

**ALM 45**

Advanced Lectures in Mathematics

# **Tsinghua Lectures in Mathematics**

清华数学讲义

Editors: Lizhen Ji • Yat-Sun Poon • Shing-Tung Yau



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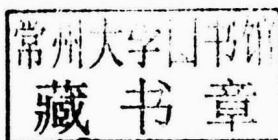
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# Preface

Since it was established in 2009, the Yau Mathematical Sciences Center has become a leading mathematics platform for students and mathematicians to learn mathematics and exchange their ideas. A closely related facility is the Tsinghua Sanya International Mathematics Forum.

There have been many lectures and talks by distinguished mathematicians at both places. This book contains a selection of lecture notes of these lectures and survey papers based on some talks. Mathematics is very diversified and broad, but there are some basic underlying themes. We hope that the writings in this book convey both the richness and unity of mathematics.

Lizhen Ji  
Yat-Sun Poon  
Shing-Tung Yau

October 2017

ADVANCED LECTURES IN MATHEMATICS

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# Part I Course Notes



# Lectures on the Geometry of the Scalar Curvature

Vlad Moraru\*

## Acknowledgements

These lecture notes are an extended version of the course I gave in the Spring of 2014 at the Yau Mathematical Science Center of Tsinghua University.

The main purposes of this course was to present the relationship between minimal hypersurfaces and the scalar curvature of the ambient manifold, and the role this connection plays in general relativity. The novelty here consists mainly in the exposition, the results being presented from the point of view of comparison geometry. But the lectures also contain a few results and remarks, perhaps known to the community, which, to the best of my knowledge, have not yet appeared in print.

These notes are roughly divided in two parts:

- (1) The first one forms the major part of the course. In it, I explain how the sign of the scalar curvature of an  $n$ -manifold  $M$  has consequences on both the geometry and the topology of stable minimal hypersurfaces contained in  $M$ . Moreover, I explain how the presence of certain area minimising hypersurfaces forces some rigidity on such manifolds.
- (2) In the second part I show how some of the ideas and results from the first part play a crucial role in the proof of the Positive Mass Theorem in general relativity.

The course was delivered to a mixture of undergraduate-, graduate- and post-graduate students having different background. Therefore, as much as the topics

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\*Yau Mathematical Science Center, Tsinghua University, Beijing. E-mail: vlad@math.tsinghua.edu.cn

allowed, I tried to assume as background only a basic course in Riemannian geometry and PDE's, as well as some basic concepts from general relativity. Emphasis was placed on the geometric side of the results rather than the analytic one. This, of course, has effected my choice of material. I therefore have suppressed the proofs of some of the analytic issues, yet I have provided proofs for several well known facts for which I have found no relevant or satisfactory reference.

Readers are invited to send comments and corrections to:

vlad@math.tsinghua.edu.cn

## 1 The geometric meaning of curvature

Let  $(M, g)$  be an  $n$ -dimensional, smooth, Riemannian manifold. When there will be no danger of confusion we will write  $M$  instead of  $(M, g)$ . We will denote by  $\nabla$  the Levi-Civita connection associated to  $g$ . If  $p \in M$  then  $T_pM$  will be the tangent space of  $M$  at  $p$ .

### 1.1 The Riemannian curvature tensor

If  $X, Y, Z, W$  are smooth tangent vector fields in a neighbourhood of  $p$ , then the  $(1, 3)$ -Riemannian curvature tensor is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where  $[X, Y] := \nabla_X Y - \nabla_Y X$  is the Lie bracket. The  $(0, 4)$ -Riemannian curvature tensor is given by

$$R(X, Y, Z, W) := g(R(X, Y)Z, W).$$

A Riemannian manifold is called *flat* if  $R$  vanishes identically.

The geometric meaning of the Riemannian curvature tensor is that it measures the failure of small geodesic parallelograms to close. For a detailed and very clear description see [54, Chapters 10–11].

**Remark 1.1.** Throughout the literature the definition of the curvature tensor sometimes differs by a sign. This is the case, for example, of [24] and [57], where  $R(X, Y)Z := \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z - \nabla_X \nabla_Y Z$ . However, with either of the two definitions, the geometric meaning is not altered.

In practice, the full curvature tensor is not very easy to handle. A more useful concept of curvature (and perhaps a more natural one from a geometric point of view) is the *sectional curvature*.

### 1.2 The sectional curvature

Let  $\sigma \in T_pM$  be a two dimensional subspace of  $T_pM$  and let  $u, v \in \sigma$  be two linear independent vectors. We define

$$K(u, v) := \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2},$$

which is called the *sectional curvature* of  $M$  at  $p$  for the section  $(u, v)$ . If  $\{e_i\}_{i=1, \dots, n}$  is an orthonormal basis of  $T_p M$  then

$$K(e_i, e_j) = K_{ij}$$

for all  $i \neq j$ .

**Proposition 1.2.** *The sectional curvatures of  $M$  at  $p$  completely determines the full curvature tensor.*

*Proof.* See [24, Corollary 3.5] for an abstract proof and [23] for an explicit formula on how to recover the full curvature tensor from the sectional curvatures.  $\square$

**Remark 1.3.** What the previous proposition is saying is that, from information point of view, there is no difference in knowing all sectional curvatures at  $p$  or the Riemannian curvature tensor. The former one, however, turns out to be more easily to handle than the later one.

*What is the geometric meaning of the sectional curvature?*

When  $n = 2$  there is only one tangent plane at  $p \in M$  which coincides with the tangent space of  $M$  at  $p$ . Hence the sectional curvature for this plane reduces to the Gauss curvature. For  $n \geq 3$  there is an useful geometric interpretation of the sectional curvature in terms of spreading of geodesics. If  $J(t)$  is variation vector of a variation through geodesics (i.e. a Jacobi vector field, which measures how geodesics emanating from  $p \in M$  diverge or converge) then one can prove that the length of  $J(t)$  has the following Taylor expansion:

$$\|J(t)\|^2 = t^2 - \frac{1}{3}K(X(0), J(0))t^4 + \mathcal{O}(t^5), \quad (1.1)$$

where  $X \perp J$ . Therefore positive sectional curvature will make geodesics to converge, whereas negative curvature will make them to diverge. For more details see [24, Proposition 2.6 and Corollary 2.10].

### 1.3 The Ricci curvature

Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis for  $T_p M$ . Then for any  $v, w \in T_p M$ , the *Ricci curvature* is defined by

$$Ric(v, w) := \sum_{i=1}^n g(R(e_i, v)w, e_i).$$

In particular we have

$$\begin{aligned} Ric(v, v) &=: Ric(v) \\ &= \sum_{i=1}^n g(R(e_i, v)v, e_i) \\ &= \sum_{i=1}^n K(e_i, v). \end{aligned}$$

Hence the Ricci curvature in the  $v \in T_p M$  direction is the sum of all sectional curvature for sections containing  $v$ .

**Remark 1.4.** This definition of the Ricci curvature is not universal throughout the literature. Yet, our definition seems to be the one mostly used. There are authors (for example [24]) who define the Ricci curvature as the average of the sectional curvature and not just the sum.

*What is the geometric meaning of the Ricci curvature?*

Once can think of the Ricci curvature as measuring the *average* curvature of  $M$  in a neighbourhood of a point  $p \in M$ . Or, in the light of (1.1),  $Ric(v)$  measures how geodesics emanating from  $p$  in the  $v$  direction, diverge or converge on average. For example, let  $n = 3$  and let  $\{e_1, e_2, e_3\}$  an orthonormal basis of  $T_p M$ . Then we have only three sectional curvatures  $K(e_i, e_j) =: K_{ij}$ , where  $i, j = 1, 2, 3$  and  $i < j$  since  $K_{ii} = 0$ . If we have for example  $Ric(e_1) > 0$  than, by definition, this means  $K_{12} + K_{13} > 0$ . Therefore if  $K_{12} < 0$  then we must have  $K_{13} > |K_{12}|$ . By our previous discussion, this means that geodesic emanating from  $p$  in the  $e_1$  direction will converge more than they diverge. Therefore, intuitively, this means that a geodesic ball with a center *near*  $p$  will have volume *less* than a geodesic ball of same radius centered at  $p$ . This intuition is actually made precise by the following fundamental comparison theorem.

**Theorem 1.5** (Bishop-Gromov Volume Comparison). *Let  $(M, g)$  be a complete,  $n$ -dimensional Riemannian manifold with  $Ric(v) \geq 0$  for all  $v$ . Then, for all  $p \in M$  and all  $R > 0$ ,*

$$Vol_M(B_p(R)) \leq Vol_{\mathbb{R}^n}(B(R)),$$

where  $B(p, R)$  is the geodesic ball centered at  $p$  of radius  $R$ , and  $Vol_{\mathbb{R}^n}$  is the volume measured in  $\mathbb{R}^n$  with respect to the flat metric.

*Proof.* See for example [60, Chapter 9]. □

Therefore, in light of this theorem, the Ricci curvature controls the volume of geodesic balls in  $M$ .

**Convention.** We will say that a Riemannian manifold  $(M, g)$  “has Ricci curvature bounded below by  $k \in \mathbb{R}$ ” if  $Ric(v, w) \geq kg(v, w)$  for all  $v, w \in T_p M$ , and all  $p \in M$ .

**Remark 1.6** (Recovering the sectional curvature from the Ricci one). When  $n = 2$  then  $Ric(v)$  is just the Gauss curvature, since  $K(u, v) = K(v, u)$  and  $K(u, u) = 0$ . The case  $n = 3$  is also a very special. Although, by definition, the Ricci curvature is the average of sectional curvatures and, hence, contains less information than the sectional curvature itself, it turns out that in three dimensions all sectional curvatures can be recovered from the Ricci ones. Indeed, there are three sectional curvatures at  $p \in M$  and hence, by the definition of the Ricci curvature we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} K_{12} \\ K_{23} \\ K_{13} \end{pmatrix} = \begin{pmatrix} Ric(e_1) \\ Ric(e_2) \\ Ric(e_3) \end{pmatrix}.$$

Since the first matrix is invertible, we can multiply the equation from the left with its inverse to recover the sectional curvatures from the Ricci curvatures.

The comparison geometry for the Ricci curvature is a vast topic, with still many open problems. See [60, Chapter 9], [70] and [74] for some important results and questions.

## 1.4 The scalar curvature

We arrived now at the scalar curvature. Manifolds with a lower bound on the scalar curvature will be the main focus of these lectures.

**Definition 1.** *The scalar curvature of Riemannian  $n$ -manifold  $(M, g)$  is the function  $S : M \rightarrow \mathbb{R}$  given by*

$$S := \text{trace}_g(\text{Ric}).$$

Hence we have that

$$\begin{aligned} S &= \sum_{i=1}^n \text{Ric}(e_i, e_i) \\ &= \sum_{j=1}^n \sum_{i=1}^n g(R(e_j, e_i)e_i, e_j) \\ &= 2 \sum_{i < j} K(e_i, e_j) \\ &= 2 \sum_{i < j} K_{ij}. \end{aligned}$$

In particular, when  $n = 2$  then  $S = 2K_{12} = 2K$ , where  $K$  is the Gauss curvature of  $M$ . When  $n = 3$  we have that  $\frac{1}{2}S = K_{12} + K_{13} + K_{23}$ .

**Remark 1.7.** By definition of the sectional-, Ricci- and scalar curvature we have that

$$K \geq 0 \implies \text{Ric} \geq 0 \implies S \geq 0,$$

and easy examples show that the converse is not generally true. For example, consider the manifold  $M := \mathbb{S}^n \times (-\varepsilon, \varepsilon)$  with the warped product metric  $(1 + t^4)ds^2 + dt^2$ , where  $(\mathbb{S}^n, ds^2)$  is the standard  $n$ -sphere of constant curvature. The manifold  $M$  has positive scalar curvature, for small enough  $|t| < \varepsilon$ , yet the Ricci curvature in the direction normal to  $\mathbb{S}^n \times \{0\}$  will be negative, as one can easily check using the formulas from the Appendix.

*What is the geometric meaning of the scalar curvature?*

The same calculation that is used to prove (1.1) can also be used to prove the following formula (see for example [28, 3.H.4] for a brief description and [31] for more details):

$$\text{Vol}_M(B(p, \varepsilon)) = \text{Vol}_{\mathbb{R}^n}(B(\varepsilon)) \left( 1 - \frac{S(p)}{6(n+2)} \varepsilon^2 + \mathcal{O}(\varepsilon^4) \right), \quad (1.2)$$

where  $p \in M$  and  $\varepsilon > 0$  is small. Therefore, if  $S(p) > 0$  then the volume of *small* geodesic balls centered at  $p$  will be smaller than the volume of balls with same radius in the flat  $\mathbb{R}^n$ .

Formula (1.2) can be regarded as an infinitesimal version of Bishop-Gromov volume comparison theorem. However, unlike the case of Ricci curvature, the comparison given by formula (1.2) is valid only at a “microscopic” level; that is to say, only for geodesic balls of small radii  $\varepsilon > 0$ .

## The Sign of the Scalar Curvature

Given a (closed)  $n$ -dimensional manifold  $M$ , are there any topological restriction for existence of a Riemannian metric  $g$  in  $M$  with scalar curvature  $S(g) > 0$ ? If  $n = 2$  and  $M$  is a closed, Riemannian 2-manifold then, as we saw above,  $S = 2K$  and hence, by the Gauss-Bonnet theorem, we have

$$\int_M S dA = 8\pi(1 - \gamma),$$

where  $\gamma$  is the genus of  $M$ . Therefore if, for example,  $M = \mathbb{S}^1 \times \mathbb{S}^1$ , then the genus is one and the right-hand side vanishes. Hence there are no metrics on the 2-torus with strictly positive scalar curvature. What about for  $n$ -dimensional tori with  $n \geq 3$ ? It turns out that the answer again turns out to be negative.

**Theorem 1.8.** [Schoen-Yau  $n \leq 7$ ; Gromov-Lawson for any  $n$ ] *If  $g$  is any complete metric on  $\mathbb{T}^n$  with  $S(g) \geq 0$ , then  $g$  is flat.*

The proof of Schoen and Yau uses minimal surfaces techniques, being in the spirit of the proof of the Positive Mass Theorem that we will address in the next chapter.

At a first glance, it might seem that the sign of the scalar curvature of a Riemannian manifold is not related with the topology of the underlying manifold. Indeed, if  $N$  is any Riemannian manifold with  $S > -\infty$  and  $\mathbb{S}^2(\varepsilon)$  is a round 2-sphere of radius  $\varepsilon > 0$ , then the manifold  $M := N \times \mathbb{S}^2(\varepsilon)$ , with the product metric, has scalar curvature  $S + \frac{1}{\varepsilon^2}$ . Then, for sufficiently small  $\varepsilon > 0$ ,  $M$  will have positive scalar curvature. Yet the geometry and the topology of  $M$  will be as least as complicated as that of  $N$ . Moreover we have the following result which states that there are no topological restrictions to metrics of negative scalar curvature.

**Theorem 1.9** ([7, p. 400]). *On every closed Riemannian manifold of dimension  $n \geq 3$  there exists a metric of negative scalar curvature.*

It was a major breakthrough when, in 1960s, Lichnerowicz proved [50] the existence of closed manifolds which admit no metric of positive scalar curvature. Such an example is provided by the Fermat surface, defined by the equation  $x^4 + y^4 + z^4 + w^4 = 0$  in  $\mathbb{C}\mathbb{P}^3$ . This is further an example of a K3 surface which, by Lichnerowicz’s result, do not admit metrics of positive scalar curvature. See, for example, [13, 6.72]. His discovery was the starting point of understanding to what extent does the underlying topological structure of a manifold determine the sign of the scalar curvature. This is a very subtle problem and great progress has

been made during the second half of the 20<sup>th</sup> century. For a survey on this topic see, for example, [11], [12, Ch.12.3.3] and [42]. Surprisingly enough, there are no topological restrictions for metrics of negative *Ricci curvature* for dimensions greater than or equal to three [11].

The next major result in this direction came from Schoen and Yau in 1979. We will use this result later, in the second part of these lecture notes, when we will prove the Positive Mass Theorem.

**Theorem 1.10** ([64]). *Let  $M$  be a closed, oriented 3-manifold whose fundamental group  $\pi_1(M)$  contains a subgroup isomorphic to the fundamental group of a surface  $\Sigma$  of genus  $\gamma \geq 1$ . Then  $M$  admits no metric of positive scalar curvature.*

In particular, if  $\Sigma$  is any closed, oriented, genus  $\gamma \geq 1$  surface. Then  $M := \Sigma_\gamma \times \mathbb{S}^1$  admits no metrics of positive scalar curvature. These theorems illustrate that the interplay between the scalar curvature of a Riemannian structure and the topology of the underlying manifold is very subtle. For an ample discussion on the sign of the scalar curvature, as well as other curvature concepts, we refer to the articles [34] and [33].

In the next chapter we will focus our attention on the effect that the scalar curvature has on the geometry of certain submanifolds.

## 2 Comparison geometry for submanifolds

Let  $M$  be a complete, oriented,  $n$ -dimensional Riemannian manifold and let  $P$  be a closed, two-sided submanifold of  $M$ . Using the exponential map of  $M$  we can move  $P$  for a short time  $t > 0$  in some normal direction, obtaining a one-parameter family of submanifolds  $(P_t)_{t>0}$  that are all diffeomorphic to  $P$  and have volume  $V(t)$  with respect to the induced metric. In this chapter we are interested how the geometry of  $M$  controls the quantity  $V(t) - V(0)$ .

Let us first look at 1-dimensional submanifolds; i.e. geodesics. For this, let  $\gamma : [0, 1] \rightarrow M$  be a geodesic and let  $X(s)$  be a vector field along  $\gamma$  such that  $X(s) \perp \gamma'(s)$ . For some  $\varepsilon > 0$  and  $t \in [0, \varepsilon]$ , let  $\gamma_t := \gamma(t, s)$  be a variation of  $\gamma$  through geodesics in the direction  $X(s)$ . Denote by  $L(t) := \text{length}(\gamma_t)$ . In particular  $L(0) = \text{length}(\gamma)$ . We are interested in comparing the length of  $\gamma$  with that of nearby geodesics  $\gamma_t$ . More precisely, we are interested in understanding how the geometry of the ambient manifold controls the sign of the quantity

$$L(t) - L(0).$$

At an infinitesimal scale, by writing the Taylor expansion for  $L(t)$ , we have that for small enough  $|t| < \varepsilon$ ,

$$L(t) - L(0) = L'(0)t + \frac{1}{2}L''(0)t^2 + \mathcal{O}(t^3).$$

The first and the second order term can be calculated explicitly as in [60, Ch.5.4 and Ch.6.2]. The first order term depends only on the “geometry” of  $\gamma$ . And since  $\gamma$  is a geodesic is also a critical point for  $L$ , but not necessarily a minimum. Hence