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Daniel W. Stroock

A Concise Introduction to the Theory of Integration

Third Edition

简明积分导论

第 3 版（影印版）

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Daniel W. Stroock
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

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The book is dedicated to my Colorado colleagues

L. Baggett, R. Holley, and W. Fulks

Preface to the Third Edition

Aside from the ceaseless displacement of old errors to make room for new ones, the book has changed from the second edition mostly by the addition of a chapter about Fourier analysis. In the spirit of Norbert Wiener (and quantum field theorists), I have chosen to base the development on Hermite functions. In particular, I first use the Hermite functions to diagonalize the Fourier transform, and then derive Parseval's identity and the Fourier inversion as a consequence. The other substantial change results from my decision to provide solutions to *all* the problems and to publish them under the same cover.

Unfortunately, these changes have resulted in a book which appears to be less "concise" than the original. Thus, for example, it is probably no longer realistic to think that everything can be covered in a single semester. On the other hand, there is no need to cover everything. For example, many students will have seen the material in Chapter V elsewhere, and others will be exposed to Fourier analysis later. Thus, I expect that most instructors will make a selection from the topics covered in Chapters V and VII.

Finally, and significantly, in several places the presentation has benefited from feedback which I received from J.J. Duistermaat, who subjected himself and his students to the book for two consecutive years, thereby replacing Burckel as the world's leading authority on its shortcomings. In addition, it was Duistermaat who convinced me to admit that this is not the only book on the topic and to suggest an alternate source. As it turns out, there are several books which approach measure theory from a similar point of view. Among these are *Measure and Integral* by R.L. Wheeden and A. Zygmund, published by Marcel Dekker in 1977, and *Lebesgue Integration on Euclidean Space* by Frank Jones, published in 1993 by Jones & Bartlett.

Daniel W. Stroock, Cambridge, MA, July, 1998

Preface to the Second Edition

It is four years since the first edition of this book appeared, and, in that time, there has been little, if any, change in either the basic material covered or my attitude toward that material. On the other hand, experience has taught me that my presentation of several points could be considerably refined and that the inclusion of some additional topics would be desirable. Thus, although they may not be immediately apparent, changes have been made throughout. Among those which are obvious are the addition of two new sections: Section 4.2, in which I prove the isodiametric inequality and discuss Lebesgue measure from the Hausdorff measure point of view, and Section 3.4, in which I have given a proof (based on the Hardy–Littlewood maximal inequality) of Lebesgue's Differentiation Theorem for $\mathbb{R}$. These additions made it desirable to reorganize the table of contents, with the result that now product measures appear in Chapter IV and succeeding chapters have been renumbered accordingly. Besides these new sections, the exposition, particularly in what are now Chapters V and VII, has been, I hope, improved. In addition, even where substantive alterations are slight, I have made a great effort to remove some of the more egregious errors with which the first edition was riddled. (In particular, I believe that I have, at last, mastered the spelling of Lebesgue's name.) Finally, at the behest of my students, I have attempted to solve some of the exercises, and the fruits of my labor appear at the end of the book.

If I have been successfully eliminated many of the errors in the first edition, most of the credit should go to R.B. Burckel, who was kind enough to send me a (five page) list of those which he found. In addition, I am indebted to Ann Kostant at Birkhäuser for her efforts, without which this second edition would probably not have appeared.

Daniel W. Stroock, Cambridge, MA, December, 1993

Preface to the First Edition

This little book is the outgrowth of a one semester course which I have taught for each of the past four years at M.I.T. Although this class used to be one of the standard courses taken by essentially every first year graduate student of mathematics, in recent years (at least in those when I was the instructor), the clientele has shifted from first year graduate students of mathematics to more advanced graduate students in other disciplines. In fact, the majority of my students have been from departments of engineering (especially electrical engineering) and most of the rest have been economists. Whether this state of affairs is a reflection on my teaching, the increased importance of mathematical analysis in other disciplines, the superior undergraduate preparation of students coming to M.I.T in mathematics, or simply the lack of enthusiasm that these students have for analysis, I have preferred not to examine too closely. On the other hand, the situation did force me to do a certain amount of thinking about what constitutes an appropriate course for a group of non-mathematicians who are courageous (foolish?) enough to sign up for an introduction to integration theory offered by the department of mathematics. In particular, I had to figure out what to do about that vast body of material which, in standard mathematics offerings, is "assumed to have been covered in your advanced calculus course". Aspiring young mathematicians seldom challenge even the most ridiculous declarations of this sort: the good ones look it up, and the others trust that "it will not appear on the exam". On the other hand, students who are not heading into mathematics are less easily shamed into accepting such claims; in fact, as I soon discovered, many of them were attending my course for the express purpose of learning what mathematicians call "advanced calculus".

In view of the preceding comments about the origins of this text, it should come as no surprise that the contents of this book are somewhat different from that of many modern introductions to measure theory. Indeed, I believe that nothing has been done here "in complete generality"! On the other hand, greater space than usual has been given to the properties of Lebesgue's measure on $\mathbb{R}^N$. In particular, the whole of Chapter IV [now, Chapter V] is devoted to applications of Lebesgue's measure to topics which are customarily "assumed to have been covered in your advanced calculus course". As a consequence, what has emerged is a kind of hybrid in which both modern integration theory and advanced calculus are represented. Because none of the many existing books on integration theory contained precisely the mix for which I was looking, I decided to add my own version of the subject to the long list of books for the next guy to reject.

Cambridge, MA, January, 1990

Contents

Chapter I

The Classical Theory

1.1 Riemann Integration

We begin by recalling a few basic facts about the integration theory which is usually introduced in advanced calculus. We do so not only for purposes of later comparison with the modern theory but also because it is the theory with which most computations are actually performed.

Let $N \in \mathbb{Z}^+$ (throughout $\mathbb{Z}^+$ will denote the positive integers). A **rectangle** in $\mathbb{R}^N$ is a subset I of $\mathbb{R}^N$ which can be written as the Cartesian product $\prod_1^N [a_k, b_k]$ of closed intervals $[a_k, b_k]$, where it is assumed that $a_k \le b_k$ for each $1 \le k \le N$. If I is such a rectangle, we call

$$\operatorname{diam}(I) \equiv \left(\sum_{k=1}^{N} (b_k - a_k)^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \operatorname{vol}(I) \equiv \prod_{k=1}^{N} (b_k - a_k)$$

the **diameter** and the **volume** of I, respectively. For purposes of this exposition, it will be convenient to also take the empty set to be a rectangle with diameter and volume 0.

Given a collection $\mathcal{C}$, we will say that $\mathcal{C}$ is **non-overlapping** if distinct elements of $\mathcal{C}$ have disjoint interiors. The following *obvious* fact is surprisingly difficult to prove.

1.1.1 Lemma. *If $\mathcal{C}$ is a non-overlapping, finite collection of rectangles each of which is contained in the rectangle J, then $\operatorname{vol}(J) \ge \sum_{I \in \mathcal{C}} \operatorname{vol}(I)$. On the other hand, if $\mathcal{C}$ is any finite collection of rectangles and J is a rectangle which is covered by $\mathcal{C}$ (i.e., $J \subseteq \bigcup \mathcal{C}$), then $\operatorname{vol}(J) \le \sum_{I \in \mathcal{C}} \operatorname{vol}(I)$.*

PROOF: Without loss of generality, we will assume throughout that each of the rectangles $I \in \mathcal{C}$ is contained in the rectangle J. Indeed, $I \subseteq J$ has already been assumed in the first part of the lemma, and, in the second part, if $I \not\subseteq J$, then one can replace I by $I \cap J$.

We begin with the observation that, if $J = \prod_1^N [a_k, b_k]$, $a_k \le c \le b_k$ for some $1 \le k \le N$, and J^+ and J^- are the rectangles obtained by replacing the kth side of J by $[a_k, c]$ and $[c, b_k]$, respectively, then $\operatorname{vol}(J) = \operatorname{vol}(J^+) + \operatorname{vol}(J^-)$. More generally, if, for each $1 \le k \le N$, $a_k = c_{k,0} \le \cdots \le c_{k,n_k} = b_k$ and $\mathcal{S}$ is the collection of $\prod n_k$ rectangles of the form

$$R(m_1,\dots,m_N) \equiv \prod_{k=1}^{N}\left[c_{k,m_k-1},c_{k,m_k}\right]$$

with $1 \le m_k \le n_k$, then an obvious induction argument on the n_k's shows that $\operatorname{vol}(J) = \sum_{R\in\mathcal{S}} \operatorname{vol}(R)$.

Next suppose that $\mathcal{C}$ is a finite collection of rectangles I contained in $J = \prod[a_k,b_k]$. For each $1 \le k \le N$, choose $a_k = c_{k,0} \le \cdots \le c_{k,n_k} = b_k$ so that each $I \in \mathcal{C}$ has the form

$$\prod_{k=1}^{N}\left[c_{k,m_k},c_{k,m'_k}\right]$$

for some choice of $0 \le m_k < m'_k \le n_k$, and determine the collection $\mathcal{S}$ accordingly, as in the preceding paragraph. Then, by the observation just made, $\operatorname{vol}(J) = \sum_{R\in\mathcal{S}} \operatorname{vol}(R)$. For the same reason, if, for a given $I \in \mathcal{C}$, $\mathcal{S}(I)$ is the collection of those $R \in \mathcal{S}$ with $R \subseteq I$, then $\operatorname{vol}(I) = \sum_{R\in\mathcal{S}(I)} \operatorname{vol}(R)$.

Now add the assumption that $\mathcal{C}$ is a non-overlapping collection, and let I and I' be different elements of $\mathcal{C}$. Then, in order for $R = R(m_1,\dots,m_N)$ to be an element of both $\mathcal{S}(I)$ and of $\mathcal{S}(I')$, it is necessary that $c_{k,m_k-1} = c_{k,m_k}$ for at least one $1 \le k \le N$. But this means that $\operatorname{vol}(R) = 0$, and so we now know that

$$\operatorname{vol}(J) = \sum_{R\in\mathcal{S}} \operatorname{vol}(R) \ge \sum_{I\in\mathcal{C}} \sum_{R\in\mathcal{S}(I)} \operatorname{vol}(R) = \sum_{I\in\mathcal{C}} \operatorname{vol}(I).$$

Finally, if we drop the non-overlapping assumption but instead assume that $\mathcal{C}$ covers J, then every $R \in \mathcal{S}$ is in $\mathcal{S}(I)$ for at least one $I \in \mathcal{C}$. Hence, in this case,

$$\operatorname{vol}(J) = \sum_{R\in\mathcal{S}} \operatorname{vol}(R) \le \sum_{I\in\mathcal{C}} \sum_{R\in\mathcal{S}(I)} \operatorname{vol}(R) = \sum_{I\in\mathcal{C}} \operatorname{vol}(I). \quad \square$$

Given a collection $\mathcal{C}$ of rectangles I, we say that $\xi : \mathcal{C} \longrightarrow \bigcup\mathcal{C}$ is a *choice map* for $\mathcal{C}$ if $\xi(I) \in I$ for each $I \in \mathcal{C}$, and we use $\Xi(\mathcal{C})$ to denote the set of all such maps. Given a finite collection $\mathcal{C}$, a choice map $\xi \in \Xi(\mathcal{C})$, and a bounded function $f : \bigcup\mathcal{C} \longrightarrow \mathbb{R}$, we define the **Riemann sum of f over $\mathcal{C}$ relative to ξ** to be

$$\mathcal{R}(f;\mathcal{C},\xi) \equiv \sum_{I\in\mathcal{C}} f(\xi(I))\operatorname{vol}(I). \tag{1.1.2}$$

Finally, if J is a rectangle and $f : J \longrightarrow \mathbb{R}$ is a function, we say that f is **Riemann integrable on J** if there is a number $A \in \mathbb{R}$ with the property that, for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$|\mathcal{R}(f;\mathcal{C},\xi) - A| < \epsilon \tag{1.1.3}$$

whenever $\xi \in \Xi(\mathcal{C})$ and $\mathcal{C}$ is a non-overlapping, finite, **exact cover** of J (i.e., $J = \bigcup \mathcal{C}$) whose **mesh size**

$$\|\mathcal{C}\| \equiv \max\left\{\operatorname{diam}(I) : I \in \mathcal{C}\right\} < \delta.$$

When f is Riemann integrable on J, we call the associated number A in (1.1.3) the **Riemann integral of f on J** and we will use

$$(\mathrm{R}) \int_J f(x)\,dx$$

to denote A.

It is a relatively simple matter to see that any $f \in C(J)$ (the space of continuous real-valued functions on J) is Riemann integrable on J. However, in order to determine when more general functions are Riemann integrable, it is useful to introduce the **Riemann upper sum**

$$\mathcal{U}(f;\mathcal{C}) \equiv \sum_{I \in \mathcal{C}} \sup_{x \in I} f(x) \operatorname{vol}(I)$$

and the **Riemann lower sum**

$$\mathcal{L}(f;\mathcal{C}) \equiv \sum_{I \in \mathcal{C}} \inf_{x \in I} f(x) \operatorname{vol}(I).$$

Clearly, one always has

$$\mathcal{L}(f;\mathcal{C}) \le \mathcal{R}(f;\mathcal{C},\xi) \le \mathcal{U}(f;\mathcal{C})$$

for any $\mathcal{C}$ and $\xi \in \Xi(\mathcal{C})$. Also, it is clear that a bounded f is Riemann integrable if and only if

$$\varliminf_{\|\mathcal{C}\| \to 0} \mathcal{L}(f;\mathcal{C}) \ge \varlimsup_{\|\mathcal{C}\| \to 0} \mathcal{U}(f;\mathcal{C})$$

where the limits are taken over non-overlapping, finite, exact covers of J. What we want to show now is that the preceding can be replaced by the condition[1]

$$\sup_{\mathcal{C}} \mathcal{L}(f;\mathcal{C}) \ge \inf_{\mathcal{C}} \mathcal{U}(f;\mathcal{C}) \tag{1.1.4}$$

where $\mathcal{C}$'s run over all non-overlapping, finite, exact covers of J.

To this end, we partially order the covers $\mathcal{C}$ by *refinement*. That is, we say that $\mathcal{C}_2$ is **more refined** than $\mathcal{C}_1$ and write $\mathcal{C}_1 \le \mathcal{C}_2$, if, for every $I_2 \in \mathcal{C}_2$, there is an $I_1 \in \mathcal{C}_1$ such that $I_2 \subseteq I_1$. Note that, for every pair $\mathcal{C}_1$ and $\mathcal{C}_2$, the *least* common refinement $\mathcal{C}_1 \vee \mathcal{C}_2$ is given by

$$\mathcal{C}_1 \vee \mathcal{C}_2 = \{I_1 \cap I_2 : I_1 \in \mathcal{C}_1 \text{ and } I_2 \in \mathcal{C}_2 \text{ with } I_1 \cap I_2 \neq \emptyset\}.$$

1.1.5 Lemma. *For any pair of non-overlapping, finite, exact covers $\mathcal{C}_1$ and $\mathcal{C}_2$ of J and any function $f : J \longrightarrow \mathbb{R}$, $\mathcal{L}(f;\mathcal{C}_1) \le \mathcal{U}(f;\mathcal{C}_2)$. Moreover, if $\mathcal{C}_1 \le \mathcal{C}_2$, then $\mathcal{L}(f;\mathcal{C}_1) \le \mathcal{L}(f;\mathcal{C}_2)$ and $\mathcal{U}(f;\mathcal{C}_1) \ge \mathcal{U}(f;\mathcal{C}_2)$.*

[1] In many texts, this condition is adopted as the definition of Riemann integrability. Obviously, since, as we are about to show, it is equivalent to the definition which we have given, there is no harm in doing so. However, when working in the more general setting studied in § 1.2, the distinction between these two definitions does make a difference. See, Exercise 1.2.26 for a little more information.

PROOF: We begin by proving the second statement. Noting that

$$\mathcal{L}(f;\mathcal{C}) = -\mathcal{U}(-f;\mathcal{C}), \tag{1.1.6}$$

we see that it suffices to check that $\mathcal{U}(f;\mathcal{C}_1) \geq \mathcal{U}(f;\mathcal{C}_2)$ if $\mathcal{C}_1 \leq \mathcal{C}_2$. But, for each $I_1 \in \mathcal{C}_1$,

$$\sup_{x\in I_1} f(x)\mathrm{vol}\,(I_1) \geq \sum_{\{I_2\in\mathcal{C}_2:I_2\subseteq I_1\}} \sup_{x\in I_2} f(x)\mathrm{vol}\,(I_2),$$

where we have used Lemma 1.1.1 to see that

$$\mathrm{vol}\,(I_1) = \sum_{\{I_2\in\mathcal{C}_2:\,I_2\subseteq I_1\}} \mathrm{vol}\,(I_2).$$

After summing the above over $I_1 \in \mathcal{C}_1$, one arrives at the required result.

Given the preceding, the rest is immediate. Namely, for any $\mathcal{C}_1$ and $\mathcal{C}_2$,

$$\mathcal{L}(f;\mathcal{C}_1) \leq \mathcal{L}(f;\mathcal{C}_1\vee\mathcal{C}_2) \leq \mathcal{U}(f;\mathcal{C}_1\vee\mathcal{C}_2) \leq \mathcal{U}(f;\mathcal{C}_2). \quad \square$$

Lemma 1.1.5 really depends only on properties of our order relation and not on the properties of vol(I). In contrast, the next lemma depends on the continuity of volume with respect to side-lengths of rectangles.

1.1.7 Lemma. *Let $\mathcal{C}$ be a non-overlapping, finite, exact cover of the rectangle J and $f : J \longrightarrow \mathbb{R}$ a bounded function. Then, for each $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\mathcal{U}(f;\mathcal{C}') \leq \mathcal{U}(f;\mathcal{C}) + \epsilon \quad \text{and} \quad \mathcal{L}(f;\mathcal{C}') \geq \mathcal{L}(f;\mathcal{C}) - \epsilon$$

whenever $\mathcal{C}'$ is a non-overlapping, finite, exact cover of J with the property that $\|\mathcal{C}'\| < \delta$.

PROOF: In view of (1.1.6), we need only consider the Riemann upper sums.

Let $J = \prod_1^N [c_k, d_k]$. Given a $\delta > 0$ and a rectangle $I = \prod_1^N [a_k, b_k]$, define $I_k^-(\delta)$ and $I_k^+(\delta)$ to be the rectangles

$$[c_1, d_1] \times \cdots \times [a_k - \delta, a_k + \delta] \times \cdots \times [c_N, d_N]$$

and

$$[c_1, d_1] \times \cdots \times [b_k - \delta, b_k + \delta] \times \cdots \times [c_N, d_N],$$

respectively. Then, for any rectangle $I' \subseteq J$ with $\mathrm{diam}(I') < \delta$, either $I' \subseteq I$ for some $I \in \mathcal{C}$ or, for some $I \in \mathcal{C}$ and $1 \leq k \leq N$, $I' \subseteq I_k^+(\delta)$ or $I' \subseteq I_k^-(\delta)$.

Now let $\mathcal{C}'$ with $\|\mathcal{C}'\| < \delta$ be given. Then, by an application of Lemma 1.1.1, we can write

$$\mathcal{U}(f;\mathcal{C}') = \sum_{I'\in\mathcal{C}'} \sup_{I'} f \operatorname{vol}(I') = \sum_{I\in\mathcal{C}}\sum_{I'\in\mathcal{C}'} \sup_{I'} f \operatorname{vol}(I\cap I')$$
$$= \sum_{I\in\mathcal{C}}\sum_{I'\in\mathcal{C}'} \sup_{I\cap I'} f \operatorname{vol}(I\cap I') + \sum_{I\in\mathcal{C}}\sum_{I'\in\mathcal{C}'} \left(\sup_{I'} f - \sup_{I\cap I'}\right) \operatorname{vol}(I\cap I').$$

But clearly

$$\sum_{I\in\mathcal{C}}\sum_{I'\in\mathcal{C}'} \sup_{I\cap I'} f \operatorname{vol}(I\cap I') \le \sum_{I\in\mathcal{C}}\sum_{I'\in\mathcal{C}'} \sup_{I} f \operatorname{vol}(I\cap I') = \mathcal{U}(f;\mathcal{C}),$$

where, in the final step, we have again used Lemma 1.1.1. Thus, it remains to estimate

$$\sum_{I\in\mathcal{C}}\sum_{I'\in\mathcal{C}'} \left(\sup_{I'} f - \sup_{I\cap I'} f\right) \operatorname{vol}(I\cap I').$$

However, by the discussion in the preceding paragraph, for each $I'\in\mathcal{C}'$, either $I'\subseteq I$ for some $I\in\mathcal{C}$, in which case

$$\sum_{I\in\mathcal{C}} \left(\sup_{I'} f - \sup_{I\cap I'} f\right) \operatorname{vol}(I\cap I') = 0,$$

or, for some $I\in\mathcal{C}$ and $1\le k\le N$, $I'\subseteq I_k^+(\delta)$ or $I'\subseteq I_k^-(\delta)$. Thus, if we set

$$\mathcal{B}(k,I)^\pm = \{I'\in\mathcal{C} : I'\subseteq I_k^\pm(\delta)\},$$

then

$$\sum_{I\in\mathcal{C}}\sum_{I'\in\mathcal{C}'} \left(\sup_{I'} f - \sup_{I\cap I'} f\right) \operatorname{vol}(I\cap I')$$
$$\le 2\|f\|_{\mathrm{u}} \sum_{k=1}^{N}\sum_{I\in\mathcal{I}} \left(\sum_{I'\in\mathcal{B}(k,I)^+} \operatorname{vol}(I\cap I') + \sum_{I'\in\mathcal{B}(k,I)^-} \operatorname{vol}(I\cap I')\right).$$

(In the preceding, we have introduced the notation, to be used throughout, that $\|f\|_{\mathrm{u}}$ denotes the **uniform norm of** f: the supremum of $|f|$ over the set on which f is defined.) Finally, by Lemma 1.1.1, for each $1\le k\le N$ and $I\in\mathcal{C}$,

$$\sum_{I'\in\mathcal{B}(k,I)^\pm} \operatorname{vol}(I\cap I') \le \operatorname{vol}\big(I_k^\pm(\delta)\big) \le 2\delta\frac{\operatorname{vol}(J)}{d_k - c_k},$$

and so we now proved that, whenever $\|\mathcal{C}'\|\le\delta$,

$$\mathcal{U}(f;\mathcal{C}') - \mathcal{U}(f;\mathcal{C}) \le \sum_{I\in\mathcal{C}}\sum_{I'\in\mathcal{C}'} \left(\sup_{I'} f - \sup_{I\cap I'}\right) \operatorname{vol}(I\cap I') \le K\|f\|_{\mathrm{u}}\delta,$$

where

$$K \equiv 4N\mathrm{card}(\mathcal{C}) \max_{1\le k\le N} \frac{\mathrm{vol}(J)}{d_k - c_k}. \quad \square$$

As an essentially immediate consequence of Lemma 1.1.7, we have the following theorem.

1.1.8 Theorem. *Let $f : J \longrightarrow \mathbb{R}$ be a bounded function on the rectangle J. Then*

$$\lim_{\|\mathcal{C}\|\to 0} \mathcal{L}(f;\mathcal{C}) = \sup_{\mathcal{C}} \mathcal{L}(f;\mathcal{C}) \quad \text{and} \quad \lim_{\|\mathcal{C}\|\to 0} \mathcal{U}(f;\mathcal{C}) = \inf_{\mathcal{C}} \mathcal{U}(f;\mathcal{C}),$$

where $\mathcal{C}$ runs over non-overlapping, finite, exact covers of J. In particular, (1.1.4) *is a necessary and sufficient condition that a bounded f on J be Riemann integrable. Moreover, if* (1.1.4)*, then*

$$(\mathrm{R})\int_J f(x)\,dx = \sup_{\mathcal{C}} \mathcal{L}(f;\mathcal{C}) = \inf_{\mathcal{C}} \mathcal{U}(f;\mathcal{C}).$$

Exercises

1.1.9 Exercise: Prove Theorem 1.1.8. Next, suppose that f and g are Riemann integrable functions on J. Show that $f \vee g \equiv \max\{f,g\}$, $f \wedge g \equiv \min\{f,g\}$, and, for any α, $\beta \in \mathbb{R}$, $\alpha f + \beta g$ are all Riemann integrable on J. In addition, check that

$$(\mathrm{R})\int_J (f \vee g)(x)\,dx \ge \left((\mathrm{R})\int_J f(x)\,dx\right) \vee \left((\mathrm{R})\int_J g(x)\,dx\right),$$
$$(\mathrm{R})\int_J (f \wedge g)(x)\,dx \le \left((\mathrm{R})\int_J f(x)\,dx\right) \wedge \left((\mathrm{R})\int_J g(x)\,dx\right),$$

and

$$(\mathrm{R})\int_J (\alpha f + \beta g)(x)\,dx = \alpha\left((\mathrm{R})\int_J f(x)\,dx\right) + \beta\left((\mathrm{R})\int_J g(x)\,dx\right).$$

Conclude, in particular, that if f and g are Riemann integrable on J and $f \le g$, then $(\mathrm{R})\int_J f(x)\,dx \le (\mathrm{R})\int_J g(x)\,dx$.

1.1.10 Exercise: Show that if f is a bounded real-valued function on the rectangle J, then f is Riemann integrable if and only if, for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{\{I\in\mathcal{C}:\sup_I f - \inf_I f > \epsilon\}} \mathrm{vol}(I) < \epsilon \tag{1.1.11}$$

whenever $\|\mathcal{C}\| < \delta$. (We use $\sup_I f$ and $\inf_I f$ to denote $\sup_{x\in I} f(x)$ and $\inf_{x\in I} f(x)$, respectively.) As a consequence, show that a bounded f on J is Riemann integrable if it is continuous on J at all but a finite number of points. (See Section 4.1 for more information on this subject.)