CTM 1

Classical Topics in Mathematics

Felix Klein

Robert Fricke

Lectures on the Theory of Elliptic Modular Functions

First Volume

椭圆模函数理论讲义

第一卷

Translated by Arthur M. DuPre

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Classical Topics in Mathematics

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Classical Topics in Mathematics

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Why should one open and read Klein-Fricke and Fricke-Klein?

Much of the most important developments of number theory in the 20th and early 21st century, especially those related to algebraic number theory, modular forms, and the connexion between special values of L-functions and arithmetic, grew out of the great discoveries made by German mathematicians in the 19th century. Klein was one of the leading figures in 19th century German mathematics, and the books by Klein and Fricke give the first systematic account of the theory of modular forms.

John Coates

1 They are classics

There is no question that almost every mathematician has heard of Felix Klein, and almost every student and mathematician in geometry has also heard of the famous Erlangen program proposed by him when he became a full professor at the age of 23.

In collaboration with his former student Robert Fricke, Klein wrote two classic books: *Vorlesungen über die Theorie der elliptischen Modulfunctionen* (Lectures on the Theory of Elliptic Modular Functions), and *Vorlesungen über die Theorie der automorphen Functionen* (Lectures on the Theory of Automorphic Functions). They will be referred to as Klein-Fricke and Fricke-Klein in the following discussion in view of the different orders of their names on the book covers.

Probably many people working in number theory, automorphic forms and discrete subgroups of Lie groups have heard both Klein-Fricke and Fricke-Klein books. On the other hand, most people working in Teichmüller theory have only heard of the book Fricke-Klein in connection with Fricke spaces.

Klein-Fricke and Fricke-Klein are classic books and have exerted their influence on mathematics. On the other hand, they unfortunately fit one definition of a classic book by Mark Twain in 1900: it is "something that everybody wants to have read and nobody wants to read". Maybe a variant describes them better: A classic book is "a book which people have heard but have never read".

There are other definitions of classic books. For example, one definition is that "a classic is a book that has never finished saying what it has to say". Another says: "The idea of a classic implies something that has continuance and consistence, and which produces unity and tradition, fashions and transmits itself, and endures."

We hope that the reader will eventually agree that the books Klein-Fricke and Fricke-Klein fit these definitions and also a definition by Goethe: "Ancient works are classical not because they are old, but because they are powerful, fresh, and healthy."

Though more people in the previous generations had cited Klein-Fricke and Fricke-Klein, recent citations and explanations of Fricke-Klein also do occur. For example, one recent major exposition of a part of the book Fricke-Klein is given in a large book [MSW].

Fricke-Klein was the first systematic exposition of the uniformization theorem for Riemann surfaces and related moduli spaces and Teichmüller spaces. Lars Ahlfors referred to the Fricke-Klein in a commentary in his collected works [Ah, p. 122] and wrote:

> An incredibly patient reader of Fricke-Klein, a two volume, 1300-page book, might have been able to discern that Fricke had anticipated Teichmüller's idea of a space $\mathcal{T}_g$ of marked Riemann surfaces of genus g with a $3g-3$ dimensional complex structure.

Lipman Bers also cited it in his papers when he discussed Teichmüller space and mapping class groups (see for example [Ber] and [BG]).

When I was working with Armand Borel on compactifications of locally symmetric spaces, he told me that the idea of adding rational boundary points (or filling in cusp points) and putting a Satake topology on the partial compactification of the upper half-plane must be in Fricke-Klein, though I did not and could not check due to the language barrier.

To many people in number theory and group theory, the famous congruence subgroup problem is natural and important. Once I asked an expert what was the exact motivation for raising and studying this problem, he said that it must be explained in Klein-Fricke. Various papers and books also cited Klein-Fricke as the first place where a non-congruent finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ appeared.

When I became interested in moduli space and Teichmüller space, the name of Fricke space appears often in books and papers (for example, [BG] [Ber] [FM] [Bes]), and some people said that Teichmüller space should be called Fricke space. I asked experts for more detail, and they always said that it must be in Fricke-Klein.

On the other hand, these books of Fricke and Klein were written in German and never translated, unlike many classical books written in German which were translated into English shortly after publication. Since the German dominance in mathematics ended with the World War II, fewer people can read German outside Germany now.

Now both Klein-Fricke and Fricke-Klein are available for the first time in English translation. As Kazhdan said, "I definitely agree that it is important to know such important works as lectures by Fricke and Klein." There are many good reasons to read these classic books. One of the best reasons is the curiosity: To see what is really inside them, to unearth unknown treasure, and to understand the origin or history of many modern topics.

For example, Kleinian groups are central in the low dimensional topology and one dimensional complex dynamics.

The question then arises: *How did it start? How was it related to the famous Felix Klein?*

Though it is not easy to name specific major results in the work of Klein, most people will agree that he was a great mathematician and one of the most influential mathematicians in the nineteenth century.

Then another question arises: *What kind of mathematics did Klein do?*

Many people have heard the sad story that Klein competed with Poincaré, and this competition ruined Klein's ability to do any more research.

Then the question is: *What did Klein and Poincaré compete exactly? Was it really important mathematics? Are they still relevant now?*

To me and hopefully to some other people, these questions and the above quote from Ahlfors give enough motivation to *open and read Fricke-Klein and Klein-Fricke.*

If these questions are not enough or irrelevant to some readers, probably a really good reason is to follow a Chinese saying: "*It is always beneficial to open any book!*"

When I was a student, I heard Shing-Tung Yau giving this advice many times, and he practiced it. (He read books rows after rows at the mathematics library at University of California at Berkeley during his graduate student days and is stilling reading all kinds of books now.)

Therefore, it is simply rewarding to open these heavy volumes written by two great mathematicians and see what is inside them. And it is also simply fun to trace out the history of some parts of mathematics we practice and love.

Some people like to say "History is now". So it is perhaps also important to heed the warning of Weyl in the preface to his famous book on Riemann surfaces [We, p. IX]:

> The younger generation is only too inclined to forget the connection of the new with the old!

In a similar spirit, Chern wrote in the foreword to the Chinese edition of Atiyah's collected works:

> No matter how refined or improved a new account is, the original papers on a subject are usually more direct and to the point. When I was young, I was benefited by the advice to read Henri Poincaré, David Hilbert, Felix Klein, Adolf Hurwitz, etc. I did better with Wihelm Blaschke, Elie Cartan and Heinz Hopf. This has also been in the Chinese tradition, when we were told to read Confucius, Han Yu in prose, and Tu Fu in poetry.

See more discussion about reading classics in the commentary by Toledo near the end of this book.

Unity of and connections between different subjects are essential characteristics of beautiful and deep mathematics. With the rapid development of mathematics, there are very few, if any, who could command all subjects of mathematics. People often say that Hilbert, Poincaré and Weyl were the last universalists. Klein should belong there too, but he was also more special since he wrote extensively and carefully with a grand vision in mind. His writing can show his unified view of mathematics. See more discussion about this in the commentary by Mazur in this book.

The biggest exposition project related to Klein was Klein's encyclopedia: A German mathematical encyclopedia published in six volumes from 1898 to 1933 with a total length of 20,000 pages.

But the writings which showed most clearly the trademark of Klein are these four volumes of Klein-Fricke and Fricke-Klein together with Klein's book on the icosahedron. Therefore, if you want to see mathematics through the eyes of Klein, open and read these books!

2 The main work of Klein and his competition with Poincaré

To understand better the books Klein-Fricke and Fricke-Klein, it is probably important to understand better the work of Klein which led to these books, and how the appearance of Poincaré influenced him and led to their competition.

No one knows the work of Klein better than himself. Let us quote from Klein in the late stage of his career when he wrote his famous book on the history of mathematics in the 19th century [Kl1, p. 325]:

> We now turn to the theory of *automorphic functions!* This is the main area of my own work, so that I am much concerned to illuminate the exposition with personal memories and to bring the report up to the time, 1882/1883, when I was prevented from working further because of illness. Since I will often have to refer to them, perhaps I may first name the three works on this subject that were composed or instigated by me:
>
> 1. Klein: *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade* [Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree] 1884.
> 2. Klein-Fricke: *Vorlesungen über die Theorie der elliptischen Modulfunctionen* [Lectures on the Theory of Elliptic Modular Functions] Vol. 1, 1890; Vol. 2, 1892.
> 3. Fricke-Klein: *Vorlesungen die Theorie der automorphen Functionen* [Lectures on the Theory of Automorphic Functions] Vol. 1, 1897; Vol. 2, 1901, 1911, 1912.
>
> Item 1 is essentially intended as a textbook; Item 2 and Item 3 have the character of monographs.

Later Klein described his interaction with Poincaré [Kl1, p. 355]:

> It is now time for me to tell of the appearance of H. Poincaré and of the personal relations which developed between us and which laid the foundations for the further development of the whole subject [of automorphic forms].
> First I must make some remarks on the terminology. When Poincaré began, his knowledge of the German work was very faulty, and what he called the groups with limit-circles "groupes fuchsiens", an undeserved name. When I made him aware of the general functions, he called them "kleinéennes". Thus a great historical confusion arose here.

Klein continued [Kl1, p. 359]:

> Now I may ... tell of my own work on this subject. ... My first note on this, which was published at New Year's 1882 .... In this paper I published that

fundamental theorem on the limit-circle case, which I above called the *Central Theorem* ... because of its special simplicity; namely, that one can, in one and only essentially only one way, uniformize any Riemann surface with $p \geq 2$ without branch points by means of automorphic functions with a limit-circle. This theorem ... was brought forth under the most difficult external circumstances....
The proof was in fact very difficult. I used the so-called *continuity method*, which compares the manifold of Riemann surfaces with a given p to the corresponding manifold of automorphic groups with limit-circles. I never doubted that the method of proof was correct, but everywhere I ran into gaps in my knowledge of function theory or in function theory itself. I could only postulate the resolution of these difficulties, which were in fact completely resolved only 30 years later (in 1912) by Koebe.
This could not stop me from stating still more general theorems ... As imperfect and unresolved as much of that paper is, the structure of the train of thought as a whole has remained, and was not even displaced by Poincaré's subsequent papers in Volumes 1, 3, 4, 5 of the then founded *Acta Mathematica.*
In fact, I was again able to precede Poincaré by a little, for my offprints were sent off at the end of November 1882; while the first issue of the *Acta*, which contains Poincaré's first paper, appeared at the beginning of December....
The price I had to pay for my work was extraordinarily high—my health completely collapsed. In the following years I had to take long leaves and to renounce all productive activity. Things did not go well again until the autumn 1884, but I have never regained my earlier level of productivity. I never returned to elaborate my earlier ideas. And later, when I was at Göttingen, I turned to extending the domain of my work and to general tasks of organizing our science. Thus one can understand that since then I have only occasionally touched on automorphic functions. My real productive activity in theoretical mathematics perished in 1882. All that has followed, insofar as it is not been purely expository, has been merely a matter of working out details.
Thus Poincaré had a free field and, until 1884, went on to publish his five great papers on the new functions.

3 Vision of Klein and his grand book project

Klein was a mathematician of vision. His Erlangen program has had changed people's perspective of geometry and of the role of transformation groups. One of the fundamental contributions of Klein and Lie is their emphasis on the important role played by group theory and their initiation of such a point of view. See [Ya] and [JP] and references there. These books by Klein and Fricke fit into his grand book project which formed his lasting contribution to mathematics.

Felix Klein's famous Erlangen program made the theory of group actions into a central part of mathematics. In the spirit of this program, Klein set out to write a grand series of books which unified many different subjects of mathematics including number theory, geometry, complex analysis, discrete subgroups, complex analysis etc.

The first book on the icosahedron and the solution of equations of the fifth degree showed close relations between three seemingly different subjects: the symmetries of the icosahedron, the solution to fifth degree algebraic equations, and the differential equation of hypergeometric functions. (See the commentary by Jeremy Gray in this book for more detail). It was translated into English in 1888, four years after its original German version was published in 1884. It was followed by two volumes on elliptic modular functions by Klein and Fricke, and two more volumes on automorphic functions by Fricke and Klein. These four volumes are vast generalizations of the first book and contain the highly original works of Poincaré and Klein on automorphic forms. They have been very influential in the development of discrete subgroups of Lie groups and automorphic forms.

In the preface to [Kl2], Klein explained relations between these books:

> The theory of Icosahedron has during the last few years obtained a place of such importance for nearly all departments of modern analysis, that it seemed expedient to publish a systematic exposition of the same. Should this prove acceptable, I propose to continue in the same course and to treat in a similar manner the subject of Elliptic Modular Functions, and the general investigations newly made of Single-valued Functions, with linear transformations into themselves. Thus a treatise of several volumes would grow, in which I should expect to promote science, at least in so far as it might introduce many to realms of modern mathematics rich in far-stretching vistas.

These five books contain many original ideas, striking examples, explicit computations and details which are not available at other places. They will be very valuable references for people at all levels and allow the reader to see the unity of mathematics through the eyes of one of the most influential mathematicians, Felix Klein. Indeed, by reading them, the reader can learn more about the unity of mathematics in a concrete situation.

For example, it is still very valuable to read his book on the icosahedron. Most modern literature concentrate on its part on singularities. Arnold[1] promoted strongly to understand better the singularities first studied by Klein in this book:

> Unexpectedly enough, the classification of the simplest singularities in all these problems turns out to be related to the Lie, Coxeter, and Weyl groups A_k, D_k, E_h to Artin's and Brieskorn's braid groups, and to the classification of the platonics in Euclidean three space.
>
> The occurrence of the diagrams A, D, E and of Coxeter groups in such different situations as the simple Lie algebra theory, the classification of simple categories of linear spaces (Gabriel, Gel'fand-Ponomarev, Roiter-Nasarova), the

[1] Quoted from page 20 of the paper *Critical Points of Smooth Functions* by Arnold in *Proceedings of the International Congress of Mathematicians*, Vancouver, 1974. See also Problem VIII posed by Arnold on page 46 of the book *Mathematical Developments Arising from Hilbert Problems*. Proceedings of Symposia in Pure Mathematics, Vol. XXVIII. American Mathematical Society, Providence, 1976.

Kodaira classification of elliptic curves degenerations, the theory of platonics, and the theory of simple singularities gives an impression of a wonderful chain of coincidences of the results of independent classifications (certain relations between some of them being known, others suspected). As we will now see, the classification of more complex singularities provides new wonderful coincidences, where Lobatchevski triangles and automorphic functions take part.

But the action of finite groups on sphere and fundamental domains discussed in this book of Klein also have had a far reaching consequence in the contemporary mathematics. Indeed, as Klein wrote [Kl1, p. 315]:

> The special subject of *group theory* extends through all of modern mathematics. As an ordering and clarifying principle, it intervenes in the most various domains.... When Lie and I then undertook to work out the significance of group theory for various domains of mathematics, ...

In the books [Kl2], Klein-Fricke and Fricke-Klein, discrete groups play an essential role. Klein made this clear on page 2 of the preface of [Kl2]:

> My indebtedness to Professor Lie dates back to the years 1869–1870, when we were spending that last period of our student-life in Berlin and Paris together in intimate comradeship. At that time we jointly conceived the scheme of investigating geometric or analytic forms susceptible of transformation by means of groups of changes. This purpose has been of directing influence in our subsequent labours, though these may have appeared to lie asunder. Whilst I primarily directed my attention to groups of discrete operations, and was thus led to the investigation of regular solids and their relations to the theory of equations, Professor Lie attacked the more recondite theory of continued groups of transformations, and therewith of differential equations.

Among these books planned by Klein, only the first one [Kl2] was written by Klein. The other two books Klein-Fricke and Fricke-Klein were written by Fricke with substantial contributions from Klein, as the reader can see from the prefaces of these books. It should be emphasized that contributions from both Klein and Fricke were substantial.

Given the grand scale and rich contents of these books, one natural question is: *What were people's reactions to these books?* Let us take a look at some reviews about them.

In a Math Review of the paper [Sl] which tried to explain the book [Kl2], Dieudonnie wrote:

> Klein's Lectures on the icosahedron, published in 1884, soon became one of the most famous books on mathematics for a half-century; no other book so forcefully put in the limelight the fundamental unity of mathematics. Klein showed that three apparently disjoint theories: the symmetries of the icosahedron (geometry), the resolution of fifth degree equations (algebra) and the differential equation of hypergeometric functions (analysis) were in fact dominated by the structure of a single object, the simple group A_5 of 60 elements. It is the group of rotations leaving the icosahedron invariant, the monodromy

group of a hypergeometric equation, and the commutator subgroup of the Galois group of the "general" fifth degree equation.

There have been many modern expositions on the book on the icosahedron, especially in many writings of Peter Slodowy.[2]

In his review of Klein-Fricke, Cole wrote [Co, p. 105]:

> The theory of the modular functions and the allied branches has been one of the chief series of investigations to which Professor Klein has devoted himself in the period of some twenty years over which his scientific activity now extends. It is characteristic of these investigations that they are not included as a subordinate part in any of the great mathematical theories heretofore commonly so recognized. Their distinctive tendency is in the direction of the combination and unification of the latter into a broader method of research. This idea has been developed by Klein to an extent and with an elaboration which has long since entitled it to recognition as an independent, and in the highest degree productive mathematical point of view.

He continued [Co, p. 109]:

> The value to the student of this breadth of treatment is simply inestimable. No one can study long under Klein without obtaining an intelligent comprehension of most of the great tendencies in mathematics. This is at present particularly desirable in view of the extreme degree of specialization which has come to prevail among mathematicians. Of all the great services which Klein has rendered to mathematics, there is none more valuable than his successful unification of its heretofore rapidly diverging branches.

In his review of the Klein-Fricke and an earlier version of the Fricke-Klein, Hutchinson wrote [Hu]:

> Among the various classes of special functions which have hitherto engaged the attention of mathematicians, that of most recent origin, and of by far the largest content (at least potentially) is the automorphic functions. This vast subject, the growth of the past quarter of a century, owes its rapid development to the genius and assiduity of the two eminent mathematicians, Klein and Poincaré, as well as to the comparatively high state of perfection of other mathematical disciplines which have been forced to contribute their assistance to this new field. Klein and Poincaré have each brought to this subject a breadth of knowledge and a corresponding wealth of ideas truly remarkable at the present day when multiplicity of interests hardly permits the investigator any other choice than to specialize within limits more or less narrow. Klein in particular has by precept and example urged the importance of a closer unity among all departments of mathematical thought, and the great advantage to

[2] For example, P. Slodowy, Platonic solids, Kleinian singularities, and Lie groups. *Algebraic geometry* pp. 102–138, Lecture Notes in Math., 1008, Springer, Berlin, 1983. And Groups and special singularities. *Singularity theory* (Trieste, 1991), 731–799, World Sci. Publ., River Edge, 1995.

> be derived from bringing to the aid of any one field the combined resources of all the others. It is in his work on the automorphic functions that he has given the most brilliant illustration of this mode of treatment. In this work he has been ably seconded by the efforts of the pupils and co-laborers he has been so fortunate as to gather about him at the Göttingen school, among whom Robert Fricke is especially deserving of the highest praise for the ability and industry which he has brought to the arduous labor of preparing for publication the volumes now before us, and for the wealth of material which he himself has contributed to the subject.
> These volumes form the sequel to, and final elaboration of the ideas contained in the volumes previously published, namely, the Ikosaeder (by Klein), and the Elliptische Modulfunctionen (by Klein and Fricke). The former of these works deals exhaustively with the finite groups of transformations on a single variable, and the functions associated with them. The two large volumes of the latter treat in a very elaborate manner the modular group, and the general ideas necessary to be followed out in the study of all such groups and their functions. One of the principal objects of this fullness of discussion in the Modulfunctionen is to pave the way for a subsequent discussion of the general theory of automorphic functions without the restraint of burdensome details. The advantage of thus disposing of preliminary details, and familiarizing the reader with many of the fundamental ideas as applied to a concrete example is clearly perceived when we observe how completely are the various elements in the present work fused into an organic whole, each part in strong and vital contact with every other part.

Among many reasons to read these classical books Klein-Fricke and Fricke-Klein, one reason is that some old ways can help people to understand better some key results, since the first intuitions are often the most direct and reasonable. For example, one of the main results in Fricke-Klein is a proof of the uniformization of Riemann surfaces. There are now many different and rigorous proofs. Klein and Poincaré used the method of continuity to prove it, which gives an intuitive and convincing argument for the validity of the theorem. More importantly, they explained motivations better than many modern books. Specifically, the moduli space of compact Riemann surfaces of genus g has the same dimension as the moduli space of compact hyperbolic surfaces of genus g. An injective map from the latter to the former suggests that every Riemann surface should come from a hyperbolic surface. This is reasonable since it is open and hence the map from any one point can be extended to nearby points. Furthermore, the fact that the proof for a result of individual Riemann surfaces involves deformation and the totality of Riemann surfaces is also appealing. This is probably the first important application of moduli spaces.

It is perhaps appropriate to end this section with a quote from Arnold [Li, p. 435] about Klein's books:

> One-half of the mathematics I know comes from the book of F. Klein *Development of Mathematics in the 19th Century.*

4 Sources of many contemporary things and unearthed treasure

Both books Klein-Fricke and Fricke-Klein contain a wealth of information. Since they were related to research of Klein and some of his contemporaries, they contain first systematic expositions of various topics. We mention some of them here.

Klein-Fricke gave the first comprehensive study of the modular group $SL(2, \mathbb{Z})$, its congruence subgroups and more general finite index subgroups. For example, it gave the first example of finite index subgroup of $SL(2, \mathbb{Z})$ which is not congruence subgroup. In other words, the congruence subgroup problem has a negative answer for $SL(2, \mathbb{Z})$. It also contains very detailed discussion of the elliptic modular function. See the commentary by Borcherds for many striking properties of the j-invariant of elliptic curves.

It seems that there are treasures to be found. As Deligne wrote, "I would be glad if someone could extract what is buried in it, especially the modular meaning of the functions Fricke and Klein considered."

In some sense, the earlier book Klein-Fricke dealt with more mature theories. But the latter book Fricke-Klein was a research monograph describing some new and ongoing works. It gave the first detailed discussion of relations between Riemann surfaces and hyperbolic metrics. For example, the main problem they tried to solve is the uniformization theorem of Riemann surfaces. To do this, they needed to develop theories of Fuchsian groups, moduli space of hyperbolic polygons (not identical to but related to the Teichmüller space of hyperbolic surfaces), moduli space of Fuchsian groups, which is also essentially the character variety of surface groups.

Consequently, the idea of Teichmüller space and mapping class group were there, but they were not clearly defined, and some proofs were not carried out in a well-organized way either.

To show that these quotients of the Poincaré disks are Riemann surfaces, automorphic functions were also developed.

Though they need moduli spaces of Riemann surfaces in order to apply the continuity method to prove the uniformization theorem, they did not really study moduli spaces of Riemann surfaces and hyperbolic surfaces for their own sake, but only used them as step-stones to prove the uniformization theorem of Riemann surfaces.

As most people working in Teichmüller theory know, Fenchel-Nielson coordinates are convenient for many purposes. For example, they give a clear and quick proof that the Teichmüller space $\mathcal{T}_g$ of Riemann surfaces of genus $g \geq 2$ is homeomorphic to the Euclidean space $\mathbb{R}^{6g-6}$. But it is not well-known that this idea of decomposing a general hyperbolic surface into simple pieces including pairs of pants, and hence essentially the Fenchel-Nielsen coordinates, was in Fricke-Klein. See the commentary by William Harvey about this and other results in Fricke-Klein and, topics in discrete subgroups which are still actively pursued now. There are also strong assertions about similar facts related to Teichmüller (or Fricke) spaces in books [FM] [ACG] without references. With this translation of the book Fricke-Klein, you can read and check for yourself.

It is perhaps helpful to put all five books [Kl2], Klein-Fricke, Fricke-Klein together. To solve quintic equations by using icosahedron invariants, Klein developed actions of discrete groups, fundamental domains for discrete groups. More precisely, in Remark

1 in [Kl1, pp. 321-322], Klein explained fundamental domains of the symmetry group of regular solids and then wrote [Kl1, p. 322]:

> Remark 1 on p. 321 leads to the consideration of other discontinuous but infinite groups of linear substitutions of one variable, and from this grows the general theory of single-valued automorphic functions.

The book Klein-Fricke gave a comprehensive study of the modular group $\mathrm{SL}(2, \mathbb{Z})$ and elliptic modular functions which serves as a model for Fuchsian groups and automorphic functions. See the commentary by Borcherds for a comprehensive summary of results on modular forms in Klein-Fricke.

It is fitting to mention that two important generalizations of $\mathrm{SL}(2, \mathbb{Z})$ were started in Fricke-Klein. The first consists of discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C})$, and more generally discrete subgroups of Lie groups, especially lattices and arithmetic subgroups of Lie groups. The currently active topic of discrete subgroups of semisimple Lie groups and locally symmetric spaces can be traced to Fricke-Klein. The second consists of mapping class groups of surfaces and their actions on Teichmüller spaces. Both generalizations have had huge impact on the geometric group theory.

The analogy between moduli spaces of Riemann surfaces and locally symmetric spaces has stimulated a lot of work in the past several decades. It culminated in work of Maryam Mirzakhan and her Fields Medal in 2014.

5 Why should one care about modular forms and automorphic forms

These books of Fricke and Klein gave systematic expositions of modular and automorphic forms for the first time, and their impact on the later development is huge.

To people not working in these subjects, one natural question is why it is important to study modular forms and automorphic forms?

The distinguished physicist C.N. Yang[3] asked me such a question on December 8, 2016: "Perhaps you can explain to me why number theorists are interested in modular functions."

It is impressive that a person of 95 years old is still so interested in one of the key subjects of the contemporary mathematics. The commentary by Borcherds gives a comprehensive summary of various special properties of modular functions, especially the j-function. For the convenience of the reader, I include my answer to Prof. Yang. Hope that a general answer from an amateur in number theory to an old lover of mathematics might be interesting to some readers outside the field of number theory.

Briefly, there are three kinds of reasons for the importance and application of modular forms and automorphic forms:

1. It has a good origin starting from problems in number theory studied by Lagrange, Gauss, Klein, Poincaré et al.

[3] Symmetry and group theory is an integral part of Prof. Yang's academic life. He received the Nobel Prize for his work with T.-D. Lee on violation of the law of parity conservation. He is also famous for the Yang-Mills theory and the Yang-Baxter equation.

2. The basic modular function, j-function, gives a natural explanation of certain number fields and plays a similar role like the exponential function for the field of rational numbers.
   Understanding all number fields was and still is the major problem in number theory, and it is basically the problem of solving algebraic equations.
3. Each modular form or more a general automorphic form gives a sequence of numbers which can be related to other problems in number theory and arithmetic algebraic geometry.
   Basically, the fact that a sequence of numbers comes this way is the essence of many problems, such as the solution by Wiles of Fermat's last theorem (conjecture), the Langlands program, and Borcherds' solution of the Monster Mooshine. These sequences of numbers can be incorporated into L-functions.

The theory of modular and automorphic forms has a distinguished and interesting (or complicated) history. In his book *Disquisitiones Arithmeticae*, Gauss studied binary quadratic forms and equivalence classes of binary quadratic forms, and related them to the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half plane $\mathbb{H}^2$: The quotient space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ classifies the equivalence classes of binary positive definite quadratic forms of determinant 1. This space is also the moduli space of compact Riemann surfaces of genus 1, and equivalently the moduli space of elliptic curves over $\mathbb{C}$. Consequently, people are interested in functions on the upper half plane $\mathbb{H}^2$ which are invariant under $\mathrm{SL}(2,\mathbb{Z})$.

Due to this modular interpretation, $\mathrm{SL}(2,\mathbb{Z})$ is called the *modular group*, and such functions are called *modular functions*. The first important such function is the j-invariant of elliptic curves. This gives a bijection between the above moduli space (or the quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$) and the complex plane $\mathbb{C}$.

But the invariance under $\mathrm{SL}(2,\mathbb{Z})$ is too strong, and people impose some other conditions under the action by adding automorphy factors. Then the theory of general modular forms for $\mathrm{SL}(2,\mathbb{Z})$ was born. One can also replace $\mathrm{SL}(2,\mathbb{Z})$ by a subgroup of finite index, in particular principal congruence subgroups, and Klein initiated a systematic study of them and detailed discussion was given in the book Klein-Fricke.

Slightly later, Poincaré and also Klein initiated the more general theory of automorphic forms, which made Poincaré first famous (Poincaré made connection with the hyperbolic geometry and applied it to the uniformization theorem of Riemann surfaces), and which also led to the competition between them.

One major result of modular forms and automorphic functions in number theory is the Kronecker-Weber theorem: every abelian extension of $\mathbb{Q}$ is contained within some cyclotomic field. People have tried to understand abelian extensions of other number fields. For imaginary quadratic fields, their extensions can be obtained by special values of the j-function. This is called Kronecker's Jugendtraum ("dream of youth"). See [VI] for a historical and systematic discussion. One major problem in number theory now is to generalize this to other number fields.

Many problems in number theory such as the number of ways an integer can be represented as sums of squares can be solved via modular forms (or rather theta functions). They are related to coefficients of modular/automorphic forms.

The basic point is that the coefficients of modular forms (and more general automorphic forms) are special and satisfy certain symmetry. This is reflected by the fact