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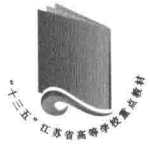
Elements of Stochastic Processes

随机过程基础

Jiashan Tang / 唐加山　编著

科学出版社

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For the Celebration of the 90th Birthday

of Professor Zikun Wang

Preface

Natural phenomena that occur in our real world can be categorized into two types: the deterministic type and the random type. As one of the subjects which are used to carry out the investigation for the random phenomenon, stochastic processes have attracted much attention since the beginning of the last century. It is not only important in scientific studies but also useful in the application areas, such as science, engineering, military science, economics, science of management, and medicine.

As more and more people have realized the importance of stochastic processes in our daily lives, stochastic processes have become a fundamental course in many majors in universities across the world. Tens of thousands of textbooks on stochastic processes have been published during the last hundred years. Chinese scholars have made great progress in the field of stochastic processes and have brought forth an important class of textbooks on stochastic processes with Chinese characteristics intertwined in them. Such kind of textbooks is particularly suitable for Chinese scholars and has played an important role in the cultivation of talents in the past.

As there have been developments in science and technology in China, more and more overseas young students have come to China to complete or expand their academic education. Textbooks of stochastic processes written in Chinese are not suitable for students who do not understand Chinese characters or have an insufficient understanding of mandarin. In addition, many western textbooks are not suitable for certain foreign students, either due to their personal field of study, background and/or environment. Evidence of this was seen when the author began to teach stochastic processes to these new students from the autumn of 2014 and found that the overseas students needed a new, more convenient textbook for learning. This made the author come to the idea of writing such a textbook.

The author has been teaching stochastic processes for undergraduate and graduate students for more than 20 years and has accumulated a wealth of teaching

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experience. The principle of writing this book is to introduce the basic concepts, methods and practices clearly by using a sufficient amount of examples and/or figures, which increases the readability of the textbook. The book is written from a non-measure theory point of view and is suitable for readers majoring in engineering, economics, management science, operations research, applied mathematics, etc. In order to save space, the author assumes that readers have the basic knowledge of advanced mathematics, linear algebra, differential equations, elements of probability, etc. Based on this idea, the book consists of seven chapters.

Chapter 1 introduces the basis of probability, which includes random events, probabilities, random variables, numerical characteristics and limiting theorems. This is just a simplified version of the probability theory.

Chapter 2 gives the definition of stochastic processes and introduces some statistical features and properties as well. After this chapter, several important stochastic processes will be introduced in different chapters in the rest of the book.

Chapter 3 presents the introduction of Poisson processes, which includes definitions and measurements, random sequences of waiting time and interarrival time, conditional distributions and several extensions as well.

Chapter 4 is on the discrete-time Markov chains. The main content consists of definitions, finite-dimensional distributions, properties of a single state, classifications of state spaces, asymptotic behaviors of transition probabilities, stationary distributions and reversible Markov chains.

Chapter 5 is a continuous counterpart of Chapter 4. Besides the contents, which are similar to that in Chapter 4, the main attention is focused on the different parts, which include Q-matrixes, Kolmogorov differential equations, asymptotic behaviors and birth and death processes.

Chapter 6 deals with simple Markovian queueing models, which can be regarded as a part of stochastic processes or one class of its application. The content includes terminologies, Little's law, PASTA property, and some queueing models, such as $M/M/1$, $M/M/n$, $M^X/M/1$ and $M/G/1$ as well.

Chapter 7 is very important for engineering readers, especially for communication guys. The main content covers: definitions of strict/weak-sense stationary processes,

analytic properties of the weak-sense stationary processes, correlation functions and their spectra, ergodicities and stationary processes passing through a linear time-invariant system.

This book is suitable for senior undergraduate students, junior graduate students and related teachers in engineering, economics, management science, operations research and applied mathematics. It is also suitable for scientific workers who are interested in stochastic processes. Moreover, various parts of chapters can be selected to be used independently for different objectives. Due to the limitation of time and the author's knowledge level, there must exist many improper things in this book. Any comments and suggestions are warmly welcomed and appreciated!

When the author was a master and a PhD student studying in Beijing Normal University, his supervisors were Prof. Zikun Wang (Chinese academician of CAS) and Prof. Zhanbing Li. During his visit to Carleton University in Canada as a postdoc (Postal doctorial fellow), his supervisors were Prof. Donald A. Dawson (fellow of the Royal Society) and Prof. Yiqiang Q. Zhao. The author thanks his mentors for their guidance into the field of stochastic processes and its applications and extends warm thanks for their invaluable guidance in his life. The author thanks many predecessors and his colleges for their valuable concerns and supports. Thanks are also given to two master students, Ying Tang in Southeast University and Heidi Gabrielle Pluntke in Nanjing University of Posts and Telecommunications, for their pre-reading to improve the English level of the current version. During the procedure of publishing the book, the chief responsible editor Zhongxing Zhang in the Science Press has given a lot of help. Additionally, the book has been elected as a key textbook for universities in Jiangsu province and Nanjing University of Posts and Telecommunications as well, and has received financial support simultaneously. The author would like to take this opportunity to say thanks to them.

The author gives special thanks to his parents and his family members. The author was born and grew up in the countryside of Anhui province. His mother never went to school and his father had finished only the fourth grade of primary school. From the 1970s to the 1980s in the last century, people lived in a difficult environment in China. During those hard times, the author's parents, as peasants, still insisted on

providing chances for their children to go to school and further their education, which was an extremely difficult task, but a worthy one. To their delight, after many years of hard work, the author became to be known amongst his neighbours in the village as the first person to become a university student. The author wishes to express deep thanks and appreciation to his parents and family members for their motivation for continuing to improve his knowledge and providing encouragement in pushing his career forward. They are the most important people in the author's life and have had the most influence on his destiny.

Jiashan Tang
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25 July, 2018

Contents

Preface

Chapter 1 Basis of probability 1

1.1 Random events 1

1.1.1 Random experiments 2

1.1.2 Random events 3

1.1.3 Calculation among random events 6

1.1.4 σ-algebras 10

1.2 Probability 12

1.2.1 What is probability 12

1.2.2 Properties of probability 14

1.2.3 Two fundamental probability models 16

1.2.4 Conditional probability and three important formulas 19

1.2.5 Independence of events 23

1.3 Random variables 25

1.3.1 Definition 25

1.3.2 Cumulative distribution functions 27

1.3.3 Classification 30

1.3.4 Multiple-dimensional random variables 34

1.3.5 Distributions of functions of random variables 38

1.3.6 Conditional distributions and independences 44

1.3.7 Order statistics 47

1.4 Numerical characteristics 50

1.4.1 Definition of mathematical expectations 50

1.4.2 Properties 56

1.4.3 Other numerical characteristics of random variables 57

1.4.4 Conditional mathematical expectations 73

1.5 Limiting theorems 76

1.5.1 Convergence of random variables 76

1.5.2 Law of large numbers 80

1.5.3 Central limit theorems 84

Exercise 1 86
Answers or tips for Exercise 1 88
Chapter 2 Stochastic processes 95
2.1 Definition and classification 95
2.1.1 Definition 96
2.1.2 Classification 98
2.1.3 Examples 99
2.2 Statistical laws of stochastic processes 102
2.2.1 Finite dimensional distribution functions 102
2.2.2 Kolmogorov's theorem 103
2.3 Measurements of stochastic processes 104
2.3.1 Measurements for one stochastic process 104
2.3.2 Measurements for two stochastic processes 107
2.4 Further comments 108
Exercise 2 109
Answers or tips for Exercise 2 111
Chapter 3 Poisson processes 112
3.1 Definition and measurements 112
3.1.1 Definition 112
3.1.2 Measurements 123
3.2 Waiting times and interarrival times 126
3.2.1 Waiting times 127
3.2.2 Interarrival times 130
3.3 Conditional distributions 131
3.4 Extensions of Poisson processes 136
3.4.1 Non-homogeneous Poisson processes 137
3.4.2 Compound Poisson processes 142
3.4.3 Conditional Poisson processes 145
3.4.4 Renewal processes 147
Exercise 3 152
Answers or tips for Exercise 3 154
Chapter 4 Discrete-time Markov chains 157
4.1 Definition of discrete-time Markov chains 157
4.1.1 Definition 159
4.1.2 Chapman-Kolmogorov equations 163

4.2 Finite-dimensional distributions 165
4.3 Properties of a single state 168
4.3.1 Periods 168
4.3.2 Transience and recurrence 169
4.4 Decomposition of state space 177
4.4.1 Equivalence relation 177
4.4.2 Decomposition of state space 181
4.5 Asymptotic behaviors of transition probabilities $p_{ij}^{(n)}$ 184
4.5.1 Case one: state j is transient or null-recurrent 185
4.5.2 Case two: state j is positive-recurrent 186
4.6 Stationary distributions 196
4.6.1 Definition of stationary distributions 196
4.6.2 How many stationary distributions a Markov chain may have? 199
4.6.3 Rates of convergence to stationary distributions 201
4.6.4 Stationary distributions of a censored Markov chain 201
4.6.5 Quasi-stationary distributions 203
4.7 Reversible Markov chains 206
Exercise 4 209
Answers or tips for Exercise 4 211
Chapter 5 Continuous-time Markov chains 213
5.1 Definition of continuous-time Markov chains 213
5.1.1 Definition 213
5.1.2 Chapman-Kolmogorov equations 218
5.2 Finite-dimensional distributions 218
5.3 Q-matrices 220
5.4 Kolmogorov differential equations 227
5.5 Asymptotic behaviors 236
5.5.1 Transience and recurrence 237
5.5.2 Limiting results 239
5.5.3 Stationary distributions 240
5.6 Birth and death processes 243
Exercise 5 249
Answers or tips for Exercise 5 251
Chapter 6 Simple Markovian queueing models 256
6.1 Terminology and notation 256

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6.2 Little's law and PASTA property ····· 259
6.2.1 Little's law ····· 259
6.2.2 PASTA property ····· 260
6.3 $M/M/1$ queueing model ····· 261
6.3.1 Stationary distribution of queue length ····· 261
6.3.2 Distributions of sojourn times and waiting times ····· 266
6.3.3 Busy period distribution ····· 270
6.3.4 Departure process ····· 273
6.4 $M/M/n$ and state dependent $M/M/1$ queueing model ····· 275
6.4.1 $M/M/n$ queueing model ····· 275
6.4.2 State dependent $M/M/1$ queueing model ····· 276
6.5 $M^X/M/1$ queueing model ····· 279
6.6 $M/G/1$ queueing model ····· 281
6.6.1 Embedded Markov chain ····· 281
6.6.2 $M/E_r/1$ queueing model ····· 286
Exercise 6 ····· 290
Answers or tips for Exercise 6 ····· 292
Chapter 7 Stationary processes ····· 295
7.1 Definition ····· 295
7.1.1 Strict-sense stationary processes ····· 295
7.1.2 Wide-sense stationary processes ····· 298
7.2 Analytic properties of wide-sense stationary processes ····· 302
7.3 Correlation functions and their spectra ····· 312
7.3.1 Properties of correlation functions ····· 312
7.3.2 Spectral density functions ····· 314
7.3.3 Properties of spectral density functions ····· 323
7.3.4 Continuous-time white noise processes ····· 325
7.4 Ergodicity ····· 327
7.5 Passing through a linear time-invariant system ····· 335
Exercise 7 ····· 345
Answers or tips for Exercise 7 ····· 346
References ····· 354
Index ····· 358

Chapter 1

Basis of probability

Main knowledge

1. Probability space.
2. Random variables.
3. Numerical measurements.
4. Limiting theorems.

1.1 Random events

In our natural world, there are many phenomena, which can be categorized into two cases. One is certain/deterministic phenomena, in which case, when we do something, if the condition is satisfied, then the unique result will happen. For example, water naturally flows downhill. The other is an uncertain/random case, in which, when we do something, although the condition is satisfied, the outcome still can not be forecasted.

Let's consider what the weather will be tomorrow. Although we know that it might be sunny, cloudy, rainy, etc. we do not know today what exactly the weather will be tomorrow. Similarly, the stock price tomorrow morning is also uncertain. In fact, there are many uncertain phenomena around us. In the modern scientific world, besides the classical mathematics, this phenomena can be analyzed by the disciplines, such as probability theory, statistics, stochastic processes, stochastic analysis and etc.

Among the subjects mentioned above, probability theory is a prerequisite course of stochastic processes. Therefore, before meeting the main content of this book beginning from Chapter 2, we need to revisit the knowledge of probability theory in the current chapter.

1.1.1 Random experiments

As a scientific approach, in order to investigate uncertain phenomena, we need to know the information or data of the phenomenon. There are two methods to collect those information: observation and experimentation. Apparently, observation is a passive behavior; how much information we can get only depends on the happening frequency of the phenomenon. Plenty of information can only be obtained through active experimentation.

Similar to the experiments implemented in physics and chemistry, we need to impose some conditions on the experiments in the probability theory.

(1) The experiments (or trials) can be infinitely repeated under the same conditions;

(2) All possible outcomes of the experiments are known at any time;

(3) The outcome of a particular experiment can not be predicted before the implementation.

If an experiment satisfies the above three conditions, then it is called a **random experiment**.

Remark 1.1 *Condition (1) is a constrain condition which is imposed on all scientific experiments. For example, we may have had a dream yesterday evening. However, we can not have the same dream tonight. Therefore, we can not make dreams a scientific experiment. In fact, all scientific experiments can be repeated by any person, at any time, as long as the same conditions hold.*

Condition (2) implies that the experiment or phenomenon we are considering is a simple and post-modelling case in some sense. For example, if some outcomes of the experiment can not be observed or can not be observed clearly, then we can regard part of the results as a particular outcome of the experiment.

Condition (3) indicates that there are at least two outcomes of a random experiment. Otherwise, if there is only one outcome of an experiment, then the outcome of the experiment can be forecasted before the implementation. In this case, the experiment is called a ***deterministic experiment****. A random experiment always has two or more results. Moreover, if a random experiment has exactly two possible outcomes, then it is called a* ***Bernoulli trial****.*

Example 1.1 Tossing a fair coin is a random experiment.

If we have a flat table and a fair coin, then we can do the experiment of tossing the coin. I can do it, and you can do it. Everybody can do it. We can do it today,

and of course, we can do it tomorrow. The experiment of tossing the coin can be repeated any times you want. It implies that condition (1) holds.

Before we flip the coin, we know all possible outcomes of the experiment. In this case, it maybe the head or the tail. Here we omit one case that the coin is cocked up. It implies that condition (2) holds.

For a particular flipping, the outcome can not be forecasted in advance, which means that condition (3) holds.

There are many examples of random experiment around us. For example, recording the number of incoming calls of a telephone exchange within one minute; considering the number of received calls of an ATM switch within one unit time; taking one from a batch of electronic components and testing its lifetime and etc.

In probability theory, the uncertain phenomena are investigated through the corresponding random experiments. All experiments are understood as random experiments in the rest of this chapter.

1.1.2 Random events

For random experiments introduced in the last subsection, we have the following definitions.

Definition 1.1 Sample point, sample space

For a random experiment, each simple outcome is called a **sample point**, which is denoted by ω. All sample points form a set, which is called a **sample space** and is denoted by Ω.

Example 1.2 For the random experiment of tossing a fair coin, we have two sample points: $\omega_1 = H$ and $\omega_2 = T$, where H means head and T means tail. Therefore, the sample space is $\Omega = \{\omega_1, \omega_2\} = \{H, T\}$.

Example 1.3 Consider a random experiment of tossing a fair dice which has six sides with faces numbered from 1 to 6, respectively. Apparently, we have six sample points, which are $\omega_1 = 1, \cdots, \omega_6 = 6$. Moreover, the sample space is given by $\Omega = \{1, 2, \cdots, 6\}$.

Example 1.4 We still consider the random experiment of tossing a fair dice as in Example 1.3. What we are interested in now is that the outcome for each toss is odd or even. In this case, "*odd*" is a simple outcome and "*even*" is another outcome. Therefore, the sample space is $\Omega = \{odd, even\}$.

Examples 1.3 and 1.4 indicate that even for the same random experiment, we may have different sample spaces, and the reason of which is depending on the facts that we are interested in. The cardinal numbers of the sample spaces in the above three examples all are finite. Let's have a look at examples whose cardinal numbers of the sample spaces are infinite.

Example 1.5 Let's consider the number of live fishes swimming in a section of the Yangtze River, which ranges from Wuhan City to Nanjing City in China. The worst thing is that there are no fish in this section, which means that 0 is a possible outcome. Similarly, 1 is also a possible outcome, and 2 is too. In fact, any non-negative integer number is a possible outcome. Therefore, the sample space is $\Omega = \{0, 1, 2, \cdots\}$, which has infinitely many numbers. Realistically, we know that the number of fishes within the Yangtze River can not be infinite. However, any non-negative integer can be the number of fishes in that section, which means that the number of fishes can be, theoretically, arbitrarily large.

Example 1.6 Consider the water storage in a reservoir with the unit being m^3. If there is no water, then the storage is 0, which means that 0 is a possible outcome. Suppose that the maximum storage of this reservoir is b and this number can not be reached. It's easy to see that the sample space can be written as $\Omega = \{x | 0 \leqslant x < b\}$.

In Example 1.3, we know that the sample space is $\Omega = \{1, 2, \cdots, 6\}$. In a practical situation, one person might be interested in the tossing result which is equal to or less than 3, while another guy is interested in the result which is greater than 2. All these kinds of results are very important in probability theory and have a particular name, which is given in the following.

Definition 1.2 Classical random event

Suppose that Ω is a sample space for some random experiment. Any subset of the sample space, $A \subseteq \Omega$, is called a **random event** of the sample space. A random event is always denoted by a capital letter with or without a subscription. The family of all events is denoted by $\mathscr{F}_{all} = \{A | A \subseteq \Omega\}$.

Example 1.7 Consider the sample space $\Omega = \{H, T\}$ in Example 1.2. Since that the empty set is a subset of all sets, $\varnothing$ is a random event. In fact, we totally have four random events and they are: $A_1 = \varnothing$, $A_2 = \{H\}$, $A_3 = \{T\}$ and $A_4 = \{H, T\}$. Furthermore, $\mathscr{F}_{all} = \{A_1, A_2, A_3, A_4\}$. If the sample space is not complicated as in this example, then the family of events can also be written as $\mathscr{F}_{all} = \{\varnothing, \{H\}, \{T\}, \Omega\}$.

Consider tossing a fair dice as in Example 1.3. The sample space is $\Omega = \{1, 2, \cdots, 6\}$. Apparently, both $A_1 = \{3\}$ and $A_2 = \{1, 3, 5\}$ are two random events. If in a particular performance, the outcome of the toss is 3, then we can say that the event, "A_1", occurred in this toss. Since $3 \in A_2$, we can say that the event, "A_2", occurred in this toss, too. Generally, if the outcome of a particular test is ω, then we can say that the event, which includes ω as its component, occurred in this test.

Example 1.8 If a sample space has three sample points $\Omega = \{\omega_1, \omega_2, \omega_3\}$, then all random events of this sample space are: $A_1 = \varnothing$, $A_2 = \{\omega_1\}$, $A_3 = \{\omega_2\}$, $A_4 = \{\omega_3\}$, $A_5 = \{\omega_2, \omega_3\}$, $A_6 = \{\omega_1, \omega_3\}$, $A_7 = \{\omega_1, \omega_2\}$, $A_8 = \{\omega_1, \omega_2, \omega_3\}$. Moreover, $\mathscr{F}_{all} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$.

Generally, for sample spaces with finite cardinal numbers of sample points, we have the following proposition on the number of random events.

Proposition 1.1 If a sample space has n sample points, $\Omega = \{\omega_1, \cdots, \omega_n\}$, then the total number of random events for this sample space is 2^n.

Proof Since that $\varnothing \subset \Omega$, $\varnothing$ is a random event, which corresponds to the subset that there is no sample points selected from the sample space. The number of this kind of subsets is $C_n^0 = 1$. This is the first case. For those subsets, each of which has only one sample point selected from the sample space, the total number is $C_n^1 = n$. This is the second case. The third case is that there are C_n^2 subsets, each of which has two sample points chosen from the sample space and so on. The last case is that the subset has all sample points chosen from the sample space, and the number is $C_n^n = 1$. Therefore, the total number of random events for the sample space having n sample points is

$$C_n^0 + C_n^1 + \cdots + C_n^n = (1+1)^n = 2^n.$$

Example 1.9 Let's toss a dice two times (or toss a pair of dice). In this case, the simplest outcome reads a pair of numbers like $\omega = (i, j)$, which means that i is the number which appeared in the first toss and j is in the second. Since the value of i can be 1, 2, $\cdots$, 6 and that of j is the same as that of i. Therefore, the sample space is $\Omega = \{(1,1), (1,2), \cdots, (1,6), (2,1), \cdots, (2,6), \cdots, (6,6)\}$. Consider some random events for this sample space.

(1) Random event having no sample point: $A_1 = \varnothing$;

(2) Random event with the property that at least one number is 6:

$$A_2 = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,5), (6,4), (6,3), (6,2), (6,1)\}.$$

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