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# Non-Newtonian Fluids: A Dynamical Systems Approach

（非牛顿流：动力系统方法）

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# Introduction

Non-Newtonian flow phenomenon exists in the fields of biology, physics and aviation industry. Newtonian fluids reflect the linear constitutive relation between stress tensor and the velocity gradient tensor, and the motion of incompressible viscous fluids can be described by famous Navier–Stokes equation. By contrast, the non-Newtonian fluids reflect that the stress tensor and the velocity gradient tensor of fluid movement no longer satisfy the linear relationship in a given temperature and pressure. The scientific studies have shown that the viscous fluid of the production and life generally belongs to non-Newtonian fluids; the main forms are suspended as colloid and high molecular fluid, such as various solution of polymer, blood of the human, fruit pulp and egg white are non-Newtonian fluids that satisfy these properties, because they all have the high molecular. The nature of the non-Newtonian fluids are summarized as follows by Rajagopal in 1993; the non-Newtonian fluids in the shear fluid exhibits an ability to differ from other fluids, it has the ability to cut into thin or thick fluid and the non-Newtonian has a nonzero standard stress difference in the shear fluid. In addition, it can produce creep under the action of stress.

Non-Newtonian characteristics are exhibited by numerous fluids including physiological liquids, geological suspensions, industrial tribological liquids and biotechnological liquids. To describe the viscoelastic properties of such fluids, recently, constitutive equations with ordinary and fractional time or space derivatives have been introduced. The starting point of the fractional derivative model of viscoelastic fluids is usually a classical differential equation which is modified by replacing the time derivative of an integer order with fractional order and may be formulated both in the Riemann–Liouville or Caputo sense. With the development of fractional calculus, fractional derivatives and fractional partial differential equations have been applied to the numerical solution of the complex problems in fluids and continuum mechanics.

In recent years, we collected and summarized the mathematical theories of non-Newtonian fluids evolution equations. This book introduces the latest research achievements, with particular emphasis on various modern approaches and recent advances. The specific content is not only concerned with the existence, uniqueness and stability of weak solutions to the initial boundary value problems, that is, strong solutions, periodic solutions and so on, but also with stability analysis, regularity or partial regularity analysis, the existence and dimensional estimates of global attractors, inertial manifold, approximated inertial manifolds as well. In particular, we give some numerical results for non-Newtonian generalized fluid, and a brief introduction of viscoelastic fluid with fractional derivative models. It should be noted that this includes the achievements in cooperation with Professor Yadong Shang and Professor Guoguang Lin.

The aim of this book is to give a basic understanding of recent development in this field for researchers as well as for senior students and graduate students. Our expectation is that the readers who work in the related areas can access the frontier of this study based on the reading of this book.

Due to the time and knowledge limited, errors and inadequacies of the book are inevitable. Any suggestions and comments are welcome. We express our thanks to the seminar members of the Institute of Applied Physics and Computational Mathematics. We also express our gratitude to all those unnamed here.

20. August 2017 Boling Guo

# Contents

# 1 Non-Newtonian fluids and their mathematical model

In mathematics, there is twofold important significance in non-Newtonian fluid mechanics equations. On the one hand, we need to consider the more general Navier–Stokes equations when researching the uniqueness of weak solutions of three-dimensional Navier–Stokes equations in fluid mechanics. All of the research gives rise to the study of various incompressible or compressible non-Newtonian fluid dynamic equations with nonlinear constitutive equations. On the other hand, for non-Newtonian fluid, there have been various applications in chemical industry, biological engineering, glaciology, geology, hemorheology and many others, so that considerable attention has been devoted to it. In this chapter, we will show a brief introduction for the characteristics of non-Newtonian fluid, and reveal the derivation of multipolar viscous non-Newtonian fluid mechanics equations, and make some statements outlining some non-Newtonian fluid mechanics equations in hemorheology, glaciology and earth covering dynamics.

## 1.1 Non-Newtonian fluids and their characteristics

In the past 30 years, the research of non-Newtonian fluid mechanics got great development, and has become an important branch of fluid mechanics. Schwalter [81], Huilgol [48], Böhme [14] and Rajagopal [78] have shown that the flow of high molecular weight fluid exhibit non-Newtonian flow behavior, such as polymer solution, viscoelastic fluid or viscoplastic fluids and colloidal suspension, the vast majority of biological fluids.

The physical properties of non-Newtonian fluid are reflected in: this kind of material has the ability to flow generally, which can be seen as a fluid; however, it also has some properties of a solid, such as elasticity as the most typical materials may be silicon rubber.

In 1993, Rajagopal summed up the main characteristics of non-Newtonian fluid:

(1) In the shear flow, the fluid has the ability for shear thinning or shear thickening;
(2) In the shear flow, the fluid has a nonzero normal stress difference;
(3) This fluid has the ability to produce stress;
(4) The fluid has the ability of present stress relaxation;
(5) The fluid has the ability to spread.

Then non-Newtonian fluid may have one or all of the above listed characteristics. Blood rheology research has shown that for the blood protein polymer, the flow of blood exhibits non-Newtonian flow characteristics, such as shear thinning, spread phenomenon, stress relaxation and others. The study of glaciology reveals a spreading flow of glaciers and also have non-Newtonian flow behavior.

As is known to all, Newtonian fluid is a kind of fluid with constant viscosity $\mu_0$, and the basis of its constitutive equation is

$$\boldsymbol{\tau}^{v} = 2\mu_0 \mathbf{e}(u) \tag{1.1.1}$$

where $\boldsymbol{\tau}^{v}$ is partial stress tensor; $\mathbf{e}$ is partial strain rate tensor; $u$ is velocity. The generalized Newton formula follows from equation (1.1.1), which reflects the linear constitutive relation between each component of partial stress tensor and the local velocity gradient tensor. With the Stokes' assumption of zero inflation (or second) viscosity coefficient, we combine the generalized Newton formula and the momentum equation, and then obtain the (also suitable for incompressible) viscous fluid Navier–Stokes equation. It is generally believed that air and water, other general low molecular weight gases and most liquids follow the generalized Newtonian formula and the Navier–Stokes equation.

In view of mathematics, the fluid, the relationship between the stress tensor and the strain rate tensor cannot be described by generalized Newtonian fluid formula, usually called the non-Newtonian fluid.

For non-Newtonian fluids, even in a steady shear flow, where normal stress difference might not be equal, cause many interesting phenomena different from a Newtonian fluid, for example:

(1) Weissenberg effect of viscoelastic fluid steady shear flow (Weissenberg's lecture in the British Imperial College, London, 1994).

When filled with a fluid container rotating around the vessel axis, the Newtonian fluid along the vessel wall climbs and the liquid surface is concave down, as shown in Figure 1.1.1; whereas the viscoelasticity of non-Newtonian fluid is up along the axis of rotation, as shown in Figure 1.1.2.

(2) Viscoelastic fluid jet expansion effect of steady shear flow.

This effect is also referred to as the Barus effect or Merrington effect. When the viscoelasticity of non-Newtonian fluid flow from a big container to a capillary, and

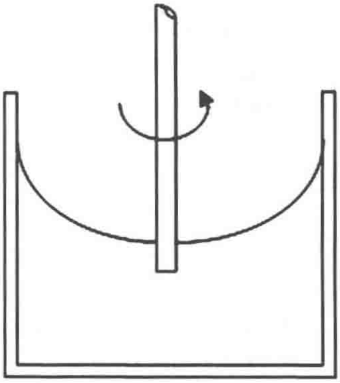

**Figure 1.1.1:** Newtonian fluid.

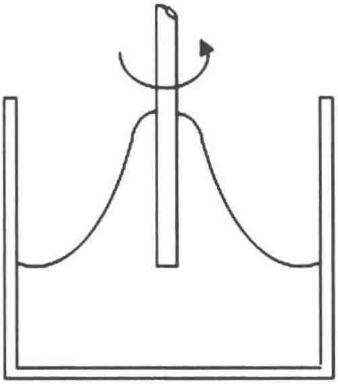

**Figure 1.1.2:** Non-Newtonian fluid.

again flow out of capillary, we can observe the viscoelasticity of the non-Newtonian fluid from the capillary mouth begin to slowly getting bigger and bigger, and endeavor to increase to the size of the container, then it slowly got smaller and smaller.

There is also an explanation: non-Newtonian fluid has memory function, and this memory is fading, as shown in Figure 1.1.3.

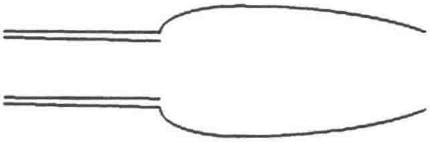

**Figure 1.1.3:** Memory function of non-Newtonian fluid.

The reader is referred to [19, 73] for more details about other properties of non-Newtonian fluid.

## 1.2 Incompressible and isothermal bipolar non-Newtonian fluids models

The fluid flow requirement generalized Newton equation has a linear relationship between partial stress tensor and the velocity gradient tensor. Thus, by relaxing the constraints of a generalized Newton's formula, to build mathematical models of the non-Newtonian fluid, is reflected in the following three areas:

(1) nonlinear constitutive relations between the viscous part of the stress tensor and velocity gradients;
(2) dependence of the viscous stress tensor on velocity gradients of order two or higher;
(3) constitutive relations for higher order partial stress tensors and higher order velocity gradients tensor.

From a mathematical point of view, Ladyzhenskaya [55, 56], Kaniel [51], Du and Gunzburger [23] and others have studied the constitutive relations of various viscous fluid models with nonlinear and higher order velocity gradient properties. For perturbation of Navier–Stokes equation with a high order velocity gradient, one can refer to Lions [59], Ou and Sritharan [73, 74] and others works.

In the applications, Green and Rivlin [33, 32] first studied the multipolar continuum theories; they considered an elastic nonstick material constitutive equation. For multipolar fluid models, please refer to Bleustein and Green's thesis [9]. Necas [6] and Silhavy [71], under the theoretical framework of Green and Rivlin, established thermodynamics theory of the multipolar viscous fluid constitutive equation. Generalized constitutive principles and material frame indifference principles of their development are constant with the second law of thermodynamics is presented by the Clausius–Duhem inequality. With special emphasis on nonlinear, isothermal and incompressible, Bellout et al. [6] in the case of bipolar, extend some conclusions of the multipolar fluid mode.

Let **u** be the velocity field, $\theta$ is temperature, $\rho$ is density, **E** is the energy per unit mass of the material, $\eta$ is entropy, $f$ is the outer physical per unit mass, **q** is heat flux vector, $r$ is outer radiative heat exchange rate, $\tau_{i,i_1,\dots,i_k,j}$ is $k = 0, 1, \dots, N-1$ space multipolar stress tensor. $x_i$, $i = 1, 2, \dots, n$ is the Euclidean space coordinate, and suppose $\tau_{i,i_1,\dots,i_k,j}$ are symmetry about indicators $i_1, \dots, i_k$.

Taken

$$\varepsilon_{i,j,k} = \begin{cases} 1, & \text{if } i, j, k \text{ are an even arrangement of } 1, 2, 3 \\ 0, & \text{if } i, j, k \text{ are the true repeatedly arrangement of } 1, 2, 3 \\ -1, & \text{if } i, j, k \text{ are an odd arrangement of } 1, 2, 3. \end{cases}$$

Using the material derivative mark,

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\partial \xi}{\partial t} + v_j \frac{\partial \xi}{\partial x_j}.$$

Followed by the mass conservation law, the law of conservation of momentum, conservation of energy, conservation of angular momentum and the second law of thermodynamics (Clausius–Duhem inequality), we can get the above functions to satisfy the following equations:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \frac{\partial u_j}{\partial x_j} = 0 \tag{1.2.1}$$

$$\rho \frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \tau_{i,j}}{\partial x_j} + \rho f_i \tag{1.2.2}$$

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(E + \frac{|\mathbf{u}|^2}{2}\right) = \frac{\partial}{\partial x_j}\left(-q_i + \sum_{k=0}^{N-1} \tau_{ji_1\dots i_k i} \frac{\partial^{k+1} u_j}{\partial x_{j_1} \cdots \partial x_{j_k} \partial x_i}\right) + \rho f_i u_i + \rho r \tag{1.2.3}$$

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon_{jkl}x_k u_l) = \frac{\partial}{\partial x_i}(\varepsilon_{ijk}x_k\tau_{li} + \varepsilon_{jkl}\tau_{lki}) + \rho\varepsilon_{jkl}x_k f_l \tag{1.2.4}$$

$$\rho\frac{\mathrm{d}\eta}{\mathrm{d}t} \geq -\frac{\partial}{\partial x_i}\left(\frac{q_i}{\theta}\right) + \rho\frac{r}{\theta}. \tag{1.2.5}$$

If $E$, $\eta$, $q$ and $\tau_{ii_1\dots i_k j}$, $k = 0, 1, 2, \dots, N-1$ are functions of $\rho, \nabla u, \dots, \nabla^k u, \theta$ and $\nabla\theta$, and when $k = N-1$, the fluid complies with the above constitutive assumptions and are called the $N$-polar fluid. Traditionally, this requires a sufficiently smooth process, and the constitutive relation satisfies the Clausius–Duhem inequality.

Set the frame transformation of the form

$$\overline{x_i} = Q_{ij}(t)x_i + C_i(t) \tag{1.2.6}$$

and $Q_{ij}(t)Q_{ik}(t) = \delta_{ik}$, then the material frame indifference theory requires a transformation of equation (1.2.6) and $\theta$, $\eta$, $\rho$, $E$ and $r$ are invariant, while $q_i$, $\tau_{ii_1\dots i_k j}$ change under the usual tensor form. In constitutive relations, we introduce the rate of deformation tensor $\mathbf{e}$ as follows,

$$\mathbf{e} = (e_{ij}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Set $\tau^0_{ii_1\dots i_k j} = \tau_{ii_1\dots i_k j}(\rho, 0, \dots, \theta, 0)$ which represents the balance part of multipolar stress $\tau_{ii_1\dots i_k j} = \tau_{ii_1\dots i_k j}(\rho, e, \dots, \nabla^k u, \theta, \nabla\theta)$, and $\tau^v_{ii_1\dots i_k j} = \tau_{ii_1\dots i_k j} - \tau^0_{ii_1\dots i_k j}$ represents the adhesive portion of the stress tensor. Let $\tau_{ii_1\dots i_k j} = 0$.

Under the condition of isothermal incompressible, assume $k \geq 1$, $\tau^0_{ii_1\dots i_k j} \equiv 0$, $\tau_{ij} = \tau_{ji}$, $\frac{\partial\tau_{lki}}{\partial x_l} = \frac{\partial\tau_{kli}}{\partial x_l}$; furthermore, assume $k = 0, 1, \dots, N-1$:

(1) $\tau^u_{ii_1\dots i_k j} = \tau^u_{ii_1\dots i_k j}(e, \dots, \nabla^k u)$;

(2) $q = -K\nabla\theta$, $K > 0$.

From equation (1.2.5), we have

$$\sum_{k=0}^{N-1}\left(\tau_{jj_1\dots j_k i} + \frac{\partial}{\partial x_l}\tau_{jj_1\dots j_k j i_l}\right)\frac{\partial^{k+1}u_j}{\partial x_{j_1}\cdots\partial x_{j_k}\partial x_i} \geq 0 \tag{1.2.7}$$

and

$$\tau^0_{ji} = -\overline{p}(\rho, \theta)\delta_{ji}$$

where $p$ is pressure. Let Helmholtz free energy be $\psi(\rho, \theta) = E(\rho, \theta) - \theta\eta(\rho, \theta)$; then condition (1.2.7) and the generalized Gibb equation

$$\rho\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\rho\eta\frac{\mathrm{d}\theta}{\mathrm{d}t} - \overline{p}(\rho, \theta)\frac{\partial u_i}{\partial x_i}$$

are equivalent to equation (1.2.5), by equation (1.2.5), we know that $E$ and $\eta$ don't depend on the gradient of $u$ and $\theta$, and

$$\eta = -\frac{\partial\psi}{\partial\theta}, \quad \overline{p} = \rho^2\frac{\partial\psi}{\partial\rho}.$$

Assuming that the fluid in constant temperature, incompressible, Bellout, Bloom and Necas investigated three basic steady flows:

(1) plane Poiseuille flow between a fixed parallel plates;

(2) Poiseuille flow of cylindrical pipes;

(3) plane Couette flow above the moving plate with constant velocity. The constitutive equation is

$$\tau_{ij} = -\overline{p}\delta_{ij} + 2\mu_0(\varepsilon + |e|^2)^{\frac{p-2}{2}} e_{ij} - 2\mu_1 \Delta e_{ij} \tag{1.2.8}$$

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k} \tag{1.2.9}$$

where, $\tau_{ij}$ is the stress tensor component; $\tau_{ijk}$ is the first multipolar stress tensor; $e_{ij}$ is the rate of deformation tensor component, that is,

$$\mathbf{e} = (e_{ij}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

there $\varepsilon$, $\mu_0$, $\mu_1$ and $\rho$ are constitutive parameters. In addition, there is a multipolar stress tensor (which affects only the higher order boundary conditions), higher order velocity gradient and also involves nonlinear viscous:

$$\gamma(\mathbf{u}) = \mu_0(\varepsilon + |e|^2)^{\frac{p-2}{2}}.$$

So to obtain the isothermal, nonlinear bipolar incompressible viscous fluid mathematical model:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla(\overline{p}) + \nabla \cdot (2\gamma e) - 2\mu_1 \nabla \cdot (\Delta e) + \rho \mathbf{f} \tag{1.2.10}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.2.11}$$

In equation (1.2.8), if $p = 2$, $\mu_1 = 0$, the constitutive relation becomes

$$\tau_{ij} = -\overline{p}\delta_{ij} + 2\mu_0 e_{ij}. \tag{1.2.12}$$

This is the generalized Newton formula under the Stokes' assumption. Then equations (1.2.10)–(1.2.11) are

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \mu_0 \Delta \mathbf{u} = -\nabla(\overline{p}) + \rho \mathbf{f} \tag{1.2.13}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.2.14}$$

They are the Navier–Stokes equation and continuity equation of incompressible flow.

If $p = 2$, $\mu_1 > 0$, the constitutive relation equation (1.2.8) becomes

$$\tau_{ij} = -\overline{p}\delta_{ij} + 2\mu_0 e_{ij} - 2\mu_1 \Delta e_{ij}. \tag{1.2.15}$$

Equations (1.2.10)–(1.2.11) are regularization Navier–Stokes equations, which Ou and Sritharan [74] discussed:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(\overline{p}) + \mu_1\Delta^2\mathbf{u} - \mu_0\Delta\mathbf{u} = \rho\mathbf{f} \tag{1.2.16}$$

and

$$\nabla\cdot\mathbf{u} = 0. \tag{1.2.17}$$

If $p \neq 2$, $\varepsilon = 0$, $\mu_1 = 0$, the constitutive relation becomes

$$\tau_{ij} = -\overline{p}\delta_{ij} + 2\mu_0|e|^{p-2}e_{ij}. \tag{1.2.18}$$

Equations (1.2.10)–(1.2.11) are modified Navier–Stokes equations that Lions [59] investigated

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(\overline{p}) - 2\mu_0\frac{\partial}{\partial_j}(|e|^{p-2}e_{ij}) = \rho\mathbf{f} \tag{1.2.19}$$

and

$$\nabla\cdot\mathbf{u} = 0. \tag{1.2.20}$$

If $p \neq 2$, $\varepsilon \neq 0$, $\mu_1 = 0$, the constitutive relation becomes

$$\tau_{ij} = -\overline{p}\delta_{ij} + 2\mu_0(\varepsilon + |e|^2)^{\frac{p-2}{2}}e_{ij}. \tag{1.2.21}$$

Equations (1.2.10)–(1.2.11) are general monopolar fluid model equations

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(\overline{p}) - 2\mu_0\frac{\partial}{\partial_j}((\varepsilon + |\mathbf{e}|^2)^{\frac{p-2}{2}}e_{ij}) = \rho\mathbf{f} \tag{1.2.22}$$

and

$$\nabla\cdot\mathbf{u} = 0. \tag{1.2.23}$$

**Definition.** The incompressible fluid is claimed as a Newtonian fluid, if its behavior can be portrayed by the generalized Newton formula; Otherwise, it is called a non-Newtoian fluid; Furthermore, the bipolar incompressible fluid can be described by constitutive relation equations (1.2.8) with $p \neq 2$, $\varepsilon > 0$, $\mu_0 > 0$, $\mu_1 > 0$. The incompressible mono-polar fluid can be described by the constitutive relation equation (1.2.8) with $p \neq 2$, $\varepsilon > 0$, $\mu_0 > 0$, $\mu_1 = 0$.

## 1.3 Isothermal compressible viscous fluids models

Let $\Omega = \bigcup_{t=0}^{T} \Omega_t$, there $\Omega_t \subset R^n$ is the region that the substance at time $t \in [0, T)$, $T > 0$, and then we have the following equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \tag{1.3.1}$$

$$\frac{(\partial \rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial(T_{ij})}{\partial x_j} + \rho f_i, 1 \le i \le n \tag{1.3.2}$$

$$\frac{(\partial \rho e)}{\partial t} + \frac{\partial(\rho e u_j)}{\partial x_j} = \frac{\partial(T_{kj} u_k)}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho r + \rho f_j u_j \tag{1.3.3}$$

represent the local form of the conservation of mass, the law of conservation of momentum and the law of conservation of energy, respectively. There $\rho$ is density; **u** is velocity field; $E$ is material energy per unit mass; **T** is symmetric stress tensor; **q** is heat flux vector; $r$ is heat exchange with the outside world by radiation; $f$ is outer physical per unit mass and $e$ is $E + \frac{|\mathbf{u}|^2}{2}$. Here, all variable assignments are in $(x, t) \in \Omega_t \times [0, T)$. In the following, for all $t \ge 0$, consider $\Omega_t = \Omega$.

$p$ is pressure; $\theta$ is temperature; $\eta$ is the entropy of fluid; $E$ is internal energy. Because these variables describe the fluid thermodynamic state and only two are independent, if we give $\rho$, $\theta$ as independent variables, then

$$p = p(\rho, \theta) \tag{1.3.4}$$

$$E = E(\rho, \theta) \tag{1.3.5}$$

$$\eta = \eta(\rho, \theta) \tag{1.3.6}$$

$$e = e(\rho, \theta, u). \tag{1.3.7}$$

If we consider that the fluid viscous effect is quite large, the stress tensor $T$ is not only dependent on pressure $p$, but also with other relevant amount, such as $\nabla \mathbf{u}$, $\nabla \boldsymbol{\theta}$, then

$$\mathbf{T} = \hat{\mathbf{T}}(\rho, \theta, \nabla\theta, \nabla\mathbf{u}).$$

If the fluid movement is isothermal, namely the temperature $\theta = \theta_0 > 0$ is constant, the tensor function $\hat{\mathbf{T}}$ does not rely on $\theta$ and $\nabla\theta$, so

$$\mathbf{T} = \hat{\mathbf{T}}(\rho, \nabla\mathbf{u}). \tag{1.3.8}$$

Thus, equations (1.3.1), (1.3.2) are not coupled with equation (1.3.3) and can be considered independent. In other words, if we determine $\rho$, **u** from equations (1.3.1), (1.3.2), then we can calculate the rest of the thermodynamic variables from equation (1.3.3). Taking into account the principle of material frame indifference, equation (1.3.8) can be simplified

$$\mathbf{T} = -p(\rho)\mathbf{I} + \hat{\mathbf{T}}(\rho, \mathbf{e}) \tag{1.3.9}$$

where $2\mathbf{e} = 2\mathbf{e}(\mathbf{u}) \equiv \nabla\mathbf{u} + (\nabla\mathbf{u})^T$ is a symmetric part of the velocity gradient $\nabla\mathbf{u}$.

Then consider the special case of equation (1.3.9)

$$\mathbf{T} = -p(\rho)\mathbf{I} + \tau^E. \tag{1.3.10}$$

$\tau^E$ is given by

$$\tau^E = \tau(\mathbf{e}) \tag{1.3.11}$$

, where $\tau : R^{n^2}_{\text{sym}} \to R^{n^2}_{\text{sym}}$ is given a continuous function, and $R^{n^2}_{\text{sym}} \equiv \{M \in R^n \times R^n; M_{ij} = M_{ji}, i, j = 1, 2, \ldots, n\}$. If we consider that the gas complies with the state equation

$$p = R\rho\theta, \tag{1.3.12}$$

$R$ is universal gas constant. In the case of isothermal, the pressure is a linear function of density $\rho$, namely

$$p(\rho) = \beta\rho, \quad \beta = R\theta_0 > 0. \tag{1.3.13}$$

Under the above assumptions of equations (1.3.10)–(1.3.13), equations (1.3.1)–(1.3.2) become

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \tag{1.3.14}$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\beta\frac{\partial\rho}{\partial x_i} + \frac{\partial\tau_{ij}(\mathbf{e})}{\partial x_j} + \rho f_i, \quad i = 1, \ldots, n. \tag{1.3.15}$$

The left-hand side of equation (1.3.15) is $\rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial(u_i)}{\partial x_j}$. If $\tau^E = \tau(\mathbf{e})$ is not a linear function of $\mathbf{e}$, equations (1.3.14)–(1.3.15) are compressible non-Newtonian fluid mathematical model equations. In order to facilitate and establish global solution theory, we assume that there exists constants $C_1 > 0$, $C_2 > 0$, and parameters $p > 1$ and $q \in [p-1, p)$ make for all $\eta \in R^{n^2}_{\text{sym}}$ and satisfy the $p$-mandatory condition

$$\tau(\eta) \cdot \eta \geq C_1|\eta|^p \tag{1.3.16}$$

and $q$-growth conditions

$$|\tau(\eta)| \geq C_2(1 + |\eta|)^q \tag{1.3.17}$$

where $|\eta| = (\eta_{ij}\eta_{ij})^{1/2}$ and for $\tau, \eta \in R^{n^2}_{\text{sym}}$, $\tau \cdot \eta = \tau_{ij}\eta_{ij}$.

## 1.4 Other related models

For nonlinear isothermal incompressible non-Newtonian fluid, from Section 1.3 it is known

$$\mathbf{T} = -p\mathbf{I} + \tau^E \tag{1.4.1}$$