

美国数学会经典影印系列

 **AMS**
AMERICAN MATHEMATICAL SOCIETY

Algebraic Curves and Riemann Surfaces

代数曲线和黎曼面

Rick Miranda



高等教育出版社

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于1888年，是国际上极具影响力的专业学术组织，目前拥有近30000会员和580余个机构成员，出版图书3500多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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2016年12月

To Jeanne

Preface

This text has evolved from lecture notes for a one-semester course which I have taught 5 times in the last 8 years as an introduction to the ideas of algebraic geometry using the theory of algebraic curves as a foundation.

There are two broad aims for the book: to keep the prerequisites to a bare minimum while still treating the major theorems seriously; and to begin to convey to the reader some of the language of modern algebraic geometry.

In order to present the material of Algebraic Curves to an initially relatively unsophisticated audience I have taken the approach that Algebraic Curves are best encountered for the first time over the complex numbers. Therefore the book starts out as a primer on Riemann surfaces, with complex charts and meromorphic functions taking center stage. In particular, one semester of graduate complex analysis should be sufficient preparation, and it is not assumed that the reader has any serious background in either algebraic topology or commutative algebra. But I try to stress that the main examples (from the point of view of algebraic geometry) come from projective curves, and slowly but surely the text evolves to the algebraic category, culminating in an algebraic proof of the Riemann-Roch theorem. After returning to the analytic side of things for Abel's theorem, the progression is repeated again when sheaves and cohomology are discussed: first the analytic, then the algebraic category.

The proof of Riemann-Roch presented here is an adaptation of the adelic proof, expressed completely in terms of solving a Mittag-Leffler problem. This is a very concrete approach, and in particular no cohomology or sheaf theory is used. However, cohomology groups clandestinely appear (as obstruction spaces to solving Mittag-Leffler problems), motivating their explicit introduction later on.

The other goal is to begin to convey, as much as possible, the language of modern algebraic geometry to the student. This language is that of rational functions, divisors, bundles, sheaves, cohomology, and the Zariski topology, to name some of the highlights presented here. I hope that a student who has read the later chapters of this book will be prepared to understand at least the first few minutes of a modern colloquium talk discussing algebraic curves and

algebraic geometry. I consider the treatment of sheaves and cohomology given here to be rather gentle; for example, most of the sheaves which are used as the initial examples were introduced in a natural way much earlier in the text. Hence by the time a sheaf is even defined the reader will actually have a decent understanding of what the technicalities entail. In addition, the zero-th and first cohomology groups will already have been seen in the proof of the Riemann-Roch theorem (without them being called that of course).

The first three chapters are introductory, discussing the basic definitions of Riemann surfaces and holomorphic maps between them. Of the 12 sections in these chapters, 5 are devoted entirely to examples of one sort or another. The main theorems here are that the sum of the orders of a meromorphic function on a compact Riemann surface is zero, and Hurwitz's Formula relating the genera of compact Riemann surfaces given a map between them. The fourth chapter on integration is meant to get to the Residue Theorem in a direct manner.

Chapters 5-8 form the technical heart of the text. Divisors and how they are used to organize forms, functions, and maps are introduced in Chapter 5, and in Chapter 6 the Riemann-Roch Theorem and Serre Duality are proved, after introducing the concept of an algebraic curve, which is defined here as a compact Riemann surface whose field of global meromorphic functions separates points and tangents. Chapter 7 is devoted to applications of Riemann-Roch. Here is found the classification of curves of low genus, Clifford's Theorem, the analysis of the canonical map, and Riemann's count of $3g - 3$ parameters for curves of genus at least two. A section on the degree of a projective curve culminates in Castelnuovo's bound on the genus. It is here most of all that the reader will feel an urge to learn more algebraic geometry, and in particular some higher-dimensional theory. The final section concerns inflection points of linear systems and Weierstrass points in particular. In Chapter 8 Abel's Theorem is proved; along the way the algebraic proof of the Residue Theorem is indicated. The final section discusses the group law on a smooth cubic curve.

The last three chapters introduce sheaves and Čech cohomology. Initially the classical topology is used, focusing in on the standard sheaves of holomorphic and meromorphic functions and forms. The Zariski topology and the algebraic sheaves are brought into the picture next, and the obstruction space for solving a Mittag-Leffler problem (seen in the proof of the Riemann-Roch Theorem) is here realized as an \check{H}^1 of an algebraic sheaf.

The last chapter is organized around the Picard group of an algebraic curve and its many manifestations: as the group of divisors modulo linear equivalence, as the group of line bundles modulo isomorphism, as the group of invertible sheaves modulo isomorphism, as the first cohomology group with values in the nowhere zero regular functions, and as the Jacobian (extended by \mathbb{Z}). Here there is an opportunity to explain why \check{H}^1 is useful to classify locally trivial objects in general, and the text closes with first-order deformations, with Riemann's count of $3g - 3$ parameters enjoying a reprise.

At the end of each chapter I have included some suggestions for further reading. These are not meant to be completely comprehensive, but simply indicate some of the sources that I am aware of which I have found illuminating.

I would like to thank Bruce Crauder, David Hahn, Luisa Paoluzzi, John Symms, and Caryn Werner for commenting on various sections; also I am greatly indebted to Ciro Ciliberto and Peter Stiller who each read through substantial portions of the text and offered many valuable suggestions.

It has been my great privilege to have been given the opportunity to study algebraic geometry in my professional life. There is no doubt that the theory of algebraic curves is the richest and deepest of the field's various roots, and I hope I have conveyed some of the special pleasure obtained in visiting this material, which serves simultaneously as one of the great jewels of classical mathematics and one of the most vital areas of modern research.

Rick Miranda
October 1994
Fort Collins, Colorado

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