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Richard Courant, Fritz John

# Introduction to Calculus and Analysis II/1

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Richard Courant • Fritz John

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**Richard Courant** was born in 1888 in a small town of what is now Poland, and died in New Rochelle, N.Y. in 1972. He received his doctorate from the legendary David Hilbert in Göttingen, where later he founded and directed its famed Mathematics Institute, a Mecca for mathematicians in the twenties. In 1933 the Nazi government dismissed Courant for being Jewish,

and he emigrated to the United States. He found, in New York, what he called “a reservoir of talent” to be tapped. He built, at New York University, a new Mathematical Sciences Institute that shares the philosophy of its illustrious predecessor and rivals it in worldwide influence.

For Courant mathematics was an adventure, with applications forming a vital part. This spirit is reflected in his books, in particular in his influential calculus text, revised in collaboration with his brilliant younger colleague, Fritz John. *(P. D. Lax)*



**Fritz John** was born on June 14, 1910, in Berlin. After his school years in Danzig (now Gdansk, Poland), he studied in Göttingen and received his doctorate in 1933, just when the Nazi regime came to power. As he was half-Jewish and his bride Aryan, he had to flee Germany in 1934. After a year in Cambridge, UK, he accepted a position at the University of Kentucky, and in 1946 joined Courant, Friedrichs and Stoker in building up at New York

University the institute that later became the Courant Institute of Mathematical Sciences. He remained there until his death in New Rochelle on February 10, 1994.

John’s research and the books he wrote had a strong impact on the development of many fields of mathematics, foremost in partial differential equations. He also worked on Radon transforms, ill-posed problems, convex geometry, numerical analysis, elasticity theory. In connection with his work in the latter field, he and Nirenberg introduced the space of the BMO-functions (bounded mean oscillations). Fritz John’s work exemplifies the unity of mathematics as well as its elegance and its beauty. *(J. Moser)*

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Richard Courant   Fritz John

# Introduction to Calculus and Analysis

Volume II

With the assistance of  
Albert A. Blank and Alan Solomon

With 120 Illustrations



Springer

Richard Courant (1888 - 1972)

Fritz John

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## Preface

Richard Courant's *Differential and Integral Calculus*, Vols. I and II, has been tremendously successful in introducing several generations of mathematicians to higher mathematics. Throughout, those volumes presented the important lesson that meaningful mathematics is created from a union of intuitive imagination and deductive reasoning. In preparing this revision the authors have endeavored to maintain the healthy balance between these two modes of thinking which characterized the original work. Although Richard Courant did not live to see the publication of this revision of Volume II, all major changes had been agreed upon and drafted by the authors before Dr. Courant's death in January 1972.

From the outset, the authors realized that Volume II, which deals with functions of several variables, would have to be revised more drastically than Volume I. In particular, it seemed desirable to treat the fundamental theorems on integration in higher dimensions with the same degree of rigor and generality applied to integration in one dimension. In addition, there were a number of new concepts and topics of basic importance, which, in the opinion of the authors, belong to an introduction to analysis.

Only minor changes were made in the short chapters (6, 7, and 8) dealing, respectively, with Differential Equations, Calculus of Variations, and Functions of a Complex Variable. In the core of the book, Chapters 1-5, we retained as much as possible the original scheme of two roughly parallel developments of each subject at different levels: an informal introduction based on more intuitive arguments together with a discussion of applications laying the groundwork for the subsequent rigorous proofs.

The material from linear algebra contained in the original Chapter 1 seemed inadequate as a foundation for the expanded calculus structure. Thus, this chapter (now Chapter 2) was completely rewritten and now presents all the required properties of  $n$ th order determinants and matrices, multilinear forms, Gram determinants, and linear manifolds.

The new Chapter 1 contains all the fundamental properties of linear differential forms and their integrals. These prepare the reader for the introduction to higher-order exterior differential forms added to Chapter 3. Also found now in Chapter 3 are a new proof of the implicit function theorem by successive approximations and a discussion of numbers of critical points and of indices of vector fields in two dimensions.

Extensive additions were made to the fundamental properties of multiple integrals in Chapters 4 and 5. Here one is faced with a familiar difficulty: integrals over a manifold  $M$ , defined easily enough by subdividing  $M$  into convenient pieces, must be shown to be independent of the particular subdivision. This is resolved by the systematic use of the family of Jordan measurable sets with its finite intersection property and of partitions of unity. In order to minimize topological complications, only manifolds imbedded smoothly into Euclidean space are considered. The notion of "orientation" of a manifold is studied in the detail needed for the discussion of integrals of exterior differential forms and of their additivity properties. On this basis, proofs are given for the divergence theorem and for Stokes's theorem in  $n$  dimensions. To the section on Fourier integrals in Chapter 4 there has been added a discussion of Parseval's identity and of multiple Fourier integrals.

Invaluable in the preparation of this book was the continued generous help extended by two friends of the authors, Professors Albert A. Blank of Carnegie-Mellon University, and Alan Solomon of the University of the Negev. Almost every page bears the imprint of their criticisms, corrections, and suggestions. In addition, they prepared the problems and exercises for this volume.<sup>1</sup>

Thanks are due also to our colleagues, Professors K. O. Friedrichs and Donald Ludwig for constructive and valuable suggestions, and to John Wiley and Sons and their editorial staff for their continuing encouragement and assistance.

FRITZ JOHN  
New York  
September 1973

<sup>1</sup>In contrast to Volume I, these have been incorporated completely into the text; their solutions can be found at the end of the volume.

# Introduction to Calculus and Analysis

## Volume II

# Contents

## Chapter 1 *Functions of Several Variables and Their Derivatives*

- 1.1 Points and Points Sets in the Plane and in Space** 1
- a. Sequences of points. Convergence, 1   b. Sets of points in the plane, 3   c. The boundary of a set. Closed and open sets, 6   d. Closure as set of limit points, 9   e. Points and sets of points in space, 9
- 1.2 Functions of Several Independent Variables** 11
- a. Functions and their domains, 11   b. The simplest types of functions, 12   c. Geometrical representation of functions, 13
- 1.3 Continuity** 17
- a. Definition, 17   b. The concept of limit of a function of several variables, 19   c. The order to which a function vanishes, 22
- 1.4 The Partial Derivatives of a Function** 26
- a. Definition. Geometrical representation, 26   b. Examples, 32   c. Continuity and the existence of partial derivatives, 34

	d. Change of the order of differentiation, 36	
<b>1.5</b>	<b>The Differential of a Function and Its Geometrical Meaning</b>	<b>40</b>
	a. The concept of differentiability, 40	
	b. Directional derivatives, 43	
	c. Geometric interpretation of differentiability, The tangent plane, 46	
	d. The total differential of a function, 49	
	e. Application to the calculus of errors, 52	
<b>1.6</b>	<b>Functions of Functions (Compound Functions) and the Introduction of New Independent Variables</b>	<b>53</b>
	a. Compound functions. The chain rule, 53	
	b. Examples, 59	
	c. Change of independent variables, 60	
<b>1.7</b>	<b>The Mean Value Theorem and Taylor's Theorem for Functions of Several Variables</b>	<b>64</b>
	a. Preliminary remarks about approximation by polynomials, 64	
	b. The mean value theorem, 66	
	c. Taylor's theorem for several independent variables, 68	
<b>1.8</b>	<b>Integrals of a Function Depending on a Parameter</b>	<b>71</b>
	a. Examples and definitions, 71	
	b. Continuity and differentiability of an integral with respect to the parameter, 74	
	c. Interchange of integrations. Smoothing of functions, 80	
<b>1.9</b>	<b>Differentials and Line Integrals</b>	<b>82</b>
	a. Linear differential forms, 82	

b. Line integrals of linear differential forms, 85 c. Dependence of line integrals on endpoints, 92

<b>1.10 The Fundamental Theorem on Integrability of Linear Differential Forms</b>	<b>95</b>
a. Integration of total differentials, 95	
b. Necessary conditions for line integrals to depend only on the end points, 96	
c. Insufficiency of the integrability conditions, 98	
d. Simply connected sets, 102	
e. The fundamental theorem, 104	

## APPENDIX

<b>A.1. The Principle of the Point of Accumulation in Several Dimensions and Its Applications</b>	<b>107</b>
a. The principle of the point of accumulation, 107	
b. Cauchy's convergence test. Compactness, 108	
c. The Heine-Borel covering theorem, 109	
d. An application of the Heine-Borel theorem to closed sets contains in open sets, 110.	
<b>A.2. Basic Properties of Continuous Functions</b>	<b>112</b>
<b>A.3. Basic Notions of the Theory of Point Sets</b>	<b>113</b>
a. Sets and sub-sets, 113	
b. Union and intersection of sets, 115	
c. Applications to sets of points in the plane, 117.	
<b>A.4. Homogeneous functions.</b>	<b>119</b>

## Chapter 2 Vectors, Matrices, Linear Transformations

- |     |   |     |
|-----|---|-----|
| 2.1 | <b>Operations with Vectors</b>  | 122 |
|     | a. Definition of vectors, 122   |     |
|     | b. Geometric representation of vectors, 124   |     |
|     | c. Length of vectors. Angles between directions, 127  |     |
|     | d. Scalar products of vectors, 131  |     |
|     | e. Equation of hyperplanes in vector form, 133  |     |
|     | f. Linear dependence of vectors and systems of linear equations, 136                              |     |
| 2.2 | <b>Matrices and Linear Transformations</b>  | 143 |
|     | a. Change of base. Linear spaces, 143   |     |
|     | b. Matrices, 146  |     |
|     | c. Operations with matrices, 150  |     |
|     | d. Square matrices. The reciprocal of a matrix. Orthogonal matrices. 153                          |     |
| 2.3 | <b>Determinants</b>   | 159 |
|     | a. Determinants of second and third order, 159  |     |
|     | b. Linear and multilinear forms of vectors, 163   |     |
|     | c. Alternating multilinear forms. Definition of determinants, 166                                 |     |
|     | d. Principal properties of determinants, 171  |     |
|     | e. Application of determinants to systems of linear equations. 175                                |     |
| 2.4 | <b>Geometrical Interpretation of Determinants</b>   | 180 |
|     | a. Vector products and volumes of parallelepipeds in three-dimensional space, 180                 |     |
|     | b. Expansion of a determinant with respect to a column. Vector products in higher dimensions, 187 |     |
|     | c. Areas of parallelograms and volumes of parallelepipeds in                                      |     |

- higher dimensions, 190 d. Orientation of parallelepipeds in  $n$ -dimensional space, 195 e. Orientation of planes and hyperplanes, 200 f. Change of volume of parallelepipeds in linear transformations, 201
- 2.5 Vector Notions in Analysis 204**
- a. Vector fields, 204 b. Gradient of a scalar, 205 c. Divergence and curl of a vector field, 208 d. Families of vectors. Application to the theory of curves in space and to motion of particles, 211

### *Chapter 3 Developments and Applications of the Differential Calculus*

- 3.1 Implicit Functions 218**
- a. General remarks, 218 b. Geometrical interpretation, 219 c. The implicit function theorem, 221 d. Proof of the implicit function theorem, 225 e. The implicit function theorem for more than two independent variables, 228
- 3.2 Curves and Surfaces in Implicit Form 230**
- a. Plane curves in implicit form, 230 b. Singular points of curves, 236 c. Implicit representation of surfaces, 238
- 3.3 Systems of Functions, Transformations, and Mappings 241**
- a. General remarks, 241 b. Curvilinear coordinates, 246 c. Extension to more than two independent variables, 249 d. Differentiation formulae for the inverse functions,

	252 e. Symbolic product of mappings,	
	257 f. General theorem on the inversion of transformations and of systems of implicit functions. Decomposition into primitive mappings, 261	
	g. Alternate construction of the inverse mapping by the method of successive approximations, 266	
	h. Dependent functions, 268	
	i. Concluding remarks, 275	
<b>3.4</b>	<b>Applications</b>	<b>278</b>
	a. Elements of the theory of surfaces, 278	
	b. Conformal transformation in general, 289	
<b>3.5</b>	<b>Families of Curves, Families of Surfaces, and Their Envelopes</b>	<b>290</b>
	a. General remarks, 290	
	b. Envelopes of one-parameter families of curves, 292	
	c. Examples, 296	
	d. Envelopes of families of surfaces, 303	
<b>3.6</b>	<b>Alternating Differential Forms</b>	<b>307</b>
	a. Definition of alternating differential forms, 307	
	b. Sums and products of differential forms, 310	
	c. Exterior derivatives of differential forms, 312	
	d. Exterior differential forms in arbitrary coordinates, 316	
<b>3.7</b>	<b>Maxima and Minima</b>	<b>325</b>
	a. Necessary conditions, 325	
	b. Examples, 327	
	c. Maxima and minima with subsidiary conditions, 330	
	d. Proof of the method of undetermined multipliers in the simplest case, 334	
	e. Generalization of the method of undetermined multipliers, 337	
	f. Examples, 340	

## APPENDIX

<b>A.1</b>	<b>Sufficient Conditions for Extreme Values</b>	<b>345</b>
<b>A.2</b>	<b>Numbers of Critical Points Related to Indices of a Vector Field</b>	<b>352</b>
<b>A.3</b>	<b>Singular Points of Plane Curves</b>	<b>360</b>
<b>A.4</b>	<b>Singular Points of Surfaces</b>	<b>362</b>
<b>A.5</b>	<b>Connection Between Euler's and Lagrange's Representation of the motion of a Fluid</b>	<b>363</b>
<b>A.6</b>	<b>Tangential Representation of a Closed Curve and the Isoperimetric Inequality</b>	<b>365</b>

*Chapter 4 Multiple Integrals*

<b>4.1</b>	<b>Areas in the Plane</b>	<b>367</b>
	a. Definition of the Jordan measure of area, 367 b. A set that does not have an area, 370 c. Rules for operations with areas, 372	
<b>4.2</b>	<b>Double Integrals</b>	<b>374</b>
	a. The double integral as a volume, 374 b. The general analytic concept of the integral, 376 c. Examples, 379 d. Notation. Extensions. Fundamental rules, 381 e. Integral estimates and the mean value theorem, 383	
<b>4.3</b>	<b>Integrals over Regions in three and more Dimensions</b>	<b>385</b>