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量子群和晶体基引论

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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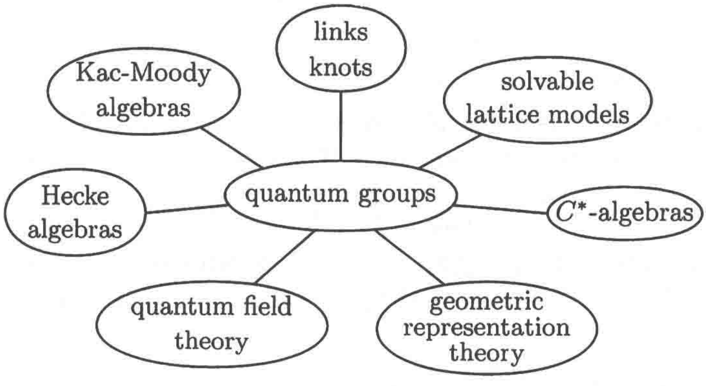
Introduction

The notion of a *quantum group* was introduced by V. G. Drinfel'd and M. Jimbo, independently, in their study of the quantum Yang-Baxter equation arising from two-dimensional solvable lattice models ([**10, 23**]). Quantum groups are certain families of Hopf algebras that are deformations of universal enveloping algebras of Kac-Moody algebras. Over the past 20 years, they turned out to be the fundamental algebraic structure behind many branches of mathematics and mathematical physics such as:

(1) solvable lattice models in statistical mechanics,
(2) topological invariant theory of links and knots,
(3) representation theory of Kac-Moody algebras,
(4) representation theory of algebraic structures, e.g., Hecke algebra,
(5) topological quantum field theory,
(6) geometric representation theory,
(7) C^*-algebras.



In particular, the theory of *crystal bases* or *canonical bases* developed independently by M. Kashiwara and G. Lusztig provides a powerful combinatorial and geometric tool to study the representations of quantum groups (**[38, 39, 48]**). The purpose of this book is to provide an elementary introduction to the theory of quantum groups and crystal bases focusing on the combinatorial aspects of the theory.

In such an introductory book, the first question to be answered would be: *What are quantum groups?* In his famous lecture given at the International Congress of Mathematicians held at Berkeley in 1986, Drinfel'd gave a *definition* of quantum groups: it was defined to be the *spectrum of a certain Hopf algebra* **[11]**. That is, Drinfel'd noted that any suitable category of groups (algebraic, topological, etc.) is antiequivalent to a suitable category of *commutative* Hopf algebras. In such a situation, one goes from the group to the algebra by considering a suitable algebra of functions, while the group can be reconstructed by taking the *spectrum* in the sense of Grothendieck. Thus, even when one has a noncommutative Hopf algebra, it becomes natural to think of the corresponding object in the opposite category as a *quantum group*, and this is the meaning of Drinfel'd's definition.

In this book, we focus on the quantum groups that appear as certain deformations of universal enveloping algebras of Kac-Moody algebras. For example, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. Choose a generic parameter q. Then, for each q, we can associate a Hopf algebra $U_q(\mathfrak{g})$, called the *quantum group* or the *quantized universal enveloping algebra*, whose structure *tends to* that of $U(\mathfrak{g})$ as q approaches 1. Therefore, we get a family of Hopf algebras $U_q(\mathfrak{g})$, and when $q = 1$, it is the same as the Hopf algebra $U(\mathfrak{g})$.

The following example shows how one can understand the above statement in a naive way. This example is not rigorous, not even mathematical, but it gives us a certain intuition. Let $\mathfrak{g} = \mathfrak{sl}_2$ be the complex Lie algebra of 2×2 matrices of trace 0. It is generated by the elements e, f, and h with defining relations

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

Thus its universal enveloping algebra $U(\mathfrak{sl}_2)$ is an associative algebra over $\mathbf{C}$ with 1 generated by the elements e, f, and h with defining relations

$$ef - fe = h, \qquad he - eh = 2e, \qquad hf - fh = -2f.$$

Now, the quantum group $U_q(\mathfrak{g}) = U_q(\mathfrak{sl}_2)$ is defined to be the associative algebra over $\mathbf{C}(q)$ with 1 generated by the elements e, f, and q^h with defining relations

$$ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}, \qquad q^h e q^{-h} = q^2 e, \qquad q^h f q^{-h} = q^{-2} f.$$

Let us look at the first of these defining relations. As q approaches 1, the left-hand side remains the same as $ef - fe$, but the right-hand side is undetermined. If we apply L'Hospital's rule (however absurd it might be), then the right-hand side is equal to

$$\lim_{q\to 1} \frac{q^h - q^{-h}}{q - q^{-1}} = \lim_{q\to 1} \frac{hq^{h-1} + hq^{-h-1}}{1 + q^{-2}} = \frac{2h}{2} = h,$$

as desired.

For the second relation, if we let $q \to 1$, then we get $e = e$, which gives nothing new. But if we *differentiate* both sides with respect to q (again, however absurd it might be), we get

$$hq^{h-1}eq^{-h} + q^h e(-h)q^{-h-1} = 2qe.$$

Thus, if we take the limit $q \to 1$, we get

$$he - eh = 2e.$$

Similarly, the last relation gives the desired relation as $q \to 1$.

Therefore, one can say that for each generic parameter q, there is a quantum group $U_q(\mathfrak{sl}_2)$ which is a Hopf algebra, so we have a family of Hopf algebras, and the structure of quantum group $U_q(\mathfrak{sl}_2)$ *tends to* that of $U(\mathfrak{sl}_2)$ as $q \to 1$. But of course this cannot be regarded as a mathematical treatment at all. So the first goal of this book is to make the above idea rigorous enough to convince ourselves.

In Chapters 1 and 2, we will give a brief review of the basic theory of Lie algebras, Hopf algebras, and Kac-Moody algebras. The notion of *universal enveloping algebras*, *highest weight modules*, and the *category* $\mathcal{O}_{\text{int}}$ will be introduced. The *Poincaré-Birkhoff-Witt theorem* and the *Weyl-Kac character formula* will be presented without proof. The readers may refer to [**1, 17, 28, 53**] for more detail and complete proofs.

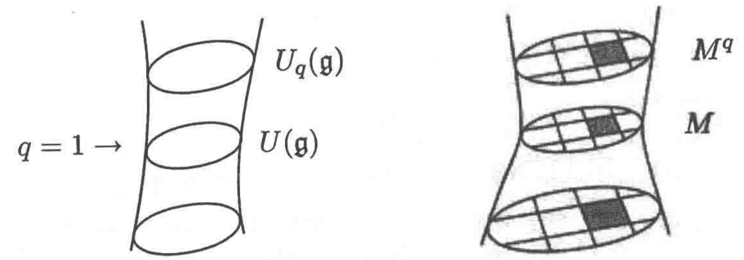
Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. In Chapter 3, we will define the *quantum group* $U_q(\mathfrak{g})$ as a certain deformation of $U(\mathfrak{g})$ with a Hopf algebra structure and show that the Hopf algebra structure of $U_q(\mathfrak{g})$ *tends to* that of $U(\mathfrak{g})$ as q approaches 1.

Moreover, we will give a rigorous proof of the statement: *The representation theory of Kac-Moody algebra $\mathfrak{g}$ is the same as the representation theory of quantum group $U_q(\mathfrak{g})$.* The essential part of this statement is a theorem proved by G. Lusztig in [**47**]:

The $\mathfrak{g}$-modules in the category $\mathcal{O}_{\text{int}}$ (= integrable modules over $\mathfrak{g}$ in the category $\mathcal{O}$) can be deformed to $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}^q_{\text{int}}$ in *such a way that the dimensions of weight spaces are invariant under the deformation.*

More precisely, let M be a $U(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$. Then it has a *weight space decomposition* $M = \bigoplus_{\lambda \in P} M_\lambda$, where M_λ is the common eigenspace for the Cartan subalgebra. Now Lusztig's theorem tells that for each generic q, there exists a $U_q(\mathfrak{g})$-module M^q in the category $\mathcal{O}^q_{\text{int}}$ with a weight space decomposition $M^q = \bigoplus_{\lambda \in P} M^q_\lambda$ such that $\dim_{\mathbf{C}(q)} M^q_\lambda = \dim_{\mathbf{C}} M_\lambda$ for all $\lambda \in P$ and that the structure of M^q *tends to* that of M as q approaches 1.

Pictorially, the results obtained in Chapter 3 can be illustrated in the following figure.



Actually, this is one of the motivations for the theory of *crystal bases*. For an integrable module M over $U(\mathfrak{g})$ in the category $\mathcal{O}_{\text{int}}$, consider the formal power series defined by

$$\operatorname{ch} M = \sum_{\lambda \in P} (\dim_{\mathbf{C}} M_\lambda) e^\lambda.$$

The formal series $\operatorname{ch} M$ is called the *character* of the $U(\mathfrak{g})$-module M. The characters of $U(\mathfrak{g})$-modules in the category $\mathcal{O}_{\text{int}}$ *characterize* the representations in the sense that if $M \cong N$, then $\operatorname{ch} M = \operatorname{ch} N$. The converse is not always true, but will hold if the two modules are both highest weight modules with one of them either a Verma module or an irreducible highest weight module. The characters often represent important and interesting mathematical quantities such as *modular forms* in number theory and *one-point functions* in solvable lattice models.

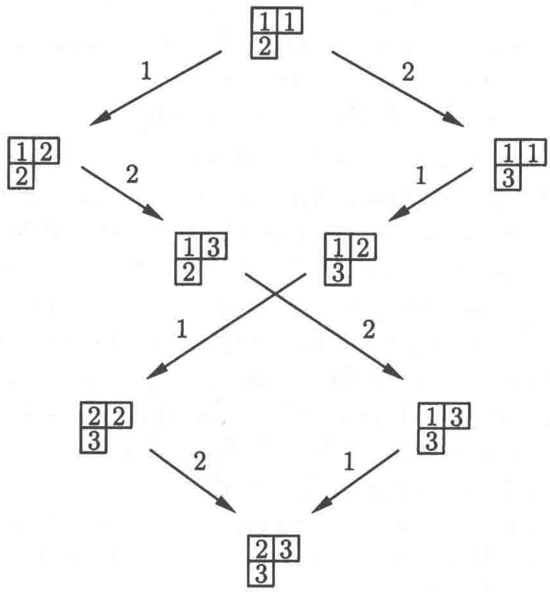
Similarly, one can define the character of a $U_q(\mathfrak{g})$-module M^q in the category $\mathcal{O}^q_{\text{int}}$ to be

$$\operatorname{ch} M^q = \sum_{\lambda \in P} (\dim_{\mathbf{C}(q)} M^q_\lambda) e^\lambda.$$

Since M^q is a quantum deformation of M, by Lusztig's theorem, $\operatorname{ch} M^q$ is the same for all generic parameter q, and it is just the character of M. So if one can calculate $\operatorname{ch} M^q$ for some special value of q, then it suffices to focus on that special case only. The natural question is: *When is the situation simple?* The crystal basis theory tells that it is so when $q = 0$.

In Chapters 4 and 5, we develop the *crystal basis theory* following the combinatorial approach given by Kashiwara [**38, 39**]. In [**48**], a more geometric approach was developed by Lusztig, and it is called the *canonical basis theory*. In [**43–45**], P. Littelmann introduced a combinatorial theory called the *path model* and obtained a colored oriented graph for irreducible highest weight modules over Kac-Moody algebras. It turned out that Littelmann's graphs coincide with Kashiwara's *crystal graphs* ([**25, 40**]).

A *crystal basis* can be understood as a basis at $q = 0$ and is given a structure of colored oriented graph, called the *crystal graph*, with arrows defined by the *Kashiwara operators*. The crystal graphs have many nice combinatorial features reflecting the internal structure of integrable modules over quantum groups. For instance, one of the major goals in combinatorial representation theory is to find an explicit expression for the characters of representations, and this goal can be achieved by finding an explicit combinatorial description of crystal bases. The following picture is the crystal graph for the adjoint representation of $U_q(\mathfrak{sl}_3)$.



Moreover, crystal bases have extremely nice behavior with respect to taking the tensor product. The action of Kashiwara operators is given by the simple *tensor product rule* and the irreducible decomposition of the tensor product of integrable modules is equivalent to decomposing the tensor product of crystal graphs into a disjoint union of connected components. Thus, the crystal basis theory provides us with a powerful combinatorial method of studying the structure of integrable modules over quantum groups.

Our exposition is based on the combinatorial approach developed by Kashiwara [**39**], and some of our arguments overlap with those given in [**21**]. The existence theorem for crystal bases will be proved using Kashiwara's *grand-loop argument* (Section 5.3). We will simplify the original argument, which consists of 14 interlocking inductive statements, to proving 7 interlocking inductive statements. Still, the spirit of the argument is the same as the original one: the fundamental properties of crystal bases for $U_q^-(\mathfrak{g})$ will play the crucial role in the proof.

The next step is to *globalize* the main idea of crystal bases. More precisely, let M^q be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\text{int}}$ with crystal basis $(\mathcal{L}, \mathcal{B})$. As we mentioned earlier, the crystal basis $\mathcal{B}$ can be regarded as a *local basis* of M^q at $q = 0$. In Chapter 6, we will show that there exists a unique *global basis* $\mathcal{G}(\mathcal{B}) = \{G(b) \mid b \in \mathcal{B}\}$ of M^q satisfying the properties

$$G(b) \equiv b \mod q\mathcal{L}, \qquad \overline{G(b)} = G(b) \quad \text{for all } b \in \mathcal{B},$$

where $^-$ denotes the automorphism on M given by (6.5). The existence theorem for global bases will be proved using the notion of a *balanced triple* and the triviality of vector bundles over $\mathbf{P}^1$. Our argument closely follows the original proof given by M. Kashiwara in [**39**].

Over the past 100 years, it has been discovered that there is a close connection between representation theory and combinatorics. We can see this in the classical works by A. Young ([**57–59**]), D. E. Littlewood and A. R. Richardson ([**46**]), D. Robinson ([**52**]), and H. Weyl ([**55**]). In Chapter 7, we study the connection between the crystal basis theory of finite dimensional $U_q(\mathfrak{gl}_n)$-modules and combinatorics of Young diagrams and Young tableaux. The notion of *admissible reading* (e.g., *Far-Eastern reading* and *Middle-Eastern reading*) lies at the heart of this connection. The crystal graph of a finite dimensional irreducible $U_q(\mathfrak{gl}_n)$-module will be realized as the set of semistandard Young tableaux of a given shape. Moreover, using the tensor product rule for Kashiwara operators, we will give a combinatorial rule (*Littlewood-Richardson rule*) for decomposing the tensor product of finite dimensional $U_q(\mathfrak{gl}_n)$-modules into a direct sum of irreducible components. One may refer to [**46**] for the classical approach.

In Chapter 8, we will extend the above idea to the study of crystal graphs for classical Lie algebras. The crystal graph of a finite dimensional irreducible module over a classical Lie algebra will be realized as the set of semistandard Young tableaux satisfying certain additional conditions depending on the type of the Lie algebra. We will also give a combinatorial rule (*generalized Littlewood-Richardson rule*) for decomposing the tensor product of crystal graphs. Most of the results in Chapters 7 and 8 can be found in [**41**] and [**50**].

As the theory of quantum groups originated from the study of the quantum Yang-Baxter equation, the theory of solvable lattice models can be best explained in the language of representation theory of *quantum affine algebras* (which are the quantum groups corresponding to the affine Kac-Moody algebras). In Chapter 9, we will describe the very basic theory of solvable lattice models and discuss its connection with the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ (see, for example, [**24, 36**]). In particular, the *one-point function* for the 6-vertex model will be expressed as the quotient of the string function by the character of the basic representation of $U_q(\widehat{\mathfrak{sl}}_2)$.

In Chapter 10, we will develop the theory of *perfect crystals* for quantum affine algebras (see [**36, 37**]), which has a lot of important applications to the representation theory of quantum affine algebras and vertex models (see, for example, [**7, 24**] and the references therein). We will first study the properties of *vertex operators* and then prove a fundamental crystal isomorphism theorem. Using this crystal isomorphism, the crystal graph of an irreducible highest weight module over a quantum affine algebra will be realized as the set of certain *paths*.

The final chapter will be devoted to the study of crystal bases for basic representations of classical quantum affine algebras using some new combinatorial objects which we call the *Young walls* (see [**34**]). The Young walls consist of colored blocks with various shapes that are built on the given *ground-state wall* and can be viewed as generalizations of Young diagrams. The rules for building Young walls and the action of Kashiwara operators will be given explicitly in terms of combinatorics of Young walls. (They are quite similar to playing with LEGO® blocks and the Tetris® game.) The crystal graph of a basic representation will be characterized as the set of all *reduced proper Young walls.* We expect that there exist interesting and important algebraic structures whose irreducible representations (at some specializations) are parameterized by reduced proper Young walls. It still remains to extend the results in this chapter to the quantum affine algebras of type $C_n^{(1)}$ $(n \geq 3)$.

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Contents

Chapter 1

Lie Algebras and Hopf Algebras

In this chapter, we will briefly review some of the basic definitions and facts about Lie algebras and Hopf algebras. The readers may refer to [**17, 19**] and [**53**] for more detail and complete proofs on these subjects. Throughout this book, $\mathbf{F}$ will denote an arbitrary field of characteristic zero. We denote by $\mathbf{Z}$ the ring of integers and by $\mathbf{Q}$ the field of rational numbers inside $\mathbf{F}$.

1.1. Lie algebras

Lie algebras originally developed as means of studying the local properties of *Lie groups*. Roughly speaking, a Lie group is a manifold with a group structure satisfying certain smoothness and compatibility conditions, and the Lie algebra appears as the tangent space to this manifold at the identity with a bilinear product which is neither commutative nor associative. There exists a good correspondence between the subgroups of a Lie group and the subalgebras of its Lie algebra. That is, the structure of a Lie group G is reflected in the structure of its Lie algebra $L = \mathrm{Lie}(G)$. Moreover, since the Lie algebra L is a vector space, a linear object, it is much easier to deal with a Lie algebra L than with a Lie group G. Thus, to understand a Lie group, a *geometric* object, we study its Lie algebra, an *algebraic* object. We will start with an algebraic definition of Lie algebras.

Definition 1.1.1. A vector space L over $\mathbf{F}$ with a bilinear operation $L \times L \to L$, denoted by $(x, y) \mapsto [x, y]$ and called the ***bracket***, is a ***Lie algebra*** if