

美国数学会经典影印系列



# Representations of Algebraic Groups

Second Edition

代数群表示论

第二版

Jens Carsten Jantzen



高等教育出版社

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 **AMS**  
AMERICAN MATHEMATICAL SOCIETY

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## 出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

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我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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## Introduction

**I** This book is meant to give its reader an introduction to the representation theory of such groups as the general linear groups  $GL_n(k)$ , the special linear groups  $SL_n(k)$ , the special orthogonal groups  $SO_n(k)$ , and the symplectic groups  $Sp_{2n}(k)$  over an algebraically closed field  $k$ . These groups are algebraic groups, and we shall look only at representations  $G \rightarrow GL(V)$  that are homomorphisms of algebraic groups. So any  $G$ -module (vector space with a representation of  $G$ ) will be a space over the same ground field  $k$ .

Many different techniques have been introduced into the theory, especially during the last thirty years. Therefore, it is necessary (in my opinion) to start with a general introduction to the representation theory of algebraic group schemes. This is the aim of Part I of this book, whereas Part II then deals with the representations of reductive groups.

**II** The book begins with an introduction to schemes (Chapter I.1) and to (affine) group schemes and their representations (Chapter I.2). We adopt the “functorial” point of view for schemes. For example, the group scheme  $SL_n$  over  $\mathbf{Z}$  is the functor mapping each commutative ring  $A$  to the group  $SL_n(A)$ . Almost everything about these matters can also be found in the first two chapters of [DG]. I have tried to enable the reader to understand the basic definitions and constructions independently of [DG]. However, I refer to [DG] for some results that I feel the reader might be inclined to accept without going through the proof. Let me add that the reader (of Part I) is supposed to have a reasonably good knowledge of varieties and algebraic groups. For example, he or she should know [Bo] up to Chapter III, or the first seventeen chapters of [Hu2], or the first six ones of [Sp2]. (There are additional prerequisites for Part II mentioned below.)

In Chapter I.3, induction functors are defined in the context of group schemes, their elementary properties are proved, and they are used to construct injective modules and injective resolutions. These in turn are applied in Chapter I.4 to the construction of derived functors, especially to that of the Hochschild cohomology groups and of the derived functors of induction. In contrast to the situation for finite groups, the induction from a subgroup scheme  $H$  to the whole group scheme  $G$  is (usually) not exact, only left exact. The values of the derived functors of induction can also be interpreted (and are so in Chapter I.5) as cohomology groups of certain associated bundles on the quotient  $G/H$  (at least for algebraic schemes over a field). Before doing that, we have to understand the construction of the quotient  $G/H$ . The situation gets simpler and has some additional features if  $H$  is normal in  $G$ . This is discussed in Chapter I.6.

One can associate to any group scheme  $G$  an (associative) algebra  $\text{Dist}(G)$  of distributions on  $G$  (called the hyperalgebra of  $G$  by some authors). When working over a field of characteristic 0, it is just the universal enveloping algebra of the Lie

algebra  $\text{Lie}(G)$  of  $G$ . In general, it reflects the properties of  $G$  much better than  $\text{Lie}(G)$  does. This is described in Chapter I.7.

A group scheme  $G$  (say over a field) is called finite if the algebra of regular functions on  $G$  is finite dimensional. For such  $G$  the representation theory is equivalent to that of a certain finite dimensional algebra and has additional features (Chapter I.8). For us, the most important cases of finite group schemes arise as Frobenius kernels (Chapter I.9) of algebraic groups over an algebraically closed field  $k$  of characteristic  $p \neq 0$ . For example, for  $G = GL_n(k)$  the map  $F : G \rightarrow G$  sending any matrix  $(a_{ij})$  to  $(a_{ij}^p)$  is a Frobenius endomorphism. The kernel of  $F^r$  (in the sense of group schemes) is the  $r^{\text{th}}$  Frobenius kernel  $G_r$  of  $G$ . The representation theory of  $G_1$  (for any  $G$ ) is equivalent to that of  $\text{Lie}(G)$  regarded as a  $p$ -Lie algebra.

In order to apply our rather extensive knowledge of the representation theory of groups like  $SL_n(\mathbf{C})$  to that of  $SL_n(k)$ , where  $k$  is a field of prime characteristic, one uses the group scheme  $SL_n$  over  $\mathbf{Z}$ . One chooses  $SL_n$ -stable lattices in  $SL_n(\mathbf{C})$ -modules and tensors with  $k$  in order to get  $SL_n(k)$ -modules. Some general properties of this procedure are proved in Chapter I.10.

From Part I, the contents of Chapters 1 (until 1.6), 2, 3, 4 (until 4.18), 5 (mainly 5.8–5.13), and 6 (until 6.9) are fundamental for everything to follow. The other sections are used less often.

In Part II, the reader is supposed to know the structure theory of reductive algebraic groups (over an algebraically closed field) as to be found in [Bo], [Hu2], [Sp2]. The reader is invited (in Chapter II.1) to believe that there is for each possible root datum a (unique) group scheme over  $\mathbf{Z}$  that yields for every field  $k$  (by extension of the base ring) a split reductive group defined over  $k$  having the prescribed root datum. Furthermore, he or she has to accept that all “standard” constructions (like root subgroups, parabolic subgroups, etc.) can be carried out over  $\mathbf{Z}$ . (The sceptical reader should turn to [SGA 3] for proof.) I have included a proof (following Takeuchi) of the uniqueness of an algebraic group with a given root datum (over an algebraically closed field) that does not use case-by-case considerations.

**III** Let me describe a selection of the contents of the remaining chapters in more detail. Assume from now on (in this introduction) that  $k$  is an algebraically closed field and that  $G$  is a (connected) reductive algebraic group over  $k$  with a Borel subgroup  $B \subset G$  and a maximal torus  $T \subset B$ . Let  $X(T)$  denote the group of characters of  $T$ .

In case  $\text{char}(k) = 0$  the representation theory of  $G$  is well understood. Each  $G$ -module is semi-simple. The simple  $G$ -modules are classified (as in the case of compact Lie groups or of complex semi-simple Lie algebras) by their highest weights. Furthermore, one has a character formula for these simple modules. In fact, Weyl’s formula for the compact groups holds when interpreted in the right way. (For us, the character of a finite dimensional  $G$ -module will always be the family of the dimensions of its weight spaces with respect to  $T$ . As the semi-simple elements in  $G$  are dense in  $G$  and as each semi-simple element is conjugate to one in  $T$ , the character determines the trace of any  $g \in G$  on the  $G$ -module.)

The situation in prime characteristic is much worse. Except for the case of a torus, there are non-semi-simple  $G$ -modules. Except for a few low rank cases, we do not know a character formula for the simple modules, and Weyl’s formula

will certainly not carry over. Only one property survives: The simple modules are still classified by their highest weights, and the possible highest weights are the “dominant” weights in  $X(T)$ . (The notion of dominant depends on the choice of an ordering of  $X(T)$ . We shall always work with an ordering for which the weights of  $T$  on  $\text{Lie}(B)$  are negative.) This classification is due to Chevalley, cf. [SC]. Let  $L(\lambda)$  denote the simple module with highest weight  $\lambda$ .

The difference of the situations in zero and prime characteristic can be observed already in the case  $G = SL_2(k)$ . Let  $H(n)$  be the  $n^{\text{th}}$  symmetric power of the natural representation of  $G$  on  $k^2$ . If  $\text{char}(k) = 0$ , then  $H(n) = L(n)$  for all  $n \in \mathbf{N}$ . (For  $SL_2$  we identify  $X(T) \simeq \mathbf{Z}$  in such a way that the dominant weights correspond to  $\mathbf{N}$ .) If  $\text{char}(k) = p \neq 0$ , then obviously not all  $H(n)$  can be simple: For all positive  $r, n \in \mathbf{N}$  the image of the map  $f \mapsto f^{p^r}$  from  $H(n)$  to  $H(p^r n)$  is a proper submodule of  $H(p^r n)$ , so  $H(p^r n)$  is not simple. It is not too difficult to show for any  $n$  that  $H(n)$  contains  $L(n)$  as its unique simple submodule, and that  $H(n) = L(n)$  if and only if  $n = ap^r - 1$  for some  $a, r \in \mathbf{N}$  with  $0 < a \leq p$ . So for all other  $n$  the module  $H(n)$  is not semi-simple.

For arbitrary  $G$  one gets  $L(\lambda)$  as the unique simple submodule of an induced module  $H^0(\lambda)$ : One extends  $\lambda \in X(T)$  to a one dimensional representation of  $B$  such that the unipotent radical of  $B$  acts trivially. Then  $H^0(\lambda)$  is the  $G$ -module induced by this  $B$ -module. It is nonzero if and only if  $\lambda$  is dominant. (In the case  $G = SL_2(k)$  the  $H^0(\lambda)$  are just the  $H(n)$  from above.) This is the main content of Chapter II.2.

The case  $G = SL_2(k)$  with  $\text{char}(k) = p \neq 0$  can serve to illustrate other general results also. For any vector space  $V$  over  $k$  let  $V^{(r)}$  be the vector space that is equal to  $V$  as an additive group and where any  $a \in k$  acts as  $a^{p^{-r}}$  does on  $V$ . Then the map  $f \mapsto f^{p^r}$  is linear when regarded as a map  $H(n)^{(r)} \rightarrow H(p^r n)$ , hence a homomorphism of  $G$ -modules. It is not difficult to show: If  $n = \sum_{i=0}^r a_i p^i$  with  $0 \leq a_i < p$  for all  $i$ , then  $f_0 \otimes f_1 \otimes \cdots \otimes f_r \mapsto \prod_{i=0}^r f_i^{p^i}$  is an isomorphism

$$H(a_0) \otimes H(a_1)^{(1)} \otimes \cdots \otimes H(a_r)^{(r)} \xrightarrow{\sim} L(n).$$

This result was generalised in [Steinberg 2] to all  $G$ : A suitable  $p$ -adic expansion of the highest weight  $\lambda$  leads to a decomposition of  $L(\lambda)$  into a tensor product of the form  $L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \cdots \otimes L(\lambda_r)^{(r)}$ . This tensor product theorem reduces the problem of calculating the characters of all simple  $G$ -modules to a finite problem (for each  $G$ ). Steinberg’s proof relied on a theorem from [Curtis 1] on the representations of  $\text{Lie}(G)$ . In the special case of  $G = SL_2(k)$ , this theorem says: Each  $L(n)$  with  $n < p$  remains simple for  $\text{Lie}(G)$ , and each simple module of  $\text{Lie}(G)$  as a  $p$ -Lie algebra is isomorphic to exactly one  $L(n)$  with  $n < p$ . More generally, each  $L(n)$  with  $n < p^r$  is simple for the  $r^{\text{th}}$  Frobenius kernel of  $SL_2(k)$ , and we get thus each simple module for this infinitesimal group scheme. This result again has an extension to all  $G$  and then leads to a rather simple proof of Steinberg’s tensor product theorem, discovered by Cline, Parshall, and Scott. (All this is done in Chapter II.3.)

The choice of the notation  $H^0(\lambda)$  for the induced module has been influenced by the fact that  $H^0(\lambda)$  is the zeroth cohomology group of a line bundle on  $G/B$  associated to  $\lambda$ . Let  $H^i(\lambda)$  denote the  $i^{\text{th}}$  cohomology group (for any  $\lambda \in X(T)$ , not only for dominant ones). We could have constructed  $H^i(\lambda)$  also by applying the  $i^{\text{th}}$

derived functor of induction from  $B$  to  $G$  to the one-dimensional  $B$ -module defined by  $\lambda$ . Another result from characteristic zero that does not carry over to prime characteristic is the Borel-Bott-Weil theorem. It describes explicitly all  $H^i(\mu)$  with  $i \in \mathbf{N}$  and  $\mu \in X(T)$ : For each  $\mu$  there is at most one  $i$  with  $H^i(\mu) \neq 0$ , and this  $H^i(\mu)$  can then be identified with a specific  $L(\lambda)$ . We observed already that we cannot expect the  $H^i(\mu)$  to be simple in prime characteristic. But, even worse, there can be more than one  $i$  for a given  $\mu$  with  $H^i(\mu) \neq 0$ , and the character of  $H^i(\mu)$  will depend on the field. (This was first discovered by Mumford.) It is crucial for the representation theory that one special case of the Borel-Bott-Weil theorem holds over any  $k$ : If  $\lambda$  is dominant, then  $H^i(\lambda) = 0$  for all  $i > 0$ . This is Kempf's vanishing theorem from [Kempf 1]. The proof given here in Chapter II.4 is due to Haboush and Andersen (independently). In Chapter II.5, we give Demazure's proof of the Borel-Bott-Weil theorem in case  $\text{char}(k) = 0$ . Furthermore we prove (following Donkin) that Weyl's character formula yields the alternating sum (over  $i$ ) of the characters of all  $H^i(\mu)$ .

Assume from now on that  $\text{char}(k) = p \neq 0$ . Kempf's vanishing theorem implies that one can construct for any  $k$  the modules  $H^0(\lambda)$  with  $\lambda$  dominant by starting with the similar object over  $\mathbf{C}$ , taking a suitable lattice stable under a  $\mathbf{Z}$ -form of  $G$ , and then tensoring with  $k$ . To construct representations in this way has the advantage that one can carry out specific computations more easily. Several examples computed especially by Braden then led Verma in the late 1960s to several conjectures (cf. [Verma]) that had a great influence on the further development of the theory. One conjecture is the linkage principle (Chapter II.6): If  $L(\mu)$  is a composition factor of  $H^0(\lambda)$  (or, more generally, if  $L(\mu)$  and  $L(\lambda)$  are both composition factors of a given indecomposable  $G$ -module), then  $\mu \in W_p \cdot \lambda$ . Here  $W_p$  is the group generated by the Weyl group  $W$  and by all translations by  $p\alpha$  with  $\alpha$  a root. The dot is to indicate a shift in the action by  $\rho$ , the half sum of the positive roots (i.e.,  $w \cdot \lambda = w(\lambda + \rho) - \rho$ ). For large  $p$  this principle was proved in [Humphreys 1]. The result was then extended by several people to almost all cases, but a general proof appeared only in 1980 (in [Andersen 4]). It relies on an analysis of the failure of Demazure's proof (of the Borel-Bott-Weil theorem) in prime characteristic.

Another conjecture of Verma was a special case of the translation principle (Chapter II.7): If two dominant weights  $\lambda, \mu$  belong to the same "facet" with respect to the affine Weyl group  $W_p$ , then the multiplicity of any  $L(w \cdot \lambda)$  with  $w \in W_p$  as a composition factor of  $H^0(\lambda)$  should be equal to that of  $L(w \cdot \mu)$  in  $H^0(\mu)$ . This was proved (modulo the linkage principle) in [Jantzen 2].

The approach to the  $H^0(\lambda)$  via representations over  $\mathbf{Z}$  also has the advantage that it allows the construction of a certain filtration (Chapter II.8) of  $H^0(\lambda)$ . One can compute the sum of the characters of the terms in the filtration ([Jantzen 3] for large  $p$ , [Andersen 12] in general) and use this "sum formula" to get information about composition factors. For example, it leads to a computation of the characters of all simple modules for  $G = SL_4(k)$  or for  $G$  of type  $G_2$ .

If  $\lambda$  and  $\lambda + p\nu$  are weights that are "small" with respect to  $p^2$  and that are "sufficiently dominant" (see II.9.17/18 for a more precise condition), then one gets the composition factors of  $H^0(\lambda + p\nu)$  from those of  $H^0(\lambda)$  by adding  $p\nu$  to the highest weights. This was proved first in [Jantzen 4] using involved computations. Later on it was realised that it follows rather easily if one develops the representation theory of the group scheme  $G_r T$ . For  $\lambda$  as above experimental evidence (cf.

[Humphreys 10]) indicated that the  $H^i(w \cdot \lambda)$  with  $w \in W$  satisfy a weak version of the Borel-Bott-Weil theorem ( $H^i(w \cdot \lambda) \neq 0$  for at most one  $i$ ). This was then proved in [Cline, Parshall, and Scott 10] using the representation theory of the group scheme  $G_r B$ . All this is described in Chapter II.9.

Let us assume that  $G$  is semi-simple and simply connected. There is for each positive integer  $r$  a unique simple  $G$ -module that is simple and injective for  $G_r$ . It is called the  $r^{\text{th}}$  Steinberg module and was first discovered by Steinberg within the representation theory of finite Chevalley groups. We do not look at its great importance there, but discuss some applications to the representation theory of  $G$  (Chapter II.10). It plays a crucial role in Haboush's proof that  $G$  is geometrically reductive. One may wonder whether any injective  $G_r$ -module can be extended to a  $G$ -module. For large  $p$  this was proved by Ballard. We discuss this (with some applications to the representation theory of  $G$ ) in Chapter II.11.

One can write down the character of a simple  $G$ -module  $L(\lambda)$  if one knows all extension groups  $\text{Ext}_G^n(L(\lambda), H^0(\mu))$ , see II.6.21. Unfortunately, rather little is known about these groups. There has been a considerable amount of work (especially by Cline, Parshall, and Scott) to understand better the Hochschild cohomology groups  $H^n(G, M) \simeq \text{Ext}_G^n(k, M)$ . One has  $H^n(G, M) \simeq \varprojlim H^n(G_r, M)$  if  $\dim M < \infty$ . So one may hope to get information on  $G$ -cohomology from information on  $G_r$ -cohomology. Here the most remarkable result is due to Friedlander and Parshall: For large  $p$  the cohomology ring  $H^*(G_1, k)$  is isomorphic to the ring of regular functions on the nilpotent cone in  $\text{Lie}(G)$ . This can be found in Chapter II.12.

The orbits of  $B$  on  $G/B$  are isomorphic to affine spaces. They are called Bruhat cells, while their closures are called Schubert varieties. For example,  $G/B$  itself is a Schubert variety. One can extend Kempf's vanishing theorem to any Schubert variety  $Y \subset G/B$ : If one restricts to  $Y$  the line bundle on  $G/B$  corresponding to a dominant weight  $\lambda$ , then all higher cohomology groups vanish. As an application one can prove the normality of  $Y$  and a character formula for the space of global sections. These results were proved by Mehta, Ramanathan, Seshadri, Ramanan, and Andersen. One can find this in Chapter II.14, whereas Chapter II.13 provides the necessary background on Schubert varieties.

The last seven chapters mentioned above can be divided into three groups (II.8, II.9–12, II.13–14), which are independent of each other. Also, the logical interdependence of Chapters II.10–12 is rather weak.

**IV** So far this introduction has been copied (with minor modifications) from the introduction of the first edition. For this new edition I have added a few chapters that I shall discuss in a moment.

As far as the old chapters are concerned, I have tried to correct mistakes and misprints. I have added several remarks and in a few cases rearranged things. In doing so, I have tried to avoid renumbering subsections and equations so that references to the first edition would also work with the second one. However, in a few cases (in particular in Chapter II.9) this turned out to be impossible. In these cases I have summed up the changes at the end of the introductions to the chapters (see II.7–9, 11, 12).

**V** The new chapters were added to Part II. They are not identified by numbers, but by capital letters so to indicate the break between the old and the new.

Keep the general assumptions from above (III). Let  $\pi$  be a finite set of dominant weights that is “saturated”. This means that for each  $\mu \in \pi$  also all dominant weights  $\nu < \mu$  belong to  $\pi$ . Then it makes sense to consider the “truncated” category of all  $G$ -modules having only composition factors with a highest weight in  $\pi$ . Such categories are studied in Chapter II.A. Each of them is equivalent to the category of all modules over a suitable finite dimensional algebra. This allows the application of techniques from the representation theory of finite dimensional algebras to the theory of  $G$ -modules.

The categories of homogeneous polynomial  $GL_n$ -modules are special cases of truncated categories for  $G = GL_n$ . They connect the representation theory of  $GL_n$  with that of Schur algebras and of symmetric groups as well as with the theory of polynomial functors.

In Chapter II.B several cohomological results for  $G$ -modules are generalised from the case of a ground field to the case where one works over a principal ideal domain. For some of these proofs we have to use results from Chapter II.A.

In Chapters II.C and II.D we describe some consequences of Lusztig’s conjecture leading to the calculation of Ext groups and to information about submodule structures, e.g., on the layers in the radical filtration of “baby Verma modules” (induced modules for  $G_1$ ). One gets also that some of these consequences in turn imply Lusztig’s conjecture.

Tilting modules (discussed in Chapter II.E) are  $G$ -modules that have a filtration with factors of the form  $H^0(\lambda)$  as well as a filtration with factors of the form  $H^0(\mu)^*$ . The indecomposable tilting modules are classified by the dominant weights (like the simple  $G$ -modules) and as for the simple  $G$ -modules the computation of the characters of indecomposable tilting modules is a major open problem. In the case of  $G = GL_n$  these tilting modules lead to yet another connection between the representation theory of  $GL_n$  and that of the symmetric groups.

The technique of “Frobenius splitting” is a powerful method to prove vanishing results for varieties in prime characteristics. We describe this in Chapter II.F and then use it to give alternative approaches to results from Chapter II.14. In Chapter II.G we use then Frobenius splitting techniques to prove the main properties of modules with a good filtration (announced in Chapter II.4).

The final chapter II.H surveys certain parts of the representation theory of quantum groups. Using these groups one can construct a representation theory in characteristic 0 that is similar to that of  $G$  in prime characteristic. However, one can prove stronger results on the quantum groups side, e.g., on characters of simple modules or of indecomposable tilting modules. This has then applications to the characteristic  $p$  theory.

**VI** Suppose that  $\mathbf{F}_q$  is a finite field and that  $k$  is an algebraically closed extension of  $\mathbf{F}_q$ . The representation theory of groups like  $GL_n(k)$  or  $Sp_{2n}(k)$  has been developed in close interaction with that of groups like  $GL_n(\mathbf{F}_q)$  or  $Sp_{2n}(\mathbf{F}_q)$ . It would therefore have been desirable to have a third part of the book dealing with representations of finite Chevalley groups (mainly over fields of the same characteristic as that over which the groups are defined). In fact, I promised to write such a part in a preliminary foreword to a preprint version of Part I. However, I hope to be forgiven for breaking this promise, as otherwise the book would have grown to an unreasonable size. Furthermore, I suspect that people most interested in these

finite groups would prefer another book where they would not have to devour at first all of Parts I and II. Now (2003) a book on this topic is under preparation by Jim Humphreys.

**VII** In the summer of 1984, I gave a series of lectures on some topics discussed in this book at the East China Normal University in Shanghai. I had been asked in advance to provide the audience with some notes. When doing so, I was still undecided about the precise contents of my lectures. I therefore included more material than I could possibly cover in my lectures. The first edition of this book has grown out of those notes.

I should like to use this opportunity to thank the mathematicians I met in Shanghai, especially Professor Cao Xihua, for their hospitality during my stay and for the patience with which they listened to my lectures.

Thanks are also due to Henning Haahr Andersen, Rolf Farnsteiner, Burkhard Haastert, Jim Humphreys, Niels Lauritzen, Zongzhu Lin, and Jesper Funch Thomsen for useful comments on my manuscript and for providing lists of misprints, before and after the publication of the first edition and during the preparation of the second edition.

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# Part I

## General Theory



## CHAPTER 1

# Schemes

It is the purpose of this first chapter to give the necessary introduction to schemes following the functorial approach of [DG]. This approach appears to be most suitable when dealing with group schemes later on. After trying to motivate the definitions in 1.1, we discuss affine schemes in 1.2–1.6. What is done there is fundamental for the understanding of everything to follow.

As far as arbitrary schemes are concerned, it is most of the time enough to know that they form a certain class of functors that includes the affine schemes and so that over an algebraically closed field any variety gives rise to a scheme in a canonical way. Sometimes, e.g., when dealing with quotients, it is useful to know more. So we give the appropriate definitions in 1.7–1.9 and mention the comparison with other approaches to schemes and with varieties in 1.11. The elementary discussion of a base change in 1.10 is again necessary for many parts later on.

There is also a discussion of closed subfunctors and of closures (1.12–1.14). Finally, we describe the functor of morphisms between two functors (1.15) and prove some of its properties. These results are used only in few places.

A ring or an associative algebra will always be assumed to have a 1, and homomorphisms are assumed to respect this 1. Let  $k$  be a fixed commutative ring. Notations of linear algebra (like  $\text{Hom}$ ,  $\otimes$ ) without special reference to a ground ring always refer to structures as  $k$ -modules. A  $k$ -algebra is always assumed to be commutative and associative. (For noncommutative algebras we shall use the terminology: algebras over  $k$ .)

**1.1.** Before giving the definitions, I want to point out how functors arise naturally in algebraic geometry. Assume for the moment that  $k$  is an algebraically closed field.

Consider a Zariski closed subset  $X$  of some  $k^n$  and denote by  $I$  the ideal of all polynomials  $f \in k[T_1, T_2, \dots, T_n]$  (over  $k$  in  $n$  variables  $T_1, T_2, \dots, T_n$ ) with  $f(X) = 0$ . Instead of looking at the zeroes of  $I$  only over  $k$ , we can also look at the zeroes over any  $k$ -algebra  $A$ , i.e., at  $\mathcal{X}(A) = \{x \in A^n \mid f(x) = 0 \text{ for all } f \in I\}$ . The map  $A \mapsto \mathcal{X}(A)$  from  $\{k\text{-algebras}\}$  to  $\{\text{sets}\}$  is a functor: Any homomorphism  $\varphi : A \rightarrow A'$  of  $k$ -algebras induces a map

$$\varphi^n : A^n \rightarrow (A')^n, \quad (a_1, a_2, \dots, a_n) \mapsto (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n))$$

with  $f(\varphi^n(x)) = \varphi(f(x))$  for all  $x \in A^n$  and  $f \in k[T_1, T_2, \dots, T_n]$ . Therefore  $\varphi^n$  maps  $\mathcal{X}(A)$  to  $\mathcal{X}(A')$ . Denote its restriction by  $\mathcal{X}(\varphi) : \mathcal{X}(A) \rightarrow \mathcal{X}(A')$ . For another homomorphism  $\varphi' : A' \rightarrow A''$  of  $k$ -algebras, one has obviously  $\mathcal{X}(\varphi') \circ \mathcal{X}(\varphi) = \mathcal{X}(\varphi' \circ \varphi)$ , proving that  $\mathcal{X}$  is indeed a functor.

A regular map  $f$  from  $X$  to a Zariski closed subset  $Y$  of some  $k^m$  is given by  $m$  polynomials  $f_1, f_2, \dots, f_m \in k[T_1, T_2, \dots, T_n]$  as

$$f : X \rightarrow Y, \quad x \mapsto (f_1(x), f_2(x), \dots, f_m(x)).$$

The  $f_i$  define for each  $k$ -algebra  $A$  a map  $f(A) : A^n \rightarrow A^m$ . The comorphism  $f^* : k[T_1, T_2, \dots, T_m] \rightarrow k[T_1, T_2, \dots, T_n]$  maps the ideal defining  $Y$  into the ideal  $I$  defining  $X$ . This implies that any  $f(A)$  maps  $\mathcal{X}(A)$  to  $\mathcal{Y}(A)$ . The family of all  $f(A)$  defines a morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  of functors, i.e., a natural transformation. The more general discussion in 1.3 (cf. 1.3(2)) shows that the map  $f \mapsto f$  is bijective (from  $\{\text{regular maps } X \rightarrow Y\}$  to  $\{\text{natural transformations } \mathcal{X} \rightarrow \mathcal{Y}\}$ ).

Taking this for granted, we have embedded the category of all affine algebraic varieties over  $k$  into the category of all functors from  $\{k\text{-algebras}\}$  to  $\{\text{sets}\}$  as a full subcategory. This embedding can be extended to the category of all algebraic varieties, see 1.11.

One advantage of working with functors instead of varieties (i.e., of working with  $\mathcal{X}$  instead of  $X$ ) will be that it gives a natural way to work with “varieties” over other fields and over rings. Furthermore, we get new objects over  $k$  (algebraically closed) in a natural way. Instead of working with  $I$ , we might have taken any ideal  $I' \subset k[T_1, T_2, \dots, T_n]$  defining  $X$ , i.e., with  $X = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in I'\}$  or, equivalently by Hilbert’s Nullstellensatz, with  $\sqrt{I'} = I$ . Replacing  $I$  by  $I'$  in the definition of  $\mathcal{X}$ , we get a functor, say  $\mathcal{X}'$ , with  $\mathcal{X}'(A) = \mathcal{X}(A)$  whenever  $A$  is a field (or just an integral domain), but with  $\mathcal{X}'(A) \neq \mathcal{X}(A)$  for some  $A$  if  $I \neq I'$ . Such functors arise in a natural way even when we deal with varieties, and they play an important role in representation theory.

Before giving the proper definitions, let us describe the functor  $\mathcal{X}$  without using the embedding of  $X$  into  $k^n$ . For each  $k$ -algebra  $A$ , we have a bijection  $\text{Hom}_{k\text{-alg}}(k[T_1, T_2, \dots, T_n], A) \rightarrow A^n$ , sending any  $\alpha$  to  $(\alpha(T_1), \alpha(T_2), \dots, \alpha(T_n))$ . The inverse image of  $\mathcal{X}(A)$  consists of those  $\alpha$  with  $0 = f(\alpha(T_1), \alpha(T_2), \dots, \alpha(T_n)) = \alpha(f)$  for all  $f \in I$ , hence can be identified with  $\text{Hom}_{k\text{-alg}}(k[T_1, T_2, \dots, T_n]/I, A)$ . As  $k[T_1, T_2, \dots, T_n]/I$  is the algebra  $k[X]$  of regular functions on  $X$ , we have thus a bijection  $\mathcal{X}(A) \simeq \text{Hom}_{k\text{-alg}}(k[X], A)$ . If  $\varphi : A \rightarrow A'$  is a homomorphism of  $k$ -algebras, the  $\mathcal{X}(\varphi)$  corresponds to the map

$$\text{Hom}_{k\text{-alg}}(k[X], A) \rightarrow \text{Hom}_{k\text{-alg}}(k[X], A'), \quad \alpha \mapsto \varphi \circ \alpha.$$

A morphism  $f : X \rightarrow Y$  is given by its comorphism  $f^* : k[Y] \rightarrow k[X]$ . Then the morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is given by  $f(A) : \text{Hom}_{k\text{-alg}}(k[X], A) \rightarrow \text{Hom}_{k\text{-alg}}(k[Y], A)$  with  $\alpha \mapsto \alpha \circ f^*$  for any  $k$ -algebra  $A$ .

**1.2. ( $k$ -functors)** Let us assume  $k$  to be arbitrary again. In the definitions to follow, we shall be rather careless about the foundations of mathematics. Instead of working with “all”  $k$ -algebras, we should (as in [DG]) take only those in some universe. We leave the appropriate modifications to the interested reader.

A  $k$ -functor is a functor from the category of  $k$ -algebras to the category of sets.

Let  $X$  be a  $k$ -functor. A *subfunctor* of  $X$  is a  $k$ -functor  $Y$  with  $Y(A) \subset X(A)$  and  $Y(\varphi) = X(\varphi)|_{Y(A)}$  for all  $k$ -algebras  $A, A'$  and all  $\varphi \in \text{Hom}_{k\text{-alg}}(A, A')$ .

Obviously, a map  $Y$  that associates to each  $k$ -algebra  $A$  a subset  $Y(A) \subset X(A)$  is a subfunctor if and only if  $X(\varphi)Y(A) \subset Y(A')$  for each homomorphism  $\varphi : A \rightarrow A'$  of  $k$ -algebras.

For any family  $(Y_i)_{i \in I}$  of subfunctors of  $X$ , we define their *intersection*  $\bigcap_{i \in I} Y_i$  through  $(\bigcap_{i \in I} Y_i)(A) = \bigcap_{i \in I} Y_i(A)$  for each  $k$ -algebra  $A$ . This is again a subfunctor. The obvious definition of a union is not the useful one, so we shall not denote it by  $\bigcup_{i \in I} Y_i$ .