

美国数学会经典影印系列



Fine Regularity of Solutions of Elliptic Partial Differential Equations

椭圆偏微分方程的解的精细正则性

Jan Malý, William P. Ziemer



高等教育出版社

美国数学会经典影印系列



Fine Regularity of Solutions of Elliptic Partial Differential Equations

椭圆偏微分方程的解的精细正则性

Jan Malý, William P. Ziemer



高等教育出版社·北京

图字：01-2019-0853 号

Fine Regularity of Solutions of Elliptic Partial Differential Equations, by Jan Malý, William P. Ziemer,
first published by the American Mathematical Society.

Copyright © 1997 by the American Mathematical Society. All rights reserved.

This present reprint edition is published by Higher Education Press Limited Company under authority
of the American Mathematical Society and is published under license.

Special Edition for People's Republic of China Distribution Only. This edition has been authorized by
the American Mathematical Society for sale in People's Republic of China only, and is not for export therefrom.

本书原版最初由美国数学会于 1997 年出版，原书名为 *Fine Regularity of Solutions of
Elliptic Partial Differential Equations*，作者为 Jan Malý, William P. Ziemer。

美国数学会保留原书所有版权。

原书版权声明：Copyright © 1997 by the American Mathematical Society。

本影印版由高等教育出版社有限公司经美国数学会独家授权出版。

本版只限于中华人民共和国境内发行。本版经由美国数学会授权仅在中华人民共和国境内销售，不得出口。

椭圆偏微分方程的解的 精细正则性

Tuoyuan Pianweifen Fangcheng
de Jie de Jingxi Zhengzhexing

图书在版编目 (CIP) 数据

椭圆偏微分方程的解的精细正则性 = Fine Regularity of Solutions
of Elliptic Partial Differential Equations: 英文 / (捷克) 简·马利
(Jan Malý), (美) 威廉姆·P·齐默 (William P. Ziemer) 著。

— 影印本. — 北京: 高等教育出版社, 2019.5

ISBN 978-7-04-051724-8

I. ① 椭… II. ① 简… ② 威… III. ① 二阶—椭圆型

方程—偏微分方程—正则性—研究—英文 IV. ① O175.23

中国版本图书馆 CIP 数据核字 (2019) 第 067185 号

策划编辑 李鹏 责任编辑 李鹏

封面设计 张申申 责任印制 赵义民

出版发行 高等教育出版社
社址 北京市西城区德外大街 4 号
邮政编码 100120

购书热线 010-58581118

咨询电话 400-810-0598

网址 <http://www.hep.edu.cn>

<http://www.hep.com.cn>

网上订购 <http://www.hepmall.com.cn>

<http://www.hepmall.com>

<http://www.hepmall.cn>

印刷 北京中科印刷有限公司

开本 787mm×1092mm 1/16

印张 19.75

字数 500 千字

版次 2019 年 5 月第 1 版

印次 2019 年 5 月第 1 次印刷

定价 135.00 元

本书如有缺页、倒页、脱页等质量问题，

请到所购图书销售部门联系调换

版权所有 侵权必究

[物料号 51724-00]



美国数学会经典影印系列

出版者的话

近年来,我国的科学技术取得了长足进步,特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时,国内的科研队伍与国外的交流合作也越来越密切,越来越多的科研工作者可以熟练地阅读英文文献,并在国际顶级期刊发表英文学术文章,在国外出版社出版英文学术著作。

然而,在国内阅读海外原版英文图书仍不是非常便捷。一方面,这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中,普通读者借阅不甚容易;另一方面,原版书价格昂贵,动辄上百美元,购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取,间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者,同美国数学会(American Mathematical Society)合作,在征求海内外众多专家学者意见的基础上,精选该学会近年出版的数十种专业著作,组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于1888年,是国际上极具影响力的专业学术组织,目前拥有近30000会员和580余个机构成员,出版图书3500多种,冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版,能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用,也希望今后能有更多的海外优秀英文著作被介绍到中国。

高等教育出版社

2016年12月

To

Jaroslava and Suzanne

Preface

“An important problem . . . is the question concerning existence of solutions of partial differential equations when the values on the boundary of the region are prescribed. . . . Has not every regular variational problem a solution, provided certain assumptions regarding given boundary conditions are satisfied and provided also if need be that the notion of a solution shall be suitably extended?” These are the words that Hilbert used to conclude his question concerning the twentieth problem stated in an address delivered before the International Congress of Mathematicians in 1900.

Hilbert’s twentieth problem and his related nineteenth problem have opened one of the most interesting chapters in analysis and have attracted the efforts of many mathematicians. Although there are still many open questions related to these problems of Hilbert, a great deal of progress has been made, some with dazzling success.

One of the components necessary to establish regularity of certain variational problems was the need to show that a weak solution of a linear equation in divergence form with bounded, measurable coefficients is Hölder continuous. This result resisted many attempts, but finally in 1957, De Giorgi and Nash, independently of each other, provided a proof of this crucial result. The De Giorgi–Nash result stimulated a great number of related investigations, one of the most important being that of Moser who, by entirely different methods, provided another proof of their result. A critical component in Moser’s proof was the discovery that the logarithm of the solution is a function of bounded mean oscillation. He also proved the Harnack inequality, which states that locally, the supremum of the solution is bounded by its infimum.

Others generalized these results to weak solutions of quasilinear equations in divergence form with a very general structure. Ladyzhenskaya and Ural’tseva fully developed the De Giorgi method in this direction while Serrin and Trudinger extended the Moser method. In particular, Trudinger discovered that weak supersolutions possess a weak Harnack inequality. That is, the infimum of a nonnegative supersolution over a ball can be bounded below by its integral average.

The primary purpose of all of these results was to investigate the behavior of weak solutions in the interior of the domain. While these methods could be easily adapted to establish some regularity results of the solution at the boundary, a definitive one similar to the inspired result by Wiener to characterize continuity at the boundary for harmonic functions was lacking. The first step in this direction was made by Littman, Stampacchia, and Weinberger who proved that a point in the boundary of an arbitrary domain was simultaneously regular for harmonic functions and weak solutions of linear equations with bounded, measurable coefficients. In

his work concerning the local behavior of solutions of quasilinear equations, Serrin discovered that a capacity, now known as the p -capacity, was the appropriate measurement for describing removable sets for weak solutions. Later, Maz'ya discovered a Wiener-type expression involving this capacity which provided a sufficient condition for continuity at the boundary of weak solutions of equations whose structure is similar to that of the p -Laplacian. Utilizing different techniques, it was shown in [GZ3] that Maz'ya's condition was also sufficient for boundary continuity for solutions of a large class of quasilinear equations in divergence form. The methods used in [GZ3] were of interest in that they did not employ any potential theoretic techniques, thus making a distinct departure from all previous attempts in this area. Indeed, a new proof of sufficiency of Wiener's theorem can be obtained by particularizing the results in [GZ3]. By carefully analyzing the methods leading to these results Evans and Jensen, [EJ1], [EJ2], established an even simpler proof of sufficiency and also included necessity, using essentially no more than Sobolev's inequality and the mean value property of superharmonic functions. Showing that the Wiener-type expression was also necessary for the general class of quasilinear equations proved to be difficult and was established only recently in [KM4]. Prior to this, Lindqvist and Martio, [LM], established necessity in the case $p > n - 1$.

The main thrust of this book is to give a comprehensive exposition of the results surrounding the work of [GZ3] and [KM4]. This includes a "fine" analysis of Sobolev spaces and a development of the associated nonlinear potential theory. The term "fine" refers to a topology on \mathbf{R}^n , actually the p -fine topology, which is induced in a natural way by the Wiener-type condition. It also includes an analysis of variational inequalities or obstacle problems where the obstacles are allowed to be irregular, even discontinuous. The exposition on variational inequalities includes the regularity theory associated with differential operators with a differentiable structure and $C^{1,\alpha}$ obstacles.

Specifically, the book is organized in the following way. Chapter 1 contains preliminary material, including important results from measure and integration theory, the Besicovitch Covering theorem, the maximal function, basic estimates involving Riesz potentials, and Sobolev spaces. We find it convenient to introduce the space $Y^{1,p}$ which differs from the Sobolev space $W^{1,p}$ only in that its elements are required to lie in L^{p^*} ($p^* := np/(n-p)$, and not in L^p). The space $Y^{1,p}$ is associated with Riesz potentials whereas Bessel potentials are the counterparts to functions in $W^{1,p}$.

The second chapter begins with the development of p -capacity, the capacity which is used in the nonlinear version of the Wiener condition. Here it is shown that Sobolev functions possess Lebesgue points everywhere except for a set of p -capacity zero, that is, a p -polar set. The classical concepts of thinness and quasicontinuity are extended to this environment involving p -capacity. A result of Maz'ya and Shaposhnikova, [MaSh], relating to multipliers of Sobolev functions, provides a borderline case to the Sobolev-type imbedding theorem due to Adams, [Ad1], which is found in Chapter 1. Together, these two results allow a large class for the coefficients of the structure associated with our equations. The second section is focused on matters related to the classical Laplace equation, including Green potentials, classical thinness and the classical Wiener criterion. The subsection entitled "Regularity of minimizers" contains a short overview of the regularity theory associated with certain variational problems. The main purpose here is to indicate the important role that the De Giorgi-Nash result plays in this theory. We

then give a brief treatment of De Giorgi's proof as well as the proof due to Moser involving his iteration method for solutions to equations with a simple structure. Although this material will be completely subsumed by the exposition in Chapters 3 and 4, we choose to include it for the benefit of the reader who wishes to understand the fundamental ideas of these proofs in a setting that is not encumbered with the additional complexities of a very general structure. A proof of the Wiener criterion for equations with a simple structure is presented, again for the purpose of preparing the reader for the complete development that appears in Chapter 4. The chapter is concluded with a treatment of the p -fine topology as well as fine Sobolev spaces. Many of the topics in this and the previous chapter have been exposed in other books that have recently appeared. The reader is recommended to consult the work of Adams and Hedberg, [AH], as well as that of Heinonen, Kilpeläinen, and Martio, [HKM2].

Chapter 3 begins the analysis of quasilinear equations with a general structure whose coefficients are permitted to lie in the multiplier spaces mentioned above. The results include local sup-norm estimates for weak subsolutions in terms of their integral averages, certain maximum principles, and the weak Harnack inequality for weak supersolutions. The techniques used here are due to Moser, [Mos1], [Mos2], Serrin [Se1], and Trudinger [T1]. The next topic in this chapter is removable sets for weak solutions, a topic first investigated by Serrin in [Se3] and [Se4]. The removability of a set is assured if its capacity is zero, thus indicating that one might suspect capacity to play a role in the criterion for boundary regularity. The chapter is concluded with a brief summary of regularity results of equations with differentiable structure. These results will be referenced in Chapters 5 and 6.

Regularity at the boundary is the main concern of Chapter 4. This includes basic energy estimates as well as a weak Harnack inequality for a class of functions termed "supersolutions up to level l ." This Harnack inequality is crucial in establishing the Wiener criterion for boundary regularity as well as proving regularity for weak solutions of double obstacle problems. Sufficiency of the Wiener condition for boundary regularity is based on the work in [GZ3] while necessity of the condition comes from [KM4]. The methods used to establish the Wiener criterion are applied to the analysis of equations of the form

$$-\operatorname{div} \mathbf{A}(x, u, \nabla u) + \mathbf{B}(x, u, \nabla u) = \mu$$

where μ is a Radon measure satisfying suitable growth conditions.

The weak Harnack inequality for supersolutions up to level l is used in Chapter 5 to prove that weak solutions to the double obstacle problem are continuous provided the obstacles satisfy a regularity condition expressed in terms of the Wiener condition. In particular, it is shown that the solution is continuous at a point if both obstacles are continuous there. Moreover, an estimate for the modulus of continuity of the solution is obtained which implies that the solution inherits a Hölder continuity estimate from its obstacles. The last part of the chapter is devoted to the double obstacle problem for equations with a differentiable structure. In this context it is natural to assume that the obstacles are $C^{1,\alpha}$, and with this assumption, it is shown that the solution also possesses Hölder continuous derivatives.

The last chapter begins with the analysis of existence of solutions to variational inequalities. Assuming that the principle term of the differential operator satisfies a coercivity condition, existence is established with the help of pseudomonotone operators. The chapter concludes with the Dirichlet problem for equations with

differentiable structure. The main result here is that if a continuous function is prescribed on the boundary of an arbitrary bounded domain, then there exists a solution which is locally $C^{1,\alpha}$ which continuously assumes the boundary values at all points satisfying the Wiener condition. By the Kellogg property, this will be satisfied at all points of the boundary except for a p -polar set.

There are several excellent sources related to this book that the reader may wish to consult. They include the recent monographs by Adams and Hedberg [AH] on nonlinear potential theory and function spaces, and by Heinonen, Kilpeläinen, and Martio [HKM2] concerning the potential theory related to p -Laplacian type equations. Books on Sobolev spaces that are related to our treatment on this subject are by Maz'ya [Maz6], Evans and Gariepy [EG], and [Z5]. Finally, the books by Ladyzhenskaya and Ural'tseva [LU], Morrey [Mo4], Gilbarg and Trudinger [GT] and Kinderlehrer and Stampacchia provide treatments of elliptic equations and related topics that are particularly relevant for our purposes.

This book is an outgrowth of lectures given by the authors at the Spring School on Potential Theory and Analysis which was held in Paseky, Czech Republic, in June, 1993. This conference was organized by the Faculty of Mathematics and Physics, Charles University, Prague. The authors wish to thank the organizers for the opportunity to participate in this conference.

In addition to support from Charles University and Indiana University, the authors gratefully acknowledge support from the following grants: 201/93/2174, 201/94/0474, and 201/96/0311 of Czech Grant Agency (GACR), No. 354 of Charles University (GAUK), and the National Science Foundation (USA).

March, 1997

Jan Malý
William P. Ziemer

Basic Notation

\mathbf{R}^n	Euclidean n -space
E°	topological interior of E
\overline{E}	closure of E
∂E	topological boundary of E
χ_E	characteristic function of E
$\text{dist}(A, B)$	distance between sets A and B
$\text{diam}(A)$	diameter of the set A
$\text{osc}_A u$	diameter of the set $\{u(x) : x \in A\}$
$\{u > \alpha\}$	$\{u(x) : x > \alpha\}$
$U \subset\subset \Omega$	\overline{U} is compact and $\overline{U} \subset \Omega$ (for p -finely open sets see Definition 3.1)
$\mathcal{C}(E)$	the space of continuous functions on E
$\mathcal{C}_0(E)$	those functions $u \in \mathcal{C}(E)$ that satisfy $u(x_j) \rightarrow 0$ whenever $ x_j \rightarrow \infty$ or $x_j \rightarrow x \notin E$
$\mathcal{C}^k(\Omega)$	k times continuously differentiable functions on Ω
$\mathcal{C}_c^k(\Omega)$	functions in $\mathcal{C}^k(\Omega)$ compactly supported in Ω
$\mathcal{C}^{0,\alpha}(\Omega)$	locally Hölder continuous functions on Ω
$\mathcal{C}^{k,\alpha}(\Omega)$	the functions whose derivatives up to order k are locally α -Hölder continuous on Ω
$\mathcal{C}^{k,\alpha}(\overline{\Omega})$	the functions whose derivatives up to order k are (globally) α -Hölder continuous on Ω
X^*	the dual of the space X
δ_z	Dirac measure supported at z
$ E $	Lebesgue measure of $E \subset \mathbf{R}^n$
H^k	k -dimensional Hausdorff measure
$L^p(\Omega)$	p^{th} -power Lebesgue integrable functions defined on Ω
$L^p(\Omega; \mathbf{R}^n)$	p^{th} -power \mathbf{R}^n -valued Lebesgue integrable functions defined on Ω
$L_{\text{loc}}^p(\Omega)$	locally p^{th} -power Lebesgue integrable functions defined on Ω
$\ u\ _{p;\Omega}$ or $\ u\ _p$	L^p norm of u on Ω
$B(x, r)$	$\{y : x - y < r\}$
$K(x, r)$	$\{y : x - y \leq r\}$
$\alpha(n)$	$ B(0, 1) $
$u \lfloor E$	u restricted to the set E

$\int_E u(x) dx$	integration with respect to Lebesgue measure
$\bar{u}_E := \int_E u(x) dx$	$ E ^{-1} \int_E u(x) dx$
$f * g$	convolution of f and g
$f \circ g$	composition $x \mapsto f(g(x))$
$a \cdot b$	scalar product in \mathbf{R}^n
$\text{spt } u$	closure of $\{x : u(x) \neq 0\}$
u^+	$\max\{u, 0\}$
u^-	$\max\{-u, 0\}$
$D_i u$	$\frac{\partial u}{\partial x_i}$
p'	$\frac{p}{p-1}$
p^*	$\begin{cases} \frac{np}{n-p} & \text{if } 1 \leq p < n \\ \infty & \text{if } p = n \end{cases}$

Contents

Preface	ix
Basic Notation	xiii
Chapter 1. Preliminaries	1
1.1 Basic results	1
1.1.1 Covering theorems	7
1.1.2 Densities of measures	9
1.1.3 The maximal function and its applications	11
1.2 Potential estimates	14
1.3 Sobolev spaces	21
1.3.1 Inequalities	28
1.3.2 Imbeddings	31
1.3.3 Pointwise differentiability of Sobolev functions	40
1.3.4 Spaces $Y^{1,p}$	45
1.3.5 Adams' inequality	50
1.3.6 Bessel and Riesz potentials	57
1.4 Historical notes	61
Chapter 2. Potential Theory	63
2.1 Capacity	63
2.1.1 Comparison of capacities; capacities of balls	66
2.1.2 Polar sets	71
2.1.3 Quasicontinuity	73
2.1.4 Multipliers	77
2.1.5 Capacity and energy minimizers	78
2.1.6 Thinness	84
2.1.7 Capacity and Hausdorff measure	86
2.1.8 Lebesgue points for Sobolev functions	89
2.2 Laplace equation	92
2.2.1 Green potentials	98
2.2.2 Classical thinness	102
2.2.3 Dirichlet problem and the Wiener criterion	105
2.3 Regularity of minimizers	109
2.3.1 Abstract minimization	110
2.3.2 Minimizers and weak solutions	110
2.3.3 Higher regularity	115
2.3.4 The De Giorgi method	118
2.3.5 Moser's iteration technique	122

2.3.6	Removable singularities	126
2.3.7	Estimates of supersolutions	128
2.3.8	Estimates of energy minimizers	131
2.3.9	Dirichlet problem	135
2.3.10	Application of thinness: the Wiener criterion	139
2.4	Fine topology	143
2.5	Fine Sobolev spaces	148
2.6	Historical notes	157
Chapter 3.	Quasilinear Equations	161
3.1	Basic properties of weak solutions	161
3.1.1	Upper bounds for weak solutions	167
3.1.2	Weak Harnack inequality	174
3.1.3	Removable sets for weak solutions	178
3.2	Higher regularity of equations with differentiable structure	180
3.3	Historical notes	182
Chapter 4.	Fine Regularity Theory	185
4.1	Basic energy estimates	185
4.2	Sufficiency of the Wiener condition for boundary regularity	197
4.2.1	The special case of harmonic functions	204
4.3	Necessity of the Wiener condition for boundary regularity	210
4.3.1	Main estimate	211
4.3.2	Necessity of the Wiener condition	224
4.4	Equations with measure data	225
4.5	Historical notes	230
Chapter 5.	Variational Inequalities – Regularity	233
5.1	Differential operators with measurable coefficients	233
5.1.1	Continuity in the presence of irregular obstacles	234
5.1.2	The modulus of continuity	236
5.2	Differential operators with differentiable structure	238
5.3	Historical notes	251
Chapter 6.	Existence Theory	253
6.1	Existence of solutions to variational inequalities	253
6.1.1	Pseudomonotone operators	262
6.1.2	Variational problems – existence of bounded solutions	263
6.1.3	Variational problems leading to unbounded solutions	266
6.2	The Dirichlet problem for equations with differentiable structure	269
6.3	Historical notes	270
References		273
Index		283
Notation Index		289

CHAPTER 1

Preliminaries

1.1 Basic results

In this section, we use the notation Ω for an open subset of \mathbf{R}^n and X for an abstract set endowed with a measure μ . The purpose here is to review some of the well-known results of analysis that will be used throughout this book.

1.1 LEMMA. *The inequality*

$$(1.1) \quad |a + b|^p \leq \begin{cases} (1 + \varepsilon)^{p-1} |a|^p + (1 + 1/\varepsilon)^{p-1} |b|^p & \text{for } 1 \leq p < \infty \\ |a|^p + |b|^p & \text{for } 0 < p < 1 \end{cases}$$

holds for arbitrary $a, b \in \mathbf{R}$ and $\varepsilon > 0$.

PROOF. We may assume $a, b > 0$. For $p > 1$, inequality (1.1) follows from the fact that $t \mapsto t^p$ is a convex function and therefore

$$\begin{aligned} (a + b)^p &= \left[\lambda \left(\frac{a}{\lambda} \right) + (1 - \lambda) \left(\frac{b}{1 - \lambda} \right) \right]^p \\ &\leq \lambda \left(\frac{a}{\lambda} \right)^p + (1 - \lambda) \left(\frac{b}{1 - \lambda} \right)^p \end{aligned}$$

whenever $0 < \lambda < 1$. Taking $\lambda = \frac{1}{1 + \varepsilon}$ establishes the result in this case. If $0 < p < 1$ and $a, b \geq 0$, then the function $x \mapsto x^p$ dominates $x \mapsto \alpha x$ where $\alpha = (a + b)^{p-1}$. \square

1.2 LEMMA (YOUNG'S INEQUALITY). *For $a, b \geq 0$ and $1 < p < \infty$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

PROOF. Recall that $\ln(x)$ is an increasing, concave function on $(0, \infty)$ i.e.

$$\ln(\lambda x + (1 - \lambda)y) \geq \lambda \ln(x) + (1 - \lambda) \ln(y)$$

for $x, y \in (0, \infty)$ and $\lambda \in [0, 1]$.

Set $x = a^p$, $y = b^{p'}$, and $\lambda = \frac{1}{p}$ (thus $(1 - \lambda) = \frac{1}{p'}$) to obtain

$$\ln\left(\frac{1}{p}a^p + \frac{1}{p'}b^{p'}\right) \geq \frac{1}{p} \ln(a^p) + \frac{1}{p'} \ln(b^{p'}) = \ln(ab). \quad \square$$

An important application of Young's inequality is *Hölder's inequality*.

1.3 THEOREM. If $u \in L^p(X)$, and $v \in L^{p'}(X)$, $p \geq 1$, then

$$(1.2) \quad \int_X |uv| d\mu \leq \|u\|_p \|v\|_{p'}.$$

When $p = 1$, we take $p' = \infty$.

PROOF. The assertion is clear if $p = 1$. It is also clear when $1 < p < \infty$ unless $0 < \|u\|_p, \|v\|_{p'} < \infty$, in which case we define

$$\tilde{u} = \frac{u}{\|u\|_p} \quad \text{and} \quad \tilde{v} = \frac{v}{\|v\|_{p'}}.$$

Then, applying Young's inequality, we obtain

$$\frac{1}{\|u\|_p \|v\|_{p'}} \int_X |u| |v| d\mu = \int_X |\tilde{u}| |\tilde{v}| d\mu \leq \frac{1}{p} + \frac{1}{p'} = 1,$$

thus obtaining the desired result. \square

1.4 REMARK. Hölder's inequality can be extended to the case of k functions,

$$u_1 \in L^{p_1}(X), \dots, u_k \in L^{p_k}(X),$$

where

$$(1.3) \quad \sum_{i=1}^k \frac{1}{p_i} = 1.$$

By an induction argument and (1.2) it follows that

$$(1.4) \quad \int_X u_1 \dots u_k d\mu \leq \|u_1\|_{p_1} \dots \|u_k\|_{p_k}.$$

When proving that L^p spaces are normed linear spaces we need the following *Minkowski's inequality*.

1.5 THEOREM. Let $u, v \in L^p(X)$. Then $u + v \in L^p(X)$ and

$$(1.5) \quad \|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

PROOF. This is easy if $p = 1$ or $p = \infty$. If $1 < p < \infty$, we apply Hölder's inequality and get

$$\begin{aligned} \int_X |u + v|^p d\mu &\leq \int_X |u + v|^{p-1} |u| d\mu + \int_X |u + v|^{p-1} |v| d\mu \\ &\leq \left(\int_X |u + v|^p d\mu \right)^{1/p'} \left(\left(\int_X |u|^p d\mu \right)^{1/p} + \left(\int_X |v|^p d\mu \right)^{1/p} \right). \quad \square \end{aligned}$$

1.6 COROLLARY. Suppose that $1 \leq p \leq q$ and $\mu(X) < \infty$. Then

$$(1.6) \quad \left(\int_X |u|^p d\mu \right)^{1/p} \leq \left(\int_X |u|^q d\mu \right)^{1/q}.$$

The following is a simple interpolation result.

1.7 COROLLARY. Suppose $p \leq q \leq r$, and $1/q = \lambda/p + (1 - \lambda)/r$. If $u \in L^p(X) \cap L^r(X)$, then $u \in L^q(X)$ and

$$(1.7) \quad \|u\|_q \leq \|u\|_p^\lambda \|u\|_r^{1-\lambda}.$$

PROOF. Let $\alpha = \lambda q$, $\beta = (1 - \lambda)q$ and apply Hölder's inequality to obtain

$$\int_X |u|^q d\mu = \int_X |u|^\alpha |u|^\beta d\mu \leq \left(\int_X |u|^{\alpha s} d\mu \right)^{1/s} \left(\int_X |u|^{\beta t} d\mu \right)^{1/t}$$

where $s = p/\lambda q$ and $t = r/(1 - \lambda)q$. □

1.8 THEOREM. If f is nonnegative and μ -measurable on X then

$$(1.8) \quad \int_X f d\mu = \int_{[0, \infty)} \mu\{x : f(x) > t\} dt.$$

PROOF. Write

$$E_t = \{x \in X : f(x) > t\}.$$

If $\mu(E_t) = \infty$ for some $t > 0$, then obviously both sides of (1.8) equal ∞ . Otherwise $E_0 = \bigcup E_{1/k}$ with $\mu(E_{1/k}) < \infty$, so that the measure $\mu \llcorner E_0$ verifies assumptions needed for the product construction. Let λ denote the Lebesgue measure and ν the product measure $(\mu \llcorner E_0) \times \lambda$. Set

$$W = \{(x, t) : 0 < t < f(x)\} \subset X \times \mathbf{R}.$$

Since

$$W = \bigcup_{q \in \mathbf{Q}^+} \{E_q \times (0, q)\}$$

where \mathbf{Q}^+ is the set of all positive rationals, W is ν -measurable. Thus,

$$\begin{aligned} \int_{[0, \infty)} \mu(E_t) d\lambda(t) &= \int_{\mathbf{R}} \left(\int_{E_0} \chi_W(x, t) d\mu(x) \right) d\lambda(t) \\ &= \int_{E_0 \times \mathbf{R}} \chi_W(x, t) d\nu(x, t) \\ &= \int_{E_0} \lambda(\{t : 0 < t < f(x)\}) d\mu(x) \\ &= \int_X f d\mu. \end{aligned} \quad \square$$

Thus, a nonnegative measurable function f is integrable over X with respect to μ if, and only if, its distribution function $t \mapsto \mu(E_t)$ is integrable over $[0, \infty)$ with respect to one-dimensional Lebesgue measure λ .

The simple idea behind the proof of Theorem 1.8 can readily be extended as in the following theorem.