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Theta Constants, Riemann Surfaces and the Modular Group

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Hershel M. Farkas, Irwin Kra

高等教育出版社

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

高 等 教 育 出 版 社

2016 年 12 月

Introduction

The theory of compact Riemann surfaces brings together diverse areas of mathematics. Its building blocks include vast areas of analysis (including Lie theory), geometry/topology and algebra. This was our point of view in our book on Riemann surfaces [6] and it dictated the material to be included in that volume. In particular, we presented a modern approach to the theory of compact Riemann surfaces based on classical methods that prepared the reader to study the modern theories of moduli of surfaces. In this book we head in a different direction and develop another classical connection: to combinatorial number theory. We do not neglect, however, the connections to the problem of uniformizing surfaces represented by very special Fuchsian groups. Problems in number theory can be reformulated as questions about Riemann surfaces, and many of the answers to some of these questions are obtained using function theory. Even though it is an old idea to use function theory (compact Riemann surfaces and automorphic forms) to study analytic and combinatorial number theory and there are many results in these fields, we found it hard to dig out the underlying function theory in the publications of number theorists. No doubt, this is our failing. But since others may also have a deficiency in this area, we decided to organize the material from this point of view. There is new material in this book that has not previously appeared in print, and part of our aim is to present this material to as wide an audience as possible. Our more important aim, however, is to expose the reader to a beautiful chapter in function theory and its applications.

The main actors in our presentation are genus one theta functions and theta constants[1] (including the classical η-function), the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$, and some of the Riemann surfaces that arise as quotients of the action of finite index subgroups of Γ on $\mathbb{H}^2$. We are particularly interested in the principal congruence subgroups $\Gamma(k)$ and the related subgroups $\Gamma_o(k)$ for (usually small) primes k. Some very interesting combinatorial identities follow from the function theory on these surfaces.[2]

Theta functions and theta constants with integral characteristics are classical objects intimately connected with the principal congruence subgroup of level 2, $\Gamma(2)$. This theory is well understood and has as one of its consequences the theorem of Picard: every entire function which omits two values is constant. As is well known, the basic ingredients in the proof of Picard's theorem are that the holomorphic universal covering of the sphere punctured at three points is the upper half plane and that its fundamental group is $\Gamma(2)$. We use theta constants with even integral characteristics to construct the universal covering map, and in this way obtain, without using the general uniformization theorem, the hyperbolicity of the three times punctured sphere. The universal covering map is constructed here as a quotient of fourth powers of any two of the three theta constants. We noticed that in this construction the three even characteristics correspond in a natural way to the three punctures on $\mathbb{H}^2/\Gamma(2)$, and we began to wonder about natural generalizations. In this book, we present the answer to these inquiries. We uniformize the Riemann surfaces $\mathbb{H}^2/\Gamma(k)$ using theta constants with special rational characteristics, and establish a one-to-one (almost canonical) correspondence between the punctures on $\mathbb{H}^2/\Gamma(k)$ and certain equivalence classes of characteristics. For example, the four punctures on $\mathbb{H}^2/\Gamma(3)$ correspond to the characteristics

$$\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}.$$

Furthermore, the Riemann surface is uniformized by a quotient of cubes of any two of the four corresponding theta constants. Similar, but obviously more complicated, expressions uniformize the surfaces represented by the higher level congruence groups. Multiple uniformizations of the same

[1]Thus the θ-functions we study, $\theta[\chi](\zeta, \tau)$, depend on three variables: a characteristic $\chi \in \mathbb{R}^2$; a variable $\zeta \in \mathbb{C}$; and a parameter $\tau \in \mathbb{H}^2$, the upper half plane. Fixing the variable $\zeta = 0$ yields the family of theta constants, an abuse of notation since these are holomorphic functions on $\mathbb{H}^2$; as functions of the local coordinates $q = e^{2\pi\imath\tau}$ these are classically known as q-series. We will use the symbol x for the local variable, since tradition in (parts of complex analysis) reserves the letter q for the weight of an automorphic form.

[2]What is interesting is clearly in the eyes of the beholder. The identities we discuss are obviously interesting to us. The reader must decide whether or not to share our enthusiasm.

Riemann surface lead to theta identities. It is an open problem to determine uniformizations of all four punctured spheres by the methods described above.

The theta constants which appear in our constructions of the uniformizing functions for $\mathbb{H}^2/\Gamma(3)$ are closely related with the formulae used by Euler and Ramanujan in the theory of partitions. Specifically, we note that[3]

$$\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,\tau)=\exp\left(\frac{\pi\imath}{6}\right)x^{\frac{1}{24}}\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{3n^2+n}{2}}=\exp\left(\frac{\pi\imath}{6}\right)x^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-x^n),$$

where $x=\exp\left(\frac{2\pi\imath\tau}{3}\right)$. Continuing in this direction, we discover that uniformizations of the Riemann surfaces $\mathbb{H}^2/\Gamma(k)$ involve functions which appear in the Jacobi triple product. We give a function theoretic proof of this famous formula and then generalize it to the quintuple and septuple product identities, explaining along the way why the formulae obtained are natural from the point of view of the theory of N-th order theta functions. The highlights of the book are systematic studies of theta constant identities, uniformizations of surfaces represented by subgroups of the modular group, partition identities and Fourier series coefficients of automorphic functions, and identities involving the σ-function and Fourier series coefficients of automorphic forms. More detailed information on the contents of each of the chapters follows.

In Chapter 1 we explain the genesis of the modular group in our theory. This group appears naturally when one classifies compact Riemann surfaces of genus one (elliptic curves) up to conformal equivalence. We discuss the generators of this group, find all the fixed points of elements of this group and describe some of the subgroups we shall need in the sequel. Almost everything we do in this chapter is well known and covered in a standard course on complex variables. We describe the structure of the Riemann surfaces $\mathbb{H}^2/G$ for subgroups G of $\text{PSL}(2,\mathbb{Z})$. In order to show that this well known and elementary material has nontrivial consequences, we use this theory to show that factors of integers of the form N^2+1 are always the sums of two squares, and we give a geometric criterion for N^2+1 to be a prime number. The result is that N^2+1 is prime if and only if the portion in the upper half plane of the straight line joining the origin to the point $N+\imath$ in the complex plane intersects the orbit of $\imath$ under $\text{PSL}(2,\mathbb{Z})$ in exactly two points, namely $N+\imath$ and $\frac{N+\imath}{N^2+1}$. We have included some of the function theoretic prerequisites in this chapter. However, most of the prerequisites will be described when needed. In general, we provide full definitions of all concepts. We do not repeat proofs or arguments readily available in

[3]The reader may at this point conclude that the η-function is a disguised theta constant with a rational characteristic. It will also become obvious that the prime 3 plays a special role in our drama.

other books, but do reproduce, usually in modified form, proofs from many research papers. The bibliography of relevant books (after the last chapter) is followed by a set of bibliographical notes containing an (incomplete) list of research and expository notes on the material covered by this volume.

In Chapter 2 we define the theta functions and theta constants with characteristics and specialize to rational characteristics of the form $\left[\begin{matrix}\frac{m}{k}, \\ \frac{m'}{k}\end{matrix}\right]$ with m, m' and k integers[4] of the same parity in order to construct a correspondence between equivalence classes of sets of characteristics and the punctures on the surface $\mathbb{H}^2/\Gamma(k)$. In this chapter we derive a most important property of the theta functions and theta constants, the transformation formula (a significant generalization of known transformation rules) for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on the upper half plane, and we give a function theoretic proof of the Jacobi triple product formula and some generalizations. The transformation formula allows us to use theta functions to construct modular and cusp forms for subgroups of $\mathrm{PSL}(2,\mathbb{Z})$. The function theoretic proof of the Jacobi triple product formula yields new proofs of important identities of Jacobi and Euler that are needed for our presentation of partition theory in Chapter 5. We construct theta constant identities which turn out to agree with discoveries of Ramanujan. Our derivations of these identities are on the one hand quite natural, and on the other hand lead to simpler expressions of the equivalent identities discovered by Ramanujan in the sense that they do not involve irrationalities (extracting roots of single valued functions) until they are artificially introduced. It appears that the theta constants which we use are a lot richer than the ones that Ramanujan had at his disposal.

Chapter 3, in a sense, contains the most important material of the book. In it we construct automorphic forms and functions for the principal congruence subgroups and some related groups. The theory we describe is particularly well suited for the study of $\mathbb{H}^2/\Gamma(k)$, and we obtain holomorphic mappings of these Riemann surfaces into projective spaces of rather low dimensions. Some interesting geometry and topology emerges as we observe connections of the principal congruence subgroups with the Platonic solids. This phenomenon first occurs for $k = 3$, 4 and 5. In these cases, $\Gamma/\Gamma(k) \cong \mathrm{PSL}(2,\mathbb{Z}_k)$ are the symmetry groups of the regular tetrahedron, octahedron and icosahedron, respectively. This suggests a relation between the images of these curves in the projective space and the regular solids and leads to a generalization of the regular solids based on curves of (some) positive genera. While our development is most suited for the groups $\Gamma(k)$, for many of the most important applications we need to construct automorphic

[4]Assume, unless otherwise stated, for these introductory remarks that k is a (positive) prime.

forms for $\Gamma_o(k)$. Part of the extra difficulties involves the presence of torsion in these groups. We need more detailed analysis, in this and subsequent chapters, to handle these richer groups.

Chapter 4 is a systematic study of theta identities. Theta constant identities are interesting for several reasons. One reason is their inherent elegance and symmetry. There is something tantalizingly beautiful about the identities of Jacobi, for example,

$$\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0\\0\end{bmatrix}(z,\tau)$$
$$=\theta^2\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0\\1\end{bmatrix}(z,\tau)+\theta^2\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}1\\0\end{bmatrix}(z,\tau)$$

or its restriction to $z = 0$ (known as the Jacobi quartic identity)

$$\theta^4\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)=\theta^4\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}1\\0\end{bmatrix}(0,\tau).$$

Aside from the inherent beauty of the form there is a combinatorial content to the identity. It relates the number of representations of an integer as a sum of four squares to its representation as a sum of four triangular numbers. This is of course just the beginning of a chapter. As one delves deeper into the theory, one finds more and more beautiful identities with more and more combinatorial content.

We present four distinct ways of constructing such identities. In two of theses methods, we use the classical technique of constructing finite dimensional linear spaces of theta functions or modular forms or functions on certain Riemann surfaces and use linear algebra and the simple notions of independence and dependence. We present another newer technique which makes use of the fact that we can use theta functions to construct elliptic functions and the fact that the sum of the residues of an elliptic function in a period parallelogram vanishes. This technique is very powerful and succeeds in giving a rather large set of identities. The main idea here is to construct the correct elliptic function, which turns out to be more an art than a science, that leads to an interesting identity. The fourth method for constructing identities uses uniformizations of Riemann surfaces. In a very nontrivial sense the next two chapters are also studies of theta identities, this time of a very special sort with a very special purpose.

In Chapter 5 we turn to the congruences discovered by Ramanujan for the partition function and show how they follow in a rather simple way from function theory on the appropriate Riemann surfaces. The main ingredient is the construction of the same function in more than one way. Some of the constructions involve averaging operators. It turns out that the averaging processes produce in some cases constant functions. We study the many implications of these constructions, especially of the appearance of constant functions. This material may not be new to the literature. We present it in a unified way based on function theoretic foundations that most of the time remove and in general isolate the mysteries in many of the research monographs on the subject. This chapter is based almost entirely on the properties of the classical η-function, a very special case of the theory described in the previous chapters. We need to know rather detailed information on its multiplier system. The needed number theoretic arguments are found in [**16**], for example. We would like to avoid dependence on these. We have only partially succeeded in doing so and have hence not included in this volume most of the results of this effort.

In Chapter 6 we begin by reviewing some concepts from covering space theory and show how these ideas lead to beautiful identities among theta constants and their interpretation as identities among infinite products. Here the main tools are the uniformizations of the Riemann surfaces in question. We then continue by showing how many of the ideas used in the congruences related with the partition function and its generalizations carry over to other modular forms. We treat in particular the $\jmath$-function and the congruences satisfied by the coefficients of its Laurent series expansion.

In Chapter 7 we show how statements about partitions are related to other combinatorial quantities such as representations of positive integers as sums of squares or of triangular numbers, and most importantly to the divisors of an integer. In particular, we describe relations to the question of primality of integers depending on statements about partitions. This suggests that while primality is usually thought of as a subject in multiplicative number theory, it can also be viewed as a part of additive number theory.

We give some examples to show what type of results can be expected in this chapter in the expectation that these applications are the main interests of some readers of this text. We emphasize that these results were not the reason for writing this book. The list of examples is by no means exhaustive. Let $\sigma(n)$ denote the classical σ-function; that is, $\sigma(n)$ is the sum of the divisors of the positive integer n. We show that

$$\sum_{j=-\infty}^{\infty} \sigma\left(n - \frac{3j^2+j}{2}\right) = 0$$

whenever n is not of the form $\frac{3m^2+m}{2}$ with $m \in \mathbb{Z}$. A companion related result is

$$\sum_{j=0}^{\infty} \sigma\left(n - \frac{j^2+j}{2}\right) = 4\sum_{j=0}^{\infty} \sigma\left(\frac{n}{2} + \frac{j^2+j}{4}\right)$$

whenever n is not of the form $\frac{m^2+m}{2}$ with $m \in \mathbb{Z}$. We obtain a variant of Jacobi's result on the number of ways a positive integer can be written as a sum of four squares by replacing squares with triangular numbers, obtaining, in our view, a cleaner result. A last example of the type of result we will discuss in this chapter is the following. Let S be a set consisting of the positive integers with an additional copy of those positive integers congruent to zero modulo 7. Decompose S into its even and odd parts, E and O respectively. Denote by $P_E(k)$ and $P_O(k)$ the number of partitions of k with parts taken from the sets E and O respectively. We prove that for all nonnegative integers k,

$$P_E(2k) = P_O(2k+1).$$

The prerequisites for this book are a thorough understanding of material traditionally covered in first year graduate courses – especially the contents of the complex analysis course. We review, however, the most salient points about elliptic function theory portions of this course. Although a knowledge of Riemann surfaces and Fuchsian groups is helpful, it is not needed by the reader who is willing to accept the summaries of the required material (with references to the literature). Although we do not, in general, reproduce material available in other textbooks, we make an exception for material on theta functions and theta constants despite the availability of excellent sources (for example, [**23**]). We do so, not only for the convenience of the reader, but also to emphasize our point of view. We have also ignored, to a great extent, the combinatorial and special functions connection. These are discussed in [**2**], [**7**] and [**10**], for example.

A road map

While we did not intend to write an encyclopedic text, the result has been quite a large book. We take the liberty of offering the readers our suggestions for possible ways of going through this text, which was written with several different types of readers in mind. These range from the beginning graduate mathematics student through the professional mathematician whose interests are either in combinatorial mathematics (partition theory, representation as sums of squares, counting points on conic sections) or function theory (Riemann surfaces, modular forms). Theoretical physicists might be interested in portions of the material we cover.

The reader is expected to have a reasonable knowledge of the theory of functions of a complex variable, through the Riemann mapping theorem, and enough mathematical maturity to follow an argument even though unfamiliar with the proofs of all the tools used. Thus, the book can be used as a text for a topics course in either analysis or analytic number theory, and as such a reasonable approach would be to go through Chapter 1 sections 1-4.5 and sections 6 and 7. The above material is essentially background material to acquaint the reader with the domains on which we will be doing analysis to obtain the combinatorial results. This reader should then continue with Chapter 2, where the theory of the one dimensional theta function is presented. This chapter should be read in its entirety. In Chapter 3 the reader or instructor pressed for time could read quickly the first section for the definitions and then go on to a careful study of sections 2 through 8.4. Chapter 4 should be read in its entirety. The above could constitute a one semester topics course for beginning graduate students.

The above suggestion leaves out Chapters 5, 6 and 7, which occupy much of this book and are an important part of it, since they deal with the theory of congruences for the Ramanujan partition function, the congruences for the $\jmath$-function, and the combinatorial interpretations of many of the identities derived in Chapter 4. In a one-year course the material studied could include Chapter 5 through section 10.8 and section 12,[5] all of Chapters 6 and 7.

The professional mathematician who is interested in Riemann surface theory should study Chapter 1, including sections 4.6 through 5.7, in order to get a picture of where the theory can possibly go. If conversant with the theory of the Riemann theta function, the reader can skip section 1 of Chapter 2 and read the remainder of that chapter. The reader should then proceed to the beginning of Chapter 3. Some of the introductory material of this chapter can be skipped or read quickly to get acquainted with the notation used; the choice of which of the remaining sections to read should be guided by interests; we suggest that this include section 10. Chapter 4 should be read in its entirety and then Chapter 5 and Chapter 6, once again guided by the interests of the reader. Chapter 7 should also be read in its entirety.

The professional mathematician whose interests are in combinatorial mathematics may wish to begin by looking at Chapter 7 and then proceed backwards through the theory. Needless to say, Chapters 2, 3 and 4 will have to be read at some point, and if interested in Ramanujan congruences, Chapter 5 is a must. In any event section 12 of Chapter 5 should be reviewed.

There is lots of flexibility in the way the text can be studied and/or approached. We trust the various readers will find their way through the maze and enjoy the material they stop to study or, as we and others have said, will enjoy this tour of Ramanujan's garden and the flowers they pick there.

[5]In a course where all nonstandard material is included, the instructor might want to spend some time on the multiplicative properties of the η-function. These could be taken from Knopp's book [**16**].

We have included descriptions of many special cases and summarized the results of many calculations. Most of the nontrivial calculations used the symbolic manipulation programs MATHEMATICA and/or MAPLE. In order to excite the reader about the flowers in the beautiful garden we are cultivating here, we start each chapter with (what we regard as) a handsome example of what will follow. We have included a number of accessible exercises and research level problems. The latter may be quite challenging and are at times only conjectures. The reader should also approach the many special cases we have included as challenges to obtain independent solutions. They are presented in the spirit of exercises, with solutions supplied by the authors.

Numbering systems. The book consists of seven chapters and a set of bibliographical notes that will be maintained and updated on the web. Chapters are subdivided into sections; these into subsections. Definitions, lemmas, propositions, theorems, exercises, problems and remarks are labeled consecutively as a single group within each section. A typical item is Theorem section.number; number starts with 1 for the first item in the section. Thus, for example, in Chapter 2, Definition 2.32 (in section 2) in our numbering scheme is followed by Lemma 4.1 (in section 4). Equations that will be referenced subsequently in the text are labeled by a decimal: chapter.number; number starts with 1 for the first numbered equation in the chapter. Tables and figures are numbered consecutively in the book.

Acknowledgments. We thank
1. the mathematicians who have had the patience to listen to our lectures on the subject of this book: whether our students at the Hebrew University in Jerusalem or the State University of New York in Stony Brook, who at times have provided us with constructive criticism, or colleagues at various institutions who have invited us to speak about this subject;
2. our graduate students who read previous drafts of the book and pointed out errors and ambiguities (needless to say, the remaining errors and ambiguities are the authors' responsibilities);
3. our undergraduate students who helped with the MATHEMATICA calculations and the figures;
4. the editorial and technical staff of the American Mathematical Society, especially Deborah Smith and Janet Pecorelli who guided us through the final preparation of the manuscript; and most importantly,

5. our respective wives, who patiently suffered with us through this long project.

Finally we wish to acknowledge and thank the U.S. - Israel Binational Science Foundation, the U.S. National Science Foundation, The Hebrew University and the State University of New York at Stony Brook for their support over the years that helped produce the research for this book; for providing us with the needed time, facilities and equipment; and for making possible visits by each of us to the institution of the other, not only to work on this volume.

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