levinson, Edmonds, WA  
Rick Mabry, Wolfram Research, Inc., Champaign, IL  
Keith Miller, National Security Agency, Fort Meade, MD  
Mike Moody, Mathematics Dept., Harvey Mudd College, Claremont, CA  
Peter Olsen, National Security Agency, Fort Meade, MD  
Jack Robertson, Mathematics Dept., Georgia College, Milledgeville, GA  
John L. Scharf, Carroll College, Helena, MT  
Lee Seitelman, Glastonbury, CT  
Theodore H. Sweetser III, Jet Propulsion Lab, Pasadena, CA  
Robert M. Tardiff, Dept. of Mathematical Sciences,  
Salisbury State University, Salisbury, MD  
Beverly West, Cornell University, Ithaca, NY  
Daniel Zwillinger, Zwillinger & Associates, Arlington, MA

### **Discussion Groups Problem**

#### *Head Judge*

Maynard Thompson, Mathematics Dept., University of Indiana,  
Bloomington, IN

#### *Associate Judges*

Victor Adamchik, Wolfram Research, Inc., Champaign, IL  
Karen Bolinger, Mathematics Dept., Arkansas State University,  
State University, AR  
Jerry Griggs, University of South Carolina, Columbia, SC  
John Kobza, Virginia Polytechnic Institute and State University,  
Blacksburg, VA  
Daphne Liu, Dept. of Mathematics and Computer Science,  
California State University Los Angeles, Los Angeles, CA  
Vijay Mehrotra, Onward Inc., Mountain View, CA  
Veena Mendiratta, Lucent Technologies, Naperville, IL  
Don Miller, Dept. of Mathematics, St. Mary's College, Notre Dame, IN  
Cathy Roberts, Northern Arizona University, Flagstaff, AZ  
Kathleen M. Shannon, Salisbury State University, Salisbury, MD  
Michael Tortorella, Lucent Technologies, NJ  
Marie Vanisko, Carroll College, Helena, MT

## **Triage Session**

### **Velociraptor Problem**

#### *Head Triage Judge*

Theresa M. Sandifer, Southern Connecticut State University, New Haven, CT

#### *Associate Judges*

Therese Bennett, Southern Connecticut State University, New Haven, CT

Susanna D. Fishel, Southern Connecticut State University, New Haven, CT

Ross B. Gingrich, Southern Connecticut State University, New Haven, CT

Cynthia B. Gubitose, Western Connecticut State University, Danbury, CT

C. Edward Sandifer, Western Connecticut State University, Danbury, CT

Xiaodi Wang, Western Connecticut State University, Danbury, CT

### **Discussion Groups Problem**

(all judges from Carroll College, Helena, MT)

#### *Head Triage Judge*

Marie Vanisko

#### *Associate Judges*

Peter Biskis

Terence Mullen

Jack Oberweiser

Philip Rose

## **Sources of the Problems**

The Velociraptor Problem was contributed by Jack Robertson, Mathematics Dept., and William Wall, Dept. of Biological and Environmental Sciences, both of Georgia College, Milledgeville, GA. The Discussion Groups Problem was contributed by Don Miller, Dept. of Mathematics and Computer Science, St. Mary's College, Notre Dame, IN.

## **Acknowledgments**

The MCM was funded this year by the National Security Agency, whose support we deeply appreciate. We thank Dr. Gene Berg of NSA for his coordinating efforts. The MCM is also indebted to INFORMS, SIAM, and the MAA, which provided judges and prizes.

I thank the MCM judges and MCM Board members for their valuable and unflagging efforts. Harvey Mudd College, its Mathematics Dept. staff, and Prof. Borrelli were gracious hosts to the judges.

# Cautions

*To the reader of research journals:*

Usually a published paper has been presented to an audience, shown to colleagues, rewritten, checked by referees, revised, and edited by a journal editor. Each of the student papers here is the result of undergraduates working on a problem over a weekend; allowing substantial revision by the authors could give a false impression of accomplishment. So these papers are essentially *au naturel*. Light editing has taken place: minor errors have been corrected, wording has been altered for clarity or economy, and style has been adjusted to that of *The UMAP Journal*. Please peruse these student efforts in that context.

*To the potential MCM Advisor:*

It might be overpowering to encounter such output from a weekend of work by a small team of undergraduates, but these solution papers are highly atypical. A team that prepares and participates will have an enriching learning experience, independent of what any other team does.

# Pursuit–Evasion Games in the Late Cretaceous

Edward L. Hamilton

Shawn A. Menninga

David Tong

Calvin College

Grand Rapids, MI 49546

{ ehamil28, smenni23, dtong23 } @calvin.edu

Advisor: Gary W. Talsma

## Summary

Using techniques from differential game theory, we model the velociraptor hunting problem by means of a semi-discrete computer algorithm.

By defining predator and prey behaviors in terms of simple, intuitive principles, we identify a set of strategies designed to counter one another, such that no one pure predator strategy or prey strategy defines an optimal behavior pattern. Instead, the ideal strategy switches between two or more pure strategies, in an essentially unpredictable, or protean, manner. The resulting optimum behaviors show a mixture of feints, bluffs, and true turns for the thescelosaurus, and a mixture of predictive interception and simple pursuit for the velociraptors.

Finally, using these strategies, we demonstrate a conclusive advantage for velociraptors hunting in pairs over velociraptors hunting in isolation.

## Introduction

We describe hunting strategies for a velociraptor and flight strategies for its prey using a computational semi-discrete representation of a differential game of pursuit and evasion.

First, we review the formalism of traditional non-differential game theory and the extension of its principles into the analysis of differential systems, taking careful note of the unique aspects of the velociraptor problem.

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Second, we propose a minimal set of assumptions required to reduce the analytical game to a numerical time-iterated computer algorithm, and submit a small set of intuitively simple strategies for each participant to execute.

Third, we examine the implications of both the assumptions of perfect and imperfect information for the optimization of strategies for a single pursuer and single evader and then extend these conclusions to more pursuers.

Finally, we comment on the inherent limitations of this model and offer our evaluation. We conclude that for the game of imperfect information there exists no pure strategy for the prey, but that the best alternative is to behave in an unpredictable fashion with respect to the alternation of turns and feints. The predator, however, has a clearly dominant strategy of compromising between a predictive algorithm and simple pursuit.

## Mathematical Formalization in Game-Theoretic Terms

Contests of interception and avoidance between a predator and its prey fall into the broad and diverse category of linear differential games. In a differential game of pursuit and evasion, two or more opposing players attempt either to maximize or to minimize their separation, subject to certain limitations on their motion.

Unlike the games of classical game theory, differential games involve the application of methodology from differential equations and make use of continuous fluctuations that define the states and objectives of the players.

In either a traditional or differential game, players seek to maximize some value, known as the *payoff function*, by selecting among a set of alternatives, with the payoff to each player determined by some function of all the players' choices. In a zero-sum game, all players are in direct competition with one another, such that the objectives expressed by their payoff functions are exactly opposite. Games of pursuit and evasion fall within this classification; the pursuer seeks to minimize the distance between itself and the evader, and the evader seeks to maximize this distance.

In the case of the velociraptor–thescelosaur pursuit–evasion game, the payoff function is a simple binary function: The only relevant outcomes are capture or escape. Such games are referred to as *games of kind*, as opposed to *games of degree*, in which the payoff can take on a larger set of possible values. In some cases, it may be helpful to consider a game of kind as being embedded in a game of degree, with the time to capture (or the separation at evasion) as the payoff function. Although this is not necessary in purely deterministic trials, the average time to capture for a particular strategy pair may provide a useful statistical measure of the efficacy of a strategy.

There are two classes of variables in a differential game: state variables, which specify the complete configuration of the entire system at any given

point in time, and control variables, which participants in the game use to alter the state function in ways favorable to their attempts to maximize their own payoff function. Common state variables in traditional pursuit–evasion games are the spatial coordinates of the participants; control variables may include an angle of maximum turn, or acceleration vectors.

Games in which all players have access to a complete set of state variables at any given point in time are known as games of *perfect information*; games in which this is not true are referred to as games of *imperfect information*. Typically, games of imperfect information lack exact analytical methods of solution. Thus, one of the best methods of evaluating such games is to use a discrete model, which can be implemented in terms of a simple computational algorithm. Unfortunately, purely discrete models often run the risk of obscuring essential details of the system that may depend on continuity of values without quantization. A reasonable compromise that still permits a computational method of solution is a semi-discrete method, in which time is iterated discretely, but spatial values are allowed to extend over the entire domain of real numbers [Isaacs 1967, 42]. We chose to implement the raptor hunting game as a semi-discrete computational algorithm.

The velociraptor problem embodies one of the most interesting situations in a two-player pursuit–evasion game. If the maximum velocity of the evader is greater than the maximum velocity of the pursuer, then the evader has an optimal strategy of moving directly away from the pursuer at maximal velocity and will always successfully evade capture. Similarly, if the pursuer is superior in both speed and maneuverability, then the pursuer has an optimal strategy of moving directly toward the evader and is guaranteed of a successful capture. The case in which the pursuer is swifter while the evader is more agile, however, has no trivial solution and may require complex or nondeterministic strategies for optimal play.

## Assumptions and Development of the Model

### Initial Configuration

Prey animals generally ignore predators until they have moved within a well-defined flight radius; when the distance between themselves and a predator falls below this value, a flight response is triggered. We are given that the flight distance of a thescelosaurus is less than 50 m and greater than 15 m, such that the moment a velociraptor is detected within this distance, the thescelosaur will immediately flee. Similarly, we assume that the raptor has been deliberately stalking the thescelosaur with the intent of capturing it; thus, the predator will be in a set crouch position and will pursue at the first sign of flight. Although the thescelosaur may be startled or surprised, it initiates the flight and chase sequence, and thus both it and the raptor begin moving (nearly) simultaneously.

Our model allows us to specify the initial state either through selection of locations and facings for each predator and prey, or by implementing a simple probabilistic stalking model to determine the initial separation. In most cases, we set the starting separation of the predator and prey close to the minimum possible value, given to be 15 m. This is because, for most large separations, the optimal strategy of the prey is to move directly away from the predator at maximum speed until the predator is very close. Thus, differences in strategic behavior do not typically become manifest until the prey is forced to deviate from a linear path due to the proximity of the predator.

In the case of multiple predators, we assume either that they have stalked out nearly opposite sides of a circle of radius 15 m or that they start from approximately the same position.

## Sensory Acuity

For the simplest level of approximation, it is sufficient to assume that all participants in a pursuit–evasion game have complete and instantaneous access to the state function for all times, such that they are involved in a game of perfect information. However, this is a relatively inaccurate assumption for most real physical systems. The ability to estimate distances and directions is always subject to random error, and vision is limited to a field of sight subtending an angle of less than  $180^\circ$ . Being forced to rely on other senses, usually hearing, to track a moving object is less than ideal, and can lead to sizable error in the assessment of distances in particular. These limitations are particularly problematic for predators, who have a narrowly focused forward field of vision and depend heavily on being able to estimate not only the present but the future locations of their prey.

Our model functions under assumptions of perfect information but with the introduction of both random and systematic error in sensory perception. The former is enforced by multiplying the magnitude and angle of the displacement vector between predator and prey each by a different random number between 0.95 and 1.05 before passing it to the routine governing the selection of control variables. Additionally, uncertainty in these equations due to the limited field of sight (systematic errors) are estimated using a linearly increasing statistical spread proportional to the angular displacement between the direction of sight and the observed object. The distance and angle are each multiplied by a different random number in the range  $1.00 \pm s\theta/\pi$ , where  $s = 0.05$  for angular displacement and 0.25 for linear displacement. Finally, to emphasize the importance of visual contact, these factors are further increased for the predator by 50% to  $s = 0.075$  and 0.375, respectively.

A related issue is reaction time. In the perfect information case, knowledge of the state vector is imparted instantly. A more realistic assumption is that these data become available for use only after they have been psychologically processed by the brain. Ordinarily, the process of updating awareness of the environment could be considered to be immediate; but in a contest measured in

hundredths of a second, to deny the prey the ability to have an advantage over the predator in its knowledge of its own actions would obscure an essential element of the model. To prevent either dinosaur from being able to react instantaneously to changes in its environment, we delay information about the state variables of other dinosaurs by 0.05 s for the raptor and 0.037 s for the thescelosaur.

## Physical Constraints on Motion

The center of mass of each of the dinosaurs is assumed to be a point particle moving in a two-dimensional plane in accordance with Newtonian mechanics. Two available options in altering the motion of a point (subject to Newtonian kinematic equations) are to impose either a linear or an angular acceleration.

The data provided by biomechanical analysis indicate that the maximum turn radius of the raptor was 1.5 m and that of the thescelosaur was only 0.5 m. (We understand the problem statement to mean that these values apply at top speed, even though this leads to the thescelosaur being capable of a 32-g turn.) This suggests, in each case, that a maximum possible angular displacement in any given time interval may be defined as  $d\theta = a(dt)/v$ , where  $v$  is the current velocity,  $dt$  is the length of the time interval, and  $a$  is the centripetal acceleration, where  $a = v^2/r$ , and  $v$  and  $r$  here are the speed and maximum radius of curvature at maximum velocity.

Without associated data on the linear acceleration of the dinosaurs, it is necessary to invoke an argument by analogy. The African cheetah is a modern predator filling an ecological role similar to that of the raptor. Cheetahs also share many of the same strategic attributes with velociraptors (high speed, limited endurance, and a turning radius inferior to that of their primary prey). One might reasonably assume that the linear acceleration capabilities of the raptor would have been similar. A cheetah can accelerate to peak velocity (over 90 km/hr) in about 2 s. However, the velociraptor has a lower maximum speed, and is also lighter than the cheetah. This suggests that the acceleration for a raptor may be somewhat greater. Operating under the assumption that the velociraptor possesses the same relative acceleration ability compared to its body size, and recognizing that the force exerted by muscle tissue is proportional to the square of the linear dimensions of the body, while body mass is equal to the cube, a factor of  $(2/3)/(1.25)^{2/3} \approx .57$  is not an unreasonable correction to the acceleration of the lighter raptor (where 1.25 is an approximate ratio of the mass of a cheetah to the mass of a raptor).

## Strategy

An animal might be expected to behave in accordance with straightforward heuristic principles, and each of the strategies that we tested reflects such a principle. For the sake of the simulation, we assume each raptor and each