

Credibility Measure Theory

A Modern Methodology of Handling
Subjective Uncertainty

可信性测度论

——处理主观不确定性的现代方法论

Yan-Kui Liu

(刘彦奎)

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Preface

Since credibility measure was first introduced in 2002, the theory of credibility measure has grown significantly. The large number of research papers based on credibility measure theory is a testimony to the vitality of this modern mathematical discipline both from a theoretical perspective and in terms of practical applications. This book is the first monograph to systematically reflect the up-to-date state of credibility measure theory developed in recent fifteen years.

The book is orientated for two kinds of credibilists. One of them is the mathematician who views credibility measure theory as a purely mathematical discipline, such as measure theory, probability theory, stochastic processes. The other kind views credibility measure theory as the mathematical modeling of subjective uncertain phenomena in human decision processes. That is, the second kind of credibilists have a view toward practical applications, which aims at finding methods and criteria to analyze the uncertain data from the real world, with the ultimate goal of making informed decisions. As a consequence, the material of this book is suitable not only for programs in mathematics, but also for programs in other disciplines, including computer science, engineering, operations research and management science, business economics, mechanics and automation control systems.

To understand the presented materials, a solid background in calculus is required, and the book is otherwise self-contained. Knowledge of generalized measure theory is helpful but not essential. Some relevant concepts in set theory are introduced at the beginning of the first chapter.

This book consists of Chapters 1-11 to present the theory of credibility measure.

In Chapter 1, an important self-dual set function referred to as *credibility measure* is introduced. The basic properties of credibility measure are discussed. The one-to-one correspondence is established between credibility measures and credibility distribution functions on Euclidean space.

In Chapter 2, the product credibility measures are constructed via extension prin-

ciples, including finite-dimensional product credibility measure, countably infinite-dimensional product credibility measure, and arbitrarily infinite-dimensional product credibility measure.

In Chapter 3, the convergence modes are discussed for sequences of fuzzy variables. The convergence relations and convergence criteria are established. Furthermore, the convergence modes are applied to the research field of approximation schemes for fuzzy variables with continuous credibility distribution functions. Based on approximation method, the techniques of both credibility simulation and critical value simulation are addressed in this chapter.

In Chapter 4, the independence of fuzzy variables and fuzzy events are characterized by credibility measure. Several equivalent characterizations are established for the independence of fuzzy vectors and fuzzy events.

In Chapter 5, the numerical characteristics of fuzzy variables are presented, including expected value, variance, spread and moments. Two types of dominated convergence theorems are established for sequences of integrable fuzzy variables. Based on convergence modes, the technique of expected value simulation is discussed in details.

In Chapter 6, the properties of uniform integrability and absolute continuity of fuzzy variables are studied, and the completeness of L_r space is discussed.

In Chapter 7, the credibilistic orders are introduced to compare the location or the magnitude of fuzzy variables. Three useful credibilistic orders are presented in this chapter, they are usual credibilistic order, credibilistic stop-loss order and credibilistic convex order.

In Chapter 8, credibilistic risk measures are studied based on the quantile function of fuzzy variable, including credibilistic value at risk, and credibilistic conditional value at risk.

In Chapter 9, the credibilistic comonotonicity of fuzzy vector is discussed, and several characterizations are established for the comonotonic fuzzy vector.

In Chapter 10, the credibilistic laws are discussed for sequences of fuzzy variables, and a strong law of large numbers is established for a sequence of mutually \top -independent fuzzy variables.

In Chapter 11, the credibilistic processes are studied in fuzzy decision systems,

including credibilistic renewal process and credibilistic renewal reward process.

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Symbols and Abbreviations

\mathcal{A}	ample field
CDF	credibility distribution function
CVaR	conditional value at risk
E	expected value operator
E^*	upper expected value operator
E_*	lower expected value operator
EAD	expected absolute deviation
\mathcal{F}	σ -algebra
K_k	kurtosis operator
\mathfrak{R}^n	n -dimensional Euclidean space
\vee	maximum operator
\wedge	minimum operator
Var	variance operator
Sp	spread operator
S_k	skewness operator
M	moment operator
VaR	value at risk
ρ_{EAD}	expected absolute deviation operator
Γ	abstract space of generic element γ
$\mathcal{P}(\Gamma)$	power set on Γ
\triangleq	stands for “equality by definition”
w.r.t.	with respect to

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Chapter 1

Credibility Measure Space

The intent of this chapter is to introduce an important generalized measure called *credibility measure*, which is defined on an ample field. The structural characteristics of credibility measure are discussed. Fuzzy vector is defined as a measurable function on an ample space, and characterized by a credibility distribution function. The one-to-one correspondence between credibility measures and credibility distribution functions will be established.

1.1 Ample Space

In what follows, let Γ be an abstract space of generic element γ . A class \mathcal{F} on Γ is called a σ -algebra if it satisfies the following conditions:

- ($\sigma 1$) $\Gamma \in \mathcal{F}$;
- ($\sigma 2$) $A \in \mathcal{F} \implies \Gamma \setminus A \in \mathcal{F}$;
- ($\sigma 3$) $A_n \in \mathcal{F}, n = 1, 2, \dots \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

In this chapter, we pay more attention on another class called *ample field* on the abstract space Γ . The term “ample field” was first introduced by Wang^[224], then it was studied by Cooman^[26], and Cooman and Kerre^[27] in a general framework.

Definition 1.1^[224] An ample field \mathcal{A} is a collection of subsets of Γ and satisfies the following conditions:

- (A1) $\Gamma \in \mathcal{A}$;
- (A2) $A \in \mathcal{A} \implies \Gamma \setminus A \in \mathcal{A}$;
- (A3) $A_i \in \mathcal{A}, i \in I \implies \bigcup_{i \in I} A_i \in \mathcal{A}$, where I is an arbitrary index set.

It is evident that an ample field is a σ -algebra, but the converse is not true. The pair (Γ, \mathcal{A}) is called an *ample space*. A special ample field on Γ is the power set $\mathcal{P}(\Gamma)$, thus $(\Gamma, \mathcal{P}(\Gamma))$ is a special ample space.

The *atom* $[\gamma]_{\mathcal{A}}$ of \mathcal{A} containing the element γ of Γ is defined by

$$[\gamma]_{\mathcal{A}} \triangleq \cap \{A \mid \gamma \in A, A \in \mathcal{A}\}.$$

In addition, if $A \in \mathcal{A}$, then it is a union of atoms in \mathcal{A} , i.e.,

$$A = \bigcup_{\gamma \in A} [\gamma]_{\mathcal{A}}.$$

Let \mathcal{C} be a subclass of $\mathcal{P}(\Gamma)$. Since the intersection of any family of ample fields is still an ample field, the smallest ample field generated by \mathcal{C} is denoted by

$$\mathcal{A}(\mathcal{C}) \triangleq \cap \{\mathcal{A} \mid \mathcal{C} \subseteq \mathcal{A} \text{ (ample field)}\}.$$

It is easy to check that the class $\mathcal{A}(\mathcal{C})$ has the following properties:

- (i) $\mathcal{A}(\mathcal{C}) \supseteq \mathcal{C}$;
- (ii) For any ample field \mathcal{A}' , one has

$$\mathcal{A}' \supseteq \mathcal{C} \implies \mathcal{A}' \supseteq \mathcal{A}(\mathcal{C}).$$

1.2 Fuzzy Vector

Let ξ be a function from Γ to \mathfrak{R}^n . For every $B \subseteq \mathfrak{R}^n$, the following set

$$\xi^{-1}B = \{\gamma \mid \xi(\gamma) \in B\}$$

is called the *inverse image* of the set B . Let \mathcal{C} be a class of \mathfrak{R}^n . Then the following set class

$$\xi^{-1}\mathcal{C} = \{\xi^{-1}B \mid B \in \mathcal{C}\}$$

is called the *inverse image* of the class \mathcal{C} .

Proposition 1.1 *Let ξ be a function from Γ to \mathfrak{R}^n , and \mathcal{C} be an ample field of \mathfrak{R}^n . Then $\xi^{-1}\mathcal{C}$ is an ample field of Γ .*

Proof By the following relations

$$\Gamma = \xi^{-1}\mathfrak{R}^n, \quad (\xi^{-1}A)^c = \xi^{-1}A^c, \quad \text{and} \quad \bigcup_{t \in T} \xi^{-1}A_t = \xi^{-1} \bigcup_{t \in T} A_t,$$

one knows that $\xi^{-1}\mathcal{C}$ is an ample field. The proof of proposition is complete. \square

Proposition 1.2 *For any class \mathcal{C} of \mathfrak{R}^n , one has*

$$\mathcal{A}(\xi^{-1}\mathcal{C}) = \xi^{-1}(\mathcal{A}(\mathcal{C})).$$

Proof By Proposition 1.1, the class $\xi^{-1}(\mathcal{A}(\mathcal{C}))$ is an ample field, so one has

$$\mathcal{A}(\xi^{-1}\mathcal{C}) \subseteq \xi^{-1}(\mathcal{A}(\mathcal{C})).$$

On the other hand, let

$$\mathcal{A}' = \{B \subseteq \mathbb{R}^n \mid \xi^{-1}B \in \mathcal{A}(\xi^{-1}\mathcal{C})\}.$$

Then \mathcal{A}' is an ample field of \mathbb{R}^n , and $\mathcal{C} \subseteq \mathcal{A}'$. Therefore $\mathcal{A}(\mathcal{C}) \subseteq \mathcal{A}'$, which implies the following relation holds

$$\xi^{-1}(\mathcal{A}(\mathcal{C})) \subseteq \mathcal{A}(\xi^{-1}\mathcal{C}).$$

The proof of proposition is complete. \square

Fuzzy variable was first presented by Kaumann^[77], then it appeared in Zadeh^[263,264]. The axiomatic definition of fuzzy variable was defined by Nahmias^[157] on a *pattern space*, which was generalized by Wang^[224] to an ample space.

Fuzzy vector is a natural extension of fuzzy variable, and it is formally stated as follows:

Definition 1.2 Let (Γ, \mathcal{A}) be an ample space. A function $\xi : \Gamma \rightarrow \mathbb{R}^n$ is called a fuzzy vector if

$$\{\gamma \in \Gamma \mid \xi(\gamma) = t\} \in \mathcal{A}, \quad (1.1)$$

for every $t \in \mathbb{R}^n$.

Particularly, when $n = 1$, ξ is called a fuzzy variable, and usually denoted as ξ .

Theorem 1.1 Let (Γ, \mathcal{A}) be an ample space. A function $\xi : \Gamma \rightarrow \mathbb{R}^n$ is a fuzzy vector if and only if

$$\xi^{-1}\mathcal{P}(\mathbb{R}^n) \subseteq \mathcal{A}.$$

Proof Let

$$\mathcal{C} = \{(-\infty, t] \mid t \in \mathbb{R}^n\}, \quad (-\infty, t] = (-\infty, t_1] \times \cdots \times (-\infty, t_n].$$

Then $\mathcal{A}(\mathcal{C}) = \mathcal{P}(\mathbb{R}^n)$. According to Definition 1.2, $\xi^{-1}\mathcal{C} \subseteq \mathcal{A}$. By Proposition 1.2, one has

$$\mathcal{A}(\xi^{-1}\mathcal{C}) = \xi^{-1}\mathcal{P}(\mathbb{R}^n).$$

Therefore, the following relation

$$\xi^{-1}\mathcal{P}(\mathbb{R}^n) \subseteq \mathcal{A}$$

holds true. The proof of proposition is complete. \square

The class $\xi^{-1}\mathcal{P}(\mathfrak{R}^n)$ is the smallest ample field in which ξ is measurable, and denoted as $\mathcal{A}(\xi)$. Generally, $\xi^{-1}\mathcal{P}(\mathfrak{R}^n) \neq \mathcal{P}(\Gamma)$. In the following, an example is given to illustrate this fact.

Example 1.1 Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$, $\mathcal{A} = \mathcal{P}(\Gamma)$. Define a discrete fuzzy variable ξ on Γ as follows

$$\xi(\gamma_1) = \xi(\gamma_2) = 0, \quad \xi(\gamma_3) = 1, \quad \text{and} \quad \xi(\gamma_4) = \xi(\gamma_5) = 2.$$

Let $B \in \mathcal{P}(\mathfrak{R})$.

If $\{0, 1, 2\} \cap B = \emptyset$, then $\xi^{-1}B = \emptyset$;

If $\{0, 1, 2\} \subseteq B$, then $\xi^{-1}B = \Gamma$;

If $0 \in B, 1 \notin B, 2 \notin B$, then $\xi^{-1}B = \{\gamma_1, \gamma_2\}$;

If $0 \notin B, 1 \in B, 2 \notin B$, then $\xi^{-1}B = \{\gamma_3\}$;

If $0 \notin B, 1 \notin B, 2 \in B$, then $\xi^{-1}B = \{\gamma_4, \gamma_5\}$;

If $0 \in B, 1 \in B, 2 \notin B$, then $\xi^{-1}B = \{\gamma_1, \gamma_2, \gamma_3\}$;

If $0 \in B, 1 \notin B, 2 \in B$, then $\xi^{-1}B = \{\gamma_1, \gamma_2, \gamma_4, \gamma_5\}$;

If $0 \notin B, 1 \in B, 2 \in B$, then $\xi^{-1}B = \{\gamma_3, \gamma_4, \gamma_5\}$.

Therefore, one has

$$\mathcal{A}(\xi) = \{\emptyset, \Gamma, \{\gamma_1, \gamma_2\}, \{\gamma_3\}, \{\gamma_4, \gamma_5\}, \{\gamma_1, \gamma_2, \gamma_3\}, \{\gamma_1, \gamma_2, \gamma_4, \gamma_5\}, \{\gamma_3, \gamma_4, \gamma_5\}\}.$$

It is evident that $\{\gamma_1\} \in \mathcal{P}(\Gamma)$, but $\{\gamma_1\} \notin \mathcal{A}(\xi)$.

Theorem 1.2 Let ξ be an n -ary fuzzy vector. If f is a vector-valued function from \mathfrak{R}^n to \mathfrak{R}^m , then $f(\xi)$ is an m -ary fuzzy vector.

Proof Let $B \subseteq \mathfrak{R}^m$. Then one has $f^{-1}B \subseteq \mathfrak{R}^n$. It follows from Theorem 1.1 that

$$f(\xi)^{-1}B = \xi^{-1}(f^{-1}B) \in \mathcal{A},$$

that is, $f(\xi)$ is a fuzzy vector. The proof of theorem is complete. \square

1.3 Credibility Measure

Possibility and necessity measures are a pair of dual nonadditive set functions, it is well-known in the literature^[224, 264]. We next recall their definitions.

Definition 1.3 Let (Γ, \mathcal{A}) be an ample space. A set function Pos defined on the ample field \mathcal{A} is called a possibility measure if it satisfies

(P1) $\text{Pos}(\Gamma) = 1$, and $\text{Pos}(\emptyset) = 0$;

(P2) $\text{Pos}\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \text{Pos}(A_i)$ for any subclass $\{A_i \mid i \in I\}$ of \mathcal{A} .

A possibility measure Pos has the following properties:

(i) *Monotonicity*: $A \subseteq B$ implies $\text{Pos}(A) \leq \text{Pos}(B)$;

(ii) *Continuity from below*: for any nondecreasing sequence $\{A_n\} \subseteq \mathcal{A}$, one has

$$\lim_{n \rightarrow \infty} \text{Pos}(A_n) = \text{Pos}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

In addition, a possibility measure Pos has the following *strong subadditive property*.

Proposition 1.3 For any events $A, B \in \mathcal{A}$, one has

$$\text{Pos}(A \cup B) + \text{Pos}(A \cap B) \leq \text{Pos}(A) + \text{Pos}(B).$$

Proof Let $A, B \in \mathcal{A}$. If $\text{Pos}(A) \leq \text{Pos}(B)$, then one has $\text{Pos}(A \cup B) = \text{Pos}(B)$, which implies

$$\text{Pos}(A \cup B) + \text{Pos}(A \cap B) = \text{Pos}(B) + \text{Pos}(A \cap B) \leq \text{Pos}(B) + \text{Pos}(A).$$

On the other hand, if $\text{Pos}(A) \geq \text{Pos}(B)$, then one has $\text{Pos}(A \cup B) = \text{Pos}(A)$, which implies

$$\text{Pos}(A \cup B) + \text{Pos}(A \cap B) = \text{Pos}(A) + \text{Pos}(A \cap B) \leq \text{Pos}(A) + \text{Pos}(B).$$

The proof of proposition is complete. \square

Based on possibility measure, the set function Nec defined by

$$\text{Nec}(A) \triangleq 1 - \text{Pos}(A^c), \quad A \in \mathcal{A}$$

is called a *necessity measure*, where A^c is the set complement of A . Therefore Pos and Nec are a pair of dual set functions. The necessity measure Nec has the following properties.

(N1) $\text{Nec}(\Gamma) = 1$, and $\text{Nec}(\emptyset) = 0$;

(N2) $\text{Nec}\left(\bigcap_{i \in I} A_i\right) = \inf_{i \in I} \text{Nec}(A_i)$ for any subclass $\{A_i \mid i \in I\}$ of \mathcal{A} ;

(N3) *Monotonicity*: $A \subseteq B$ implies $\text{Nec}(A) \leq \text{Nec}(B)$;

(N4) *Continuity from above*: for any nonincreasing sequence $\{A_n\} \subseteq \mathcal{A}$, one has

$$\lim_{n \rightarrow \infty} \text{Nec}(A_n) = \text{Nec} \left(\bigcap_{n=1}^{\infty} A_n \right).$$

In addition, a necessary measure Nec has the following *weak superadditive property*.

Proposition 1.4 For any events $A, B \in \mathcal{A}$, one has

$$\text{Nec}(A \cup B) + \text{Nec}(A \cap B) \geq \text{Nec}(A) + \text{Nec}(B).$$

Proof The following inequality holds

$$\text{Nec}(A \cup B) + \text{Nec}(A \cap B) \geq \text{Nec}(A) + \text{Nec}(B)$$

if and only if the following inequality holds

$$\text{Pos}(A^c) + \text{Pos}(B^c) \geq \text{Pos}(A^c \cup B^c) + \text{Pos}(A^c \cap B^c),$$

which is valid by Proposition 1.3. □

More importantly, a self-dual set function, called *credibility measure* Cr, is defined as follows.

Definition 1.4 ^[100] If Pos is a possibility measure on an ample space (Γ, \mathcal{A}) , then the set function Cr defined by

$$\text{Cr}(A) \triangleq \frac{1}{2} (1 + \text{Pos}(A) - \text{Pos}(A^c)), \quad A \in \mathcal{A} \quad (1.2)$$

is called a credibility measure on the ample space (Γ, \mathcal{A}) .

The triplet $(\Gamma, \mathcal{A}, \text{Cr})$ is referred to as a *credibility measure space*. To avoid confusion by the word *fuzzy*, a fuzzy vector/variable defined on a credibility measure space $(\Gamma, \mathcal{A}, \text{Cr})$ may also be called a *credibilistic vector / variable*.

A credibility measure Cr has the following fundamental properties:

Proposition 1.5 A credibility measure Cr on an ample space (Γ, \mathcal{A}) has the following fundamental properties:

(C1) *Normality*: $\text{Cr}(\Gamma) = 1$;

(C2) *Monotonicity*: $\text{Cr}(A) \leq \text{Cr}(B)$ whenever $A, B \in \mathcal{A}$ and $A \subseteq B$;

(C3) *Self-duality*: $\text{Cr}(A) + \text{Cr}(A^c) = 1$ for any event $A \in \mathcal{A}$;

(C4) *Credibilistic Arithmetic Law*: for any family $\{A_i, i \in I\} \subseteq \mathcal{A}$ with $\text{Cr}(A_i) \leq 1/2$, one has

$$\text{Cr}\left(\bigcup_{i \in I} A_i\right) \wedge \frac{1}{2} = \sup_{i \in I} \text{Cr}(A_i).$$

Proof First, by the definition of credibility measure, one has

$$\text{Cr}(\Gamma) = \frac{1}{2}(1 + \text{Pos}(\Gamma) - \text{Pos}(\emptyset)) = 1,$$

which proves (C1).

Second, if $A, B \in \mathcal{A}$ and $A \subseteq B$, one has

$$\text{Cr}(A) = \frac{1}{2}(1 + \text{Pos}(A) - \text{Pos}(A^c)) \leq \frac{1}{2}(1 + \text{Pos}(B) - \text{Pos}(B^c)) = \text{Cr}(B),$$

which proves (C2). It follows from the following two equalities

$$\text{Cr}(A) = \frac{1}{2}(1 + \text{Pos}(A) - \text{Pos}(A^c)),$$

$$\text{Cr}(A^c) = \frac{1}{2}(1 + \text{Pos}(A^c) - \text{Pos}(A)),$$

we know that property (C3) is valid.

Finally, we prove property (C4). For any family $\{A_i, i \in I\} \subseteq \mathcal{A}$ with $\text{Cr}(A_i) \leq 1/2$, one has

$$\text{Pos}(A_i) = \min\{2\text{Cr}(A_i), 1\} = 2\text{Cr}(A_i)$$

for any $i \in I$. By the following possibilistic arithmetic law

$$\text{Pos}\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \text{Pos}(A_i),$$

one has

$$\min\left\{2\text{Cr}\left(\bigcup_{i \in I} A_i\right), 1\right\} = \sup_{i \in I} \min\{2\text{Cr}(A_i), 1\} = 2 \sup_{i \in I} \text{Cr}(A_i),$$

which proves the validness of property (C4). \square

In addition, a credibility measure has the following two types of subadditive properties.

Proposition 1.6 *Let Cr be a credibility measure on an ample space (Γ, \mathcal{A}) .*

(i) *Finite Subadditivity*: for any $A, B \in \mathcal{A}$, one has

$$\text{Cr}(A \cup B) \leq \text{Cr}(A) + \text{Cr}(B);$$