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Advanced Lectures in Mathematics

Handbook of Group Actions (Vol. III)

群作用手册（第 Ⅲ 卷）

Editors: Lizhen Ji • Athanase Papadopoulos • Shing-Tung Yau

HIGHER EDUCATION PRESS

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Preface to Volumes III and IV

Although the mathematical literature is growing at an exponential rate, with papers and books published every day on various topics and with different objectives: research, expository or historical, the editors of the present Handbook feel that the mathematical community still needs good surveys, presenting clearly the bases and the open problems in the fundamental research fields. Group actions constitute one of these great classical and always active subjects which are beyond the fashionable and non-fashionable, at the heart of several domains, from geometry to dynamics, passing by complex analysis, number theory, and many others. The present Handbook is a collection of surveys concerned with this vast theory. Volumes I and II appeared in 2015.[1] Volume III and IV are published now simultaneously.

The list of topics discussed in these two volumes is broad enough to show the diversity of the situations in which group actions appear in a substantial way.

Volume III is concerned with hyperbolic group actions, groups acting on metric spaces of non-positive curvature, automorphism groups of geometric structures (complex, projective, algebraic and Lorentzian), and topological group actions, including the Hilbert–Smith conjecture and related conjectures.

Volume IV contains surveys on the asymptotic and large-scale geometry of metric spaces, presenting rigidity results in various contexts, with applications in geometric group theory, representation spaces and representation varieties, homogeneous spaces, symmetric spaces, and several aspects of dynamics: Property T, group actions on the circle, actions on Hilbert spaces and other symmetries.

Several surveys in these two volumes include new or updated versions of interesting open problems related to group actions.

We hope that this series will be a guide for mathematicians, from the graduate student to the experienced researcher, in this vast and ever-growing field.

L. Ji (Ann Arbor)
A. Papadopoulos (Strasbourg and Providence)
S.-T. Yau (Cambridge, MA)
April 2018

[1] L. Ji, A. Papadopoulos and S. T. Yau (ed.). Handbook of Group actions, Volumes I and II, Higher Education Press and International Press, Vols. 31 and 32 of the Advanced Lectures in Mathematics, 2015.

Introduction to Volume III

Group actions appear (without the name) in the work of Galois in the early 1830s, in the setting of substitutions of roots of algebraic equations. The notion of group action grew up slowly, starting from permutation groups of finite sets, and by the last quarter of the nineteenth century, it was already playing a fundamental role in a variety of domains such as differential equations, algebraic invariants, automorphic functions, uniformization and other fields of analysis, geometry and number theory. In the 1870s, Lie and Klein equated geometry with group actions.

The present volume is the third one of the *Handbook of Group Actions*, a collection of survey articles concerned with the various aspects of this vast theory.

This volume is divided into three parts.

Part A is concerned with group actions on spaces of negative or nonpositive curvature in various senses, including manifolds of constant negative curvature, Gromov hyperbolic spaces, and others. Metrics of constant negative curvature on surfaces and three-manifolds are represented by discrete faithful representations of fundamental groups of such manifolds into $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$ respectively. The development of these theories led naturally to other group actions, e.g. actions on laminations and their dual actions on $\mathbb{R}$-trees, and more generally on Λ-trees. In dimension two, the theory of hyperbolic structures with their deformation spaces in terms of group actions on the hyperbolic plane was already extensively developed by Poincaré, in the last decades of the nineteenth century, and it continues to grow today. In dimension three, the theory was given a tremendous impetus in the 1970s by Thurston and his uniformization program for three-manifolds, and it led to a huge amount of activity which is still growing. In dimension four, very few works exist on hyperbolic manifolds, compared to the work done in the first two dimensions. In fact, it is still unclear whether in that dimension the role played by hyperbolic manifolds and hyperbolic group actions is rather special, or as overwhelming as in dimensions 2 and 3, or something in between. A few techniques of hyperbolization exist in dimension four, mostly initiated by Gromov. The latter's ideas of a broad field involving group actions and hyperbolicity are outlined in his 1981 paper *Hyperbolic manifolds, groups and actions*[1] and then in his 1987 paper *Hyperbolic groups.*[2] Part A of the present volume contains surveys related to these various notions of hyperbolicity.

Part B is a collection of surveys on automorphisms of geometric structures, in-

[1]M. Gromov, Hyperbolic manifolds, groups and actions. In: Riemann surfaces and related topics: Proc. 1978 Stony Brook Conf., Ann. Math. Stud. 97 (1981), 183–213.

[2]M. Gromov, Hyperbolic groups. In: Essays in group theory, Publ. Math. Sci. Res. Inst. 8 (1987), 75–263.

cluding complex, projective, algebraic and Lorentzian structures. The Lorentzian theory has a long history, which can be traced back to the discovery of the geometry underlying relativity theory in the first decade of the twentieth century. A recent trend in Lorentzian geometry, which is surveyed here in two different chapters, emphasizes the analogies between this theory and that of group actions on surfaces and Teichmüller spaces. Several aspects of group actions that arise in real projective geometry are generalizations of properties of hyperbolic structures, and this aspect is also surveyed in Part B.

Part C contains surveys on various group actions on topological manifolds. This includes in particular an update of many interesting open questions that occur in this setting, in particular a survey of the Hilbert–Smith Conjecture which concerns the necessary conditions on topological groups in order to act faithfully on a topological manifold, and related conjectures. This part also contains a chapter on the application of a principle of Emmy Noether to the existence and classification of surfaces of constant mean curvature in Riemannian manifolds. The principle says that when a variational problem is preserved by a continuous group of symmetries, then the solutions of the variational problem satisfy a system of conservation laws.

Let us now give a more detailed description of the content of the present volume.

Part A: Hyperbolicity and Group Actions on Spaces of Nonpositive Curvature

Group actions on the hyperbolic plane lead to hyperbolic structures on surfaces and to families of such actions, more precisely, to Riemann's moduli spaces and Teichmüller spaces. Chapter 1, by Hideki Miyachi, is concerned with group actions on Teichmüller spaces. In principle, when we think of group actions on Teichmüller spaces, the first class of groups which comes to mind is that of mapping class groups. In fact such actions are reviewed in detail in several chapters in Volume I of the present Handbook. Miyachi, in Chapter 1 of the present volume, discusses other kinds of actions on Teichmüller spaces equipped with their Teichmüller metrics, namely, actions by quasi-isometries. These actions are obtained using extremal length geometry and more particularly the Gardiner–Masur boundary of Teichmüller space. Miyachi proposes a problem whose solution, if affirmative, leads to a new approach to a rigidity result saying that the inclusion of the group of isometries of Teichmüller space into the monoid of quasi-isometries of that space induces an isomorphism from that isometry group into the group of the so-called parallel classes of quasi-isometries. The Gromov product associated to the Teichmüller metric is used in this study, despite the fact that this metric is not Gromov-hyperbolic. Incidentally, this is one of the rare instances where the Gromov product is used in a non-Gromov hyperbolic setting. Analogies and differences between such actions on Teichmüller space and actions on hyperbolic spaces $\mathbb{H}^n$ are also discussed.

Chapter 2 by Ken'ichi Ohshika is concerned with group actions on hyperbolic spaces in dimensions 2 and 3. In some sense, the setting is that of Teichmüller theory (for dimension 2) and of deformation spaces of Kleinian groups (for dimension 3). The stress is on the group actions that describe degenerations of hyperbolic structures. At the same time, Ohshika makes a comprehensive review of some important results in low-dimensional topology and geometry, including Thurston's compactification of Teichmüller space, making relations with actions on $\mathbb{R}$-trees and the notion of Gromov convergence of metric spaces, Thurston's uniformization theory of Haken manifolds, hyperbolic structures on 3-manifolds obtained by assembling ideal tetrahedra, and degenerations of such structures. Ohshika also studies deformation spaces of faithful discrete representations of fundamental groups of surfaces into $\mathrm{PSL}(2,\mathbb{C})$, the theory of pleated surfaces — in particular Thurston's Double Limit Theorem, and the works of Culler–Shalen and Morgan–Shalen giving alternative approaches to Thurston's theory of degenerations of hyperbolic structures on 3-manifolds, using valuations and Λ-tress.

Chapter 3 by Bruno Martelli is concerned with the construction of hyperbolic four-manifolds of finite volume. The author starts by reviewing the important results in this field, and he then presents several examples. Most of the examples arise from a basic construction, namely, a reflection group Γ acting on four-dimensional hyperbolic space $\mathbb{H}^4$ together with its Coxeter polytope P. The hyperbolic manifolds are then obtained either algebraically by constructing torsion-free subgroups of Γ, or geometrically by assembling copies of P. The examples are limited and, up to now, only few very symmetric polytopes P can be assembled successfully. The work done on this subject generally uses computers.

Gromov introduced in his 1987 seminal paper *Hyperbolic groups* the idea of *hyperbolization of polyhedra.* Pedro Ontaneda, in Chapter 4, surveys Gromov's original ideas on this subject, their development and the applications of hyperbolization which were worked out by various authors. He considers in particular one of the major questions related to hyperbolization, which was thoroughly studied by Davis and Januszkiewicz, namely, the construction of spaces of negative curvature using Gromov's hyperbolization process. He also discusses his own method of smoothing, the so-called Charney–Davis *strict hyperbolization* on a smooth manifold, obtaining a Riemannian metric of negative sectional curvature. He calls this process *Riemannian hyperbolization.* The original Charney–Davis hyperbolization produces a negatively curved metric which in general is not Riemannian: several singularities appear during the process. Ontaneda provides a sketch of the way in which this smoothing is accomplished and he also presents some important applications of his techniques. Gromov's ideas on hyperbolic groups are in the background of all these hyperbolization techniques.

Chapter 5 by Sang-hyun Kim is also concerned with hyperbolic groups, and more particularly with a question raised by Gromov, namely, whether every one-ended hyperbolic group contains a surface group. In some sense, a positive answer to that question would give a new and precise evidence to the general idea that the class of hyperbolic groups is a generalization of the class of fundamental groups of negatively curved manifolds. The question may also be regarded as a generalization of the Surface Subgroup Theorem in the theory of 3-manifolds (stated by Waldhausen), whose proof was obtained by Kahn and Markovic. The result says, in the hyperbolic case, that every closed hyperbolic 3-manifold group contains a subgroup isomorphic to the fundamental group of a closed hyperbolic surface.[3] Another result which is referred to in Chapter 5 and which is a special case of the above question is the Virtual Haken Conjecture, recently settled positively by Agol,[4] stating that every compact orientable aspherical 3-manifold is virtually Haken, that is, it has a finite-sheeted cover containing a properly embedded π_1-injective surface of Euler characteristic ≤ 0. Ontaneda surveys results related to Gromov's problem, more particularly in the cases of 3-manifold groups, CAT(0) groups, graphs of free groups and random groups. He also proposes several open questions related to this subject.

Bruno Duchesne, in Chapter 6, surveys the theory of groups acting on metric spaces of non-positive curvature in the CAT(0) sense and in weaker senses involving convexity properties of the distance function. He studies several examples in some detail, including Euclidean buildings and CAT(0) cube complexes. Ideas of Busemann, Alexandrov, Toponogov and Gromov are highlighted. Rigidity, amenability, the Haagerup property, property (T), amenability at infinity, superrigidity and rank-rigidity properties of such group actions are discussed. The role played by several types of boundaries at infinity of the spaces involved is emphasized. Duchesne also discusses the links between, on the one hand, algebraic and analytic properties of groups, and, on the other hand, properties of the same kind shared by the spaces on which these groups act. The chapter concludes with a series of conjectures, including a discussion of the rank rigidity conjecture and the flat closing conjecture.

Part B: Group Actions on Geometric Structures

Chapter 7 by Harish Seshadri and Kaushal Verma is concerned with the action of the group $\mathrm{Aut}(D)$ of holomorphic automorphisms of a bounded domain D in $\mathbb{C}^n$ on that domain. By a result of H. Cartan, $\mathrm{Aut}(D)$ is a Lie group of dimension $\leq n^2+2n$ whose action on D is proper. One question which the authors address is to understand the relationship between D and $D/\mathrm{Aut}(D)$. They mention work of C. L. Siegel related to this question. In the case where $D/\mathrm{Aut}(D)$ is compact, D is necessarily a domain of holomorphy. In the case where it is not, the dynamics of the action of $\mathrm{Aut}(D)$ and the boundary ∂D enters into consideration. More generally, one of Seshadri and Verma's themes in this study is the fact that the geometry of the domain D influences the group $\mathrm{Aut}(D)$ and vice versa. It is known that there exist non-biholomorphic domains with isomorphic automorphism groups. Despite this fact, a basic question remains, namely, to what extent $\mathrm{Aut}(D)$ determines D. In other words, what are possible additional constraints on $\mathrm{Aut}(D)$ so that such a group determines uniquely the domain D. The survey is built around that question. Results of various people on related questions are discussed.

Chapter 8, by Alexander Isaev, is a comprehensive survey on the known results and techniques used in the study of proper actions by biholomorphisms of high-

[3] J. Kahn, V. Markovic, Immersing almost geodesic surfaces in a closed hyperbolic three manifold, Ann. Math. 175 (2012), 1127–1190.

[4] I. Agol, The virtual Haken Conjecture, Doc. Math. 18 (2013), 1045–1087.

dimensional Lie groups on complex manifolds. The dimension of the acting group is assumed to be "high" compared to the dimension of the manifold, a condition that holds generally in the instances where the explicit determination of the action is possible. The survey includes a complete description of Kobayashi-hyperbolic manifolds with high-dimensional automorphism groups.

Chapter 9, by Karel Dekimpe, is concerned with the theory of infra-nilmanifolds. These spaces generalize flat manifolds, and their fundamental groups, the so-called almost-Bieberbach groups, generalize the Bieberbach groups. The latter are torsion-free crystallographic groups which are cocompact discrete subgroup of isometries of the Euclidean space $\mathbb{R}^n$. At the same time, Dekimpe provides a systematic survey of the theory of flat manifolds and Bieberbach groups.

Chapter 10 by Suhyoung Choi, Gye-Seon Lee and Ludovic Marquis is a survey on representations of finitely generated groups into Lie groups. The main examples of finitely generated groups that are considered are the Fuchsian groups, 3-manifold groups and Coxeter groups. From the geometrical point of view, the authors study projective structures on surfaces and on 3-manifolds equipped with their Hilbert geometry. In fact, Hilbert geometry is the metric geometry of discrete subgroups of the group of projective transformations of real projective space preserving a properly convex open subset. Classical works of Benzécri, Kuiper, Koszul, and more recent works of Goldman, Vinberg, Tits, Benoist, Choi, Lee, Marquis and others are mentioned in this chapter. The authors also survey the deformation spaces of convex real projective structures on closed manifolds and orbifolds, including works of Johnson–Millson, Thurston, Goldman–Millson, Benoist and others. Infinitesimal deformation and infinitesimal rigidity (works of Weil, Raghunathan, etc.) are also described. The reader will also find in this chapter introductions to character varieties and geometric structures on orbifolds.

Chapter 11 by Thierry Barbot is concerned with group actions on Lorentzian manifolds. More precisely, the idea is to develop an analogue of the theory of n-dimensional Kleinian groups, that is, discrete subgroups of isometries of hyperbolic n-space, in the setting of Lorentzian geometry. Expressed in purely set-theoretical words, Kleinian groups are discrete subgroups of the Lie group $\mathrm{SO}(1,n)$ whereas their Lorentzian analogues, which the author calls *Lorentzian Kleinian groups*, are discrete subgroups of the Lie group $\mathrm{SO}(2,n)$. The development of part of this theory in the Lorentzian setting relies on the introduction of causality notions, which originate in the general notion of causality in the spacetime of general relativity, and also on a theory of achronal subgroups, which are subgroups that are well behaved with respect to causality. Barbot surveys the work already done on the classification of globally hyperbolic spacetimes of constant curvature and its close connection, in the case of dimension $n = 2+1$, with Teichmüller theory. The results obtained on Teichmüller spaces by working in Lorentzian geometry are in the lineage of a ground-breaking work of Geoffrey Mess done in 1990 and published much later.[5]

[5]G. Mess, Lorentz spacetimes of constant curvature, Geometriae Dedicata 126 (2007), Issue 1, 3–45. The paper was published together with a commentary by L. Andersson, T. Barbot, R. Benedetti, F. Bonsante, W. M. Goldman, F. Labourie, K. P. Scannell and J-M. Schlenker, Notes on a paper of Mess, Geometriae Dedicata 126 (2007), Issue 1, 47–70.

Chapter 12, by François Fillastre and Graham Smith is in the lineage of work considered in Chapter 11, namely, the theory of group actions on Lorentzian manifolds in analogy with Kleinian groups. Besides showing that the Lorentzian theory leads to new results on Teichmüller spaces, the authors prove that some classical results in Teichmüller theory may be considered from a completely new point of view using Lorentzian geometry, and more precisely the theory of globally hyperbolic spacetimes. Fillastre and Smith provide a systematic survey of this field, from the perspective of a scattering theory. Their chapter also contains a new collection of open problems that replaces previous lists which needed to be updated.

Part C: Group Actions on Topological Manifolds and Symmetries

Chapter 13, by Allan Edmonds, is a survey on topological and locally linear actions of finite or compact Lie groups on topological 4-manifolds. The exposition ranges from the 1960s to the present. It covers a variety of notions of topological, algebraic, geometric and symplectic nature. It includes topics from Donaldson's Yang–Mills instantons and the more recent Seiberg–Witten theory of monopoles. The author emphasizes the aspects of the theory where things behave differently in dimension four than in dimensions two and three. He highlights the positive topological results that pose challenges for the theory of differentiable transformation groups. In particular, he displays examples of topological locally linear actions on smooth manifolds that are not equivalent to smooth actions. The chapter contains a list of problems which in part update problems that appeared in previous lists. In particular, a revision is made of the problems of Kirby's list as well as those of the Colorado 1984 Transformation Groups Conference that are related to group actions on 4-manifolds.

A deep (although obvious) general principle that was formulated by Emmy Noether back in 1918 says that when a variational problem is preserved by a continuous group of symmetries, then the solutions of the variational problem satisfy a system of conservation laws. In Chapter 14 by Christine Breiner and Stephen Kleene, this principle is applied to the existence and classification of surfaces of constant mean curvature in Riemannian geometry. The conserved quantities in this setting arise from the fact that the mean curvature is invariant by the symmetries of the ambient manifold and the diffeomorphisms of the domain. The authors survey results of several authors on this topic, showing that these conserved quantities play a central role in the classification and the gluing techniques that are involved in that theory.

Chapter 15, by Krzysztof Pawałowski, is concerned with symmetries of CW complexes. His starting point is the famous Hilbert–Smith Conjecture which says that a locally compact Lie group acting faithfully and continuously on a topological manifold is necessarily a Lie group. To this day, there remains one case to analyse in order to settle completely this conjecture, namely, the action of the additive group $\mathbb{Z}_p$ of p-adic integers; that is, one needs to show that for any prime p, such a group cannot act faithfully on a topological manifold. Pawałowski, in Chapter 15, concentrates on two problems related to this conjecture. The first one dates back to 1960 and is called the Smith Equivalence Problem. It concerns a finite group G acting smoothly on a sphere whose fixed point set consists of two points, and it asks whether the tangent G-modules at these two points are necessarily isomorphic. A complete response to this conjecture is still unknown, and Pawałowski surveys the known results for various groups G. The second problem is what the author calls the Laitinen conjecture. It also concerns actions of finite groups on topological manifolds. Its formulation came as a by-product of the work by Erkki Laitinen on the Smith equivalence problem. This conjecture implies that the answer to the Smith equivalence problem is negative if the G is a finite Oliver group with two or more real conjugacy classes of elements not of prime power order. The works done on these two problems involve techniques from topology, group theory, representation theory, index theory and number theory. Chapter 15 also contains an introductory section on Oliver groups.

We hope that the various surveys contained in this volume of the *Handbook of Group Actions* will be useful to the mathematical community.

Athanase Papadopoulos

Contents

Part A: Hyperbolicity and Group Actions on Spaces of Nonpositive Curvature

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Action at Infinity of Quasi-isometries on Teichmüller Space and the Geometry of the Gromov Product

Hideki Miyachi*

Abstract

In this paper, we discuss the action of (quasi-)isometries on Teichmüller space via extremal length geometry. We will propose a problem whose affirmative solution leads to an alternative approach to the quasi-isometric rigidity theorem proved by B. Bowditch, A. Eskin, H. Masur and K. Rafi.

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Keywords and Phrases: Teichmüller space, Teichmüller distance, Gromov hyperbolic space, Gromov product, mapping class group.

1 Introduction

In this paper, we will indicate an analog between the Teichmüller space $\mathcal{T}_g$ and the (real) hyperbolic space $\mathbb{H}^n$ in terms of the actions of the group of isometries and the monoid of quasi-isometries, after the author's researches [21] and [24]. We will also propose a problem which leads to an alternative approach to the quasi-isometric rigidity theorem which was obtained by A. Eskin, H. Masur and K. Rafi [5] and B. Bowditch [2] independently. The quasi-isometric rigidity theorem says that the inclusion of the group $\mathrm{Isom}(\mathcal{T}_g)$ of isometries of Teichmüller space into the

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monoid of quasi-isometries induces an isomorphism from $\mathrm{Isom}(\mathcal{T}_g)$ to the group of parallel classes of quasi-isometries (cf. §4.2).

Before starting the discussion, the author must confess that it is generally considered that the actions of isometries and quasi-isometries on Teichmüller space is *quite different* from those on the hyperbolic space. For example, when $g \geq 2$, the isometry group is virtually isomorphic to the mapping class group of a compact orientable surface Σ_g of genus g and acts on $\mathcal{T}_g$ properly and discontinuously, while the action of the isometry group on $\mathbb{H}^n$ is transitive. Furthermore, Teichmüller space has higher (geometric and topological) rank (cf. [2] and [4], see also [17]). Hence, from the coarse geometric point of view, Teichmüller space is extensively compared with symmetric spaces of higher rank, not with (Gromov) hyperbolic spaces (e.g. [5]). Actually, a hyperbolic space is not quasi-isometrically rigid, but so are nonflat irreducible symmetric space of noncompact type of rank at least 2 (cf. [12]). Nevertheless, besides the results in this chapter, some analogies are also observed in view of the geometry at infinity (e.g. [23]). The author hopes that this research will be informative in studying the actions of (quasi-)isometries on metric spaces with moderate properties between Gromov hyperbolic spaces and symmetric spaces.

The main tool for investigating the action of (quasi-)isometries on Teichmüller space in this chapter is the Gromov product. By definition, the *Gromov product* on a metric space (X, d_X) with base point o is

$$\langle x_1, x_2\rangle_o = \langle x_1, x_2\rangle_o^X = \frac{1}{2}(d_X(o, x_1) + d_X(o, x_2) - d_X(x_1, x_2))$$

(cf. [6]). The Gromov product is widely recognized as one of the important tools for investigating Gromov hyperbolic spaces. Indeed, since the Gromov product is nothing but the error of the triangle inequality, one can understand it to measure how three points $\{x_1, o, x_2\}$ line up straightly in this order. Hence, one may imagine that when a divergent sequence $(x_n)_n$ in X satisfies $\langle x_n, x_m\rangle_o \to \infty$ as $n, m \to \infty$, the visual angle between x_n and x_m from the base point o tends to 0, and hence the sequence could "converge" to an ideal boundary point. This observation is not justified in general, but is valid when a given metric space is Gromov hyperbolic.

One of motivations of this work is that this "imagination" can be justified in many cases for the case of Teichmüller space $(\mathcal{T}_g, d_T)$ with the Teichmüller distance d_T, although $(\mathcal{T}_g, d_T)$ is not Gromov hyperbolic (cf. [16]). In viewing this, we consider a canonical ideal boundary of $(\mathcal{T}_g, d_T)$ which is called the Gardiner-Masur boundary (cf. §4). This boundary contains the space $\mathcal{PMF}$ of projective measured folations, and admits a projectified geometric intersection number which coincides with the usual projectified geometric intersection number on $\mathcal{PMF}$ when restricted to pairs of points in that space (cf. (4) in §4). The following two facts are observed:

(a) If a sequence $(p_n)_n$ in $\mathcal{T}_g$ converges to a point in the Gardiner-Masur boundary, $\langle p_n, p_m\rangle_o \to \infty$ when $n, m \to \infty$.

(b) When a sequence $(p_n)_n$ in $\mathcal{T}_g$ satisfies $\langle p_n, p_m\rangle_o \to \infty$ $(n, m \to \infty)$, any two