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James E. Humphreys

Introduction to Lie Algebras and Representation Theory

李代数和表示论导论

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James E. Humphreys

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J. E. Humphreys
Professor of Mathematics
University of Massachusetts
Amherst, MA 01003
USA

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To the memory of my nephews

Willard Charles Humphreys III

and

Thomas Edward Humphreys

Preface

This book is designed to introduce the reader to the theory of semisimple Lie algebras over an algebraically closed field of characteristic 0, with emphasis on representations. A good knowledge of linear algebra (including eigenvalues, bilinear forms, euclidean spaces, and tensor products of vector spaces) is presupposed, as well as some acquaintance with the methods of abstract algebra. The first four chapters might well be read by a bright undergraduate; however, the remaining three chapters are admittedly a little more demanding.

Besides being useful in many parts of mathematics and physics, the theory of semisimple Lie algebras is inherently attractive, combining as it does a certain amount of depth and a satisfying degree of completeness in its basic results. Since Jacobson's book appeared a decade ago, improvements have been made even in the classical parts of the theory. I have tried to incorporate some of them here and to provide easier access to the subject for non-specialists. For the specialist, the following features should be noted:

(1) The Jordan-Chevalley decomposition of linear transformations is emphasized, with "toral" subalgebras replacing the more traditional Cartan subalgebras in the semisimple case.

(2) The conjugacy theorem for Cartan subalgebras is proved (following D. J. Winter and G. D. Mostow) by elementary Lie algebra methods, avoiding the use of algebraic geometry.

(3) The isomorphism theorem is proved first in an elementary way (Theorem 14.2), but later obtained again as a corollary of Serre's Theorem (18.3), which gives a presentation by generators and relations.

(4) From the outset, the simple algebras of types A, B, C, D are emphasized in the text and exercises.

(5) Root systems are treated axiomatically (Chapter III), along with some of the theory of weights.

(6) A conceptual approach to Weyl's character formula, based on Harish-Chandra's theory of "characters" and independent of Freudenthal's multiplicity formula (22.3), is presented in §23 and §24. This is inspired by D.-N. Verma's thesis, and recent work of I. N. Bernstein, I. M. Gel'fand, S. I. Gel'fand.

(7) The basic constructions in the theory of Chevalley groups are given in Chapter VII, following lecture notes of R. Steinberg.

I have had to omit many standard topics (most of which I feel are better suited to a second course), e.g., cohomology, theorems of Levi and Mal'cev, theorems of Ado and Iwasawa, classification over non-algebraically closed fields, Lie algebras in prime characteristic. I hope the reader will be stimulated to pursue these topics in the books and articles listed under References, especially Jacobson [1], Bourbaki [1], [2], Winter [1], Seligman [1].

A few words about mechanics: Terminology is mostly traditional, and notation has been kept to a minimum, to facilitate skipping back and forth in the text. After Chapters I–III, the remaining chapters can be read in almost any order if the reader is willing to follow up a few references (except that VII depends on §20 and §21, while VI depends on §17). A reference to Theorem 14.2 indicates the (unique) theorem in subsection 14.2 (of §14). Notes following some sections indicate nonstandard sources or further reading, but I have not tried to give a history of each theorem (for historical remarks, cf. Bourbaki [2] and Freudenthal-deVries [1]). The reference list consists largely of items mentioned explicitly; for more extensive bibliographies, consult Jacobson [1], Seligman [1]. Some 240 exercises, of all shades of difficulty, have been included; a few of the easier ones are needed in the text.

This text grew out of lectures which I gave at the N.S.F. Advanced Science Seminar on Algebraic Groups at Bowdoin College in 1968; my intention then was to enlarge on J.-P. Serre's excellent but incomplete lecture notes [2]. My other literary debts (to the books and lecture notes of N. Bourbaki, N. Jacobson, R. Steinberg, D. J. Winter, and others) will be obvious. Less obvious is my personal debt to my teachers, George Seligman and Nathan Jacobson, who first aroused my interest in Lie algebras. I am grateful to David J. Winter for giving me pre-publication access to his book, to Robert L. Wilson for making many helpful criticisms of an earlier version of the manuscript, to Connie Engle for her help in preparing the final manuscript, and to Michael J. DeRise for moral support. Financial assistance from the Courant Institute of Mathematical Sciences and the National Science Foundation is also gratefully acknowledged.

New York, April 4, 1972 J. E. Humphreys

Notation and Conventions

Z, $\mathbf{Z}^+$, **Q**, **R**, **C** denote (respectively) the integers, nonnegative integers, rationals, reals, and complex numbers

$\amalg$ denotes direct sum of vector spaces

$A \ltimes B$ denotes the semidirect product of groups A and B, with B normal

Card	= cardinality	Ker	= kernel
char	= characteristic	Im	= image
det	= determinant	Tr	= trace
dim	= dimension		

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Chapter I

Basic Concepts

In this chapter F *denotes an arbitrary (commutative) field.*

1. Definitions and first examples

1.1. The notion of Lie algebra

Lie algebras arise "in nature" as vector spaces of linear transformations endowed with a new operation which is in general neither commutative nor associative: $[x, y] = xy - yx$ (where the operations on the right side are the usual ones). It is possible to describe this kind of system abstractly in a few axioms.

Definition. A vector space L over a field F, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [xy]$ and called the **bracket** or **commutator** of x and y, is called a **Lie algebra** over F if the following axioms are satisfied:

($L1$) The bracket operation is bilinear.
($L2$) $[xx] = 0$ for all x in L.
($L3$) $[x[yz]] + [y[zx]] + [z[xy]] = 0 \quad (x, y, z \in L)$.

Axiom ($L3$) is called the **Jacobi identity.** Notice that ($L1$) and ($L2$), applied to $[x+y, x+y]$, imply anticommutativity: ($L2'$) $[xy] = -[yx]$. (Conversely, if char F $\neq 2$, it is clear that ($L2'$) will imply ($L2$).)

We say that two Lie algebras L, L' over F are **isomorphic** if there exists a vector space isomorphism ϕ: $L \to L'$ satisfying $\phi([xy]) = [\phi(x)\phi(y)]$ for all x, y in L (and then ϕ is called an **isomorphism** of Lie algebras). Similarly, it is obvious how to define the notion of (Lie) **subalgebra** of L: A subspace K of L is called a subalgebra if $[xy] \in K$ whenever $x, y \in K$; in particular, K is a Lie algebra in its own right relative to the inherited operations. Note that any nonzero element $x \in L$ defines a one dimensional subalgebra Fx, with trivial multiplication, because of ($L2$).

In this book we shall be concerned almost exclusively with Lie algebras L whose underlying vector space is *finite dimensional* over F. *This will always be assumed, unless otherwise stated.* We hasten to point out, however, that certain infinite dimensional vector spaces and associative algebras over F will play a vital role in the study of representations (Chapters V–VII). We also mention, before looking at some concrete examples, that the axioms for a Lie algebra make perfectly good sense if L is only assumed to be a module over a commutative ring, but we shall not pursue this point of view here.

1.2. Linear Lie algebras

If V is a finite dimensional vector space over F, denote by End V the set of linear transformations $V \to V$. As a vector space over F, End V has dimension n^2 ($n = \dim V$), and End V is a ring relative to the usual product operation. Define a new operation $[x, y] = xy - yx$, called the **bracket** of x and y. With this operation End V becomes a Lie algebra over F: axioms ($L1$) and ($L2$) are immediate, while ($L3$) requires a brief calculation (which the reader is urged to carry out at this point). In order to distinguish this new algebra structure from the old associative one, we write $\mathfrak{gl}(V)$ for End V viewed as Lie algebra and call it the **general linear algebra** (because it is closely associated with the **general linear group** $GL(V)$ consisting of all invertible endomorphisms of V). When V is infinite dimensional, we shall also use the notation $\mathfrak{gl}(V)$ without further comment.

Any subalgebra of a Lie algebra $\mathfrak{gl}(V)$ is called a **linear Lie algebra**. The reader who finds matrices more congenial than linear transformations may prefer to fix a basis for V, thereby identifying $\mathfrak{gl}(V)$ with the set of all $n \times n$ matrices over F, denoted $\mathfrak{gl}\,(n, \mathsf{F})$. This procedure is harmless, and very convenient for making explicit calculations. For reference, we write down the multiplication table for $\mathfrak{gl}\,(n, \mathsf{F})$ relative to the standard basis consisting of the matrices e_{ij} (having 1 in the (i, j) position and 0 elsewhere). Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that:

$$(*) \qquad [e_{ij}, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{li}\, e_{kj}.$$

Notice that the coefficients are all ± 1 or 0; in particular, all of them lie in the prime field of F.

Now for some further examples, which are central to the theory we are going to develop in this book. They fall into four families A_ℓ, B_ℓ, C_ℓ, D_ℓ ($\ell \geq 1$) and are called the **classical algebras** (because they correspond to certain classical linear Lie groups). For $\mathsf{B}_\ell - \mathsf{D}_\ell$, let char $\mathsf{F} \neq 2$.

A_ℓ: Let $\dim V = \ell + 1$. Denote by $\mathfrak{sl}(V)$, or $\mathfrak{sl}(\ell+1, \mathsf{F})$, the set of endomorphisms of V having trace zero. (Recall that the **trace** of a matrix is the sum of its diagonal entries; this is independent of choice of basis for V, hence makes sense for an endomorphism of V.) Since $Tr(xy) = Tr(yx)$, and $Tr(x+y) = Tr(x) + Tr(y)$, $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$, called the **special linear algebra** because of its connection with the **special linear group** $SL(V)$ of endomorphisms of det 1. What is its dimension? On the one hand $\mathfrak{sl}(V)$ is a proper subalgebra of $\mathfrak{gl}(V)$, hence of dimension at most $(\ell+1)^2 - 1$. On the other hand, we can exhibit this number of linearly independent matrices of trace zero: Take all e_{ij} ($i \neq j$), along with all $h_i = e_{ii} - e_{i+1,i+1}$ ($1 \leq i \leq \ell$), for a total of $\ell + (\ell+1)^2 - (\ell+1)$ matrices. We shall always view this as the standard basis for $\mathfrak{sl}(\ell+1, \mathsf{F})$.

C_ℓ: Let $\dim V = 2\ell$, with basis $(v_1, \ldots, v_{2\ell})$. Define a nondegenerate skew-symmetric form f on V by the matrix $s = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$. (It can be shown

that even dimensionality is a necessary condition for existence of a nondegenerate bilinear form satisfying $f(v, w) = -f(w, v)$.) Denote by $\mathfrak{sp}(V)$, or $\mathfrak{sp}(2\ell, \mathsf{F})$, the **symplectic algebra**, which by definition consists of all endomorphisms x of V satisfying $f(x(v), w) = -f(v, x(w))$. The reader can easily verify that $\mathfrak{sp}(V)$ is closed under the bracket operation. In matrix terms, the condition for $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ $(m, n, p, q \in \mathfrak{gl}(\ell, \mathsf{F}))$ to be symplectic is that $sx = -x^t s$ (x^t = transpose of x), i.e., that $n^t = n$, $p^t = p$, and $m^t = -q$. (This last condition forces $Tr(x) = 0$.) It is easy now to compute a basis for $\mathfrak{sp}(2\ell, \mathsf{F})$. Take the diagonal matrices $e_{ii} - e_{\ell+i,\ell+i}$ $(1 \le i \le \ell)$, ℓ in all. Add to these all $e_{ij} - e_{\ell+j,\ell+i}$ $(1 \le i \ne j \le \ell)$, $\ell^2 - \ell$ in number. For n we use the matrices $e_{i,\ell+i}$ $(1 \le i \le \ell)$ and $e_{i,\ell+j} + e_{j,\ell+i}$ $(1 \le i < j \le \ell)$, a total of $\ell + \frac{1}{2}\ell(\ell-1)$, and similarly for the positions in p. Adding up, we find $\dim \mathfrak{sp}(2\ell, \mathsf{F}) = 2\ell^2 + \ell$.

B_ℓ: Let $\dim V = 2\ell+1$ be odd, and take f to be the nondegenerate symmetric bilinear form on V whose matrix is $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}$. The **orthogonal algebra** $\mathfrak{o}(V)$, or $\mathfrak{o}(2\ell+1, \mathsf{F})$, consists of all endomorphisms of V satisfying $f(x(v), w) = -f(v, x(w))$ (the same requirement as for C_ℓ). If we partition x in the same form as s, say $x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$, then the condition $sx = -x^t s$ translates into the following set of conditions: $a = 0$, $c_1 = -b_2^t$, $c_2 = -b_1^t$, $q = -m^t$, $n^t = -n$, $p^t = -p$. (As in the case of C_ℓ, this shows that $Tr(x) = 0$.) For a basis, take first the ℓ diagonal matrices $e_{ii} - e_{\ell+i,\ell+i}$ $(2 \le i \le \ell+1)$. Add the 2ℓ matrices involving only row one and column one: $e_{1,\ell+i+1} - e_{i+1,1}$ and $e_{1,i+1} - e_{\ell+i+1,1}$ $(1 \le i \le \ell)$. Corresponding to $q = -m^t$, take (as for C_ℓ) $e_{i+1,j+1} - e_{\ell+j+1,\ell+i+1}$ $(1 \le i \ne j \le \ell)$. For n take $e_{i+1,\ell+j+1} - e_{j+1,\ell+i+1}$ $(1 \le i < j \le \ell)$, and for p, $e_{i+\ell+1,j+1} - e_{j+\ell+1,i+1}$ $(1 \le j < i \le \ell)$. The total number of basis elements is $2\ell^2 + \ell$ (notice that this was also the dimension of C_ℓ).

$\mathsf{D}_{\ell(\ell \ge 2)}$: Here we obtain another **orthogonal algebra**. The construction is identical to that for B_ℓ, except that $\dim V = 2\ell$ is even and s has the simpler form $\begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$. We leave it as an exercise for the reader to construct a basis and to verify that $\dim \mathfrak{o}(2\ell, \mathsf{F}) = 2\ell^2 - \ell$ (Exercise 8).

We conclude this subsection by mentioning several other subalgebras of $\mathfrak{gl}(n, \mathsf{F})$ which play an important subsidiary role for us. Let $\mathfrak{t}(n, \mathsf{F})$ be the set of **upper triangular matrices** (a_{ij}), $a_{ij} = 0$ if $i > j$. Let $\mathfrak{n}(n, \mathsf{F})$ be the **strictly upper triangular matrices** ($a_{ij} = 0$ if $i \ge j$). Finally, let $\mathfrak{d}(n, \mathsf{F})$ be the set of all **diagonal matrices**. It is trivial to check that each of these is closed under the bracket. Notice also that $\mathfrak{t}(n, \mathsf{F}) = \mathfrak{d}(n, \mathsf{F}) + \mathfrak{n}(n, \mathsf{F})$ (vector space direct sum), with $[\mathfrak{d}(n, \mathsf{F}), \mathfrak{n}(n, \mathsf{F})] = \mathfrak{n}(n, \mathsf{F})$, hence $[\mathfrak{t}(n, \mathsf{F}), \mathfrak{t}(n, \mathsf{F})] = \mathfrak{n}(n, \mathsf{F})$, cf. Exercise 5. (If H, K are subalgebras of L, $[H\ K]$ denotes the subspace of L spanned by commutators $[xy]$, $x \in H$, $y \in K$.)

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