普通高等教育“十三五”规划教材
陕西省精品资源共享课程（双语）配套教材

An Introduction to Complex Analysis

（复分析引论）

曹怀信 张建华 陈峥立 郭志华 编

科学出版社

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北京

内 容 简 介

本书是作者多年从事复变函数论双语教学经验的总结. 其内容设置完全适合我国现行高等院校(特别是师范院校)本科教学的教学目标与课时需要. 本书内容深入浅出、层次分明, 理论体系严谨、逻辑推导详尽, 强调“分析式”教学法, 在引入概念前, 加入了必要的分析与归纳总结, 然后提出相应的概念; 在提出问题之后, 进行推理分析、增加条件, 最后得到问题的答案, 并把前边的讨论总结成一个定理. 其次, 本书配有大量图形, 帮助读者直观理解相应的概念与论证思路.

本书可供高等院校(特别是师范院校)大学生作为复分析(复变函数论)课程的英文教材, 也可供相关科技人员参考.

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Preface

This book is intended to serve as a textbook for a one-term introductory course in the theory of complex analysis. It is assumed that the reader is acquainted with the real number system and all its properties. It is also assumed the reader is familiar with elementary concepts and results from calculus (sometimes, called mathematical analysis). Roughly, analysis may be characterized as the study of functions and their various generalizations by using limits. In calculus (or mathematical analysis), real-valued continuous functions of real variables are mainly discussed and complex analysis, which is also called theory of functions of one complex variable, is devoted to the study of analytic complex-valued functions of one complex variable.

As an introductory book on complex analysis, this book contains the following seven chapters. Based on the real number field, the complex number field is introduced in Chapter 1, where it is defined as the set of all pairs of real numbers with the addition and multiplication. In addition, the algebraic and geometric structures of the complex number system are surveyed here.

In Chapter 2, functions of a complex variable are discussed and a theory of differentiation is developed. The main goal of this chapter is to introduce analytic functions, which play the center role in complex analysis.

In Chapter 3, various elementary functions of a complex variable are considered, including exponential function, sine and cosine functions, and so on.

In Chapter 4, the theory of integration of complex-valued functions of a complex variable is studied, which is very important in complex analysis and becomes a powerful tool for dealing with complex functions.

In Chapter 5, series of complex numbers are considered first. With power series and Laurent series, two representations of analytic functions are obtained, which are called Taylor expansions and Laurent expansions of analytic functions. Continuity and differentiability of the sums of power series and Laurent series are studied.

In Chapter 6, isolated singular points of analytic functions are classified in terms

of Laurent expansions of the functions. Zeros of analytic functions are also discussed and the behavior of a function near isolated singular points is studied as well.

The last chapter is devoted to basic theory of conformal mappings, including conformal mappings, geometrical properties, unilateral functions, local inverse, fractional linear transformations.

In each chapter, most of the basic results are stated as propositions, theorems, or corollaries, followed by examples and exercises illustrating those results.

This book evolved from classes taught at Shaanxi Normal University, where I have been conducting courses in Complex Analysis, Real Analysis and Functional Analysis in a so-called "bilingual" way. The main purpose of teaching in this way is to improve the undergraduate student's ability to read mathematical books written in English and to complete exercises in English. Using a Chinese textbook in bilingual courses is not, however, helpful for training student's ability of English-thinking. Although there are many books on complex analysis in English editions written by foreign authors, the contents are not suitable for bilingual teaching in China. Consequently, it is necessary to write a textbook on complex analysis in English for Chinese undergraduate students. This is the main aim of compiling the present book.

Some of materials of this book are from the book named *Complex Variables and Applications* written by James Ward Brown and Ruel V. Churchill and the book named *Functions of one Complex Variables* written by John B. Conway. My heartfelt thanks should be extended to the authors of these books.

In the preparations of this book, continual interest and support have been provided by a number of people, including my colleagues and students. The following deserve special thanks: Jianhua Zhang, Zhengli Chen and Zhihua Guo, co-authors of this book, without whose cooperation this book could not have been completed. I would like to thank Professor Jianlin Li, Professor Guoxing Ji, Professor Baowei Wu and Professor Xinping Liu for their kindly encouragement. Especially, I should like to thank Professor Ming Yang for correcting some errors in English writing. Finally, I wish to thank the staff at Science Press, not only for their work for this book, but also for the publication of so many fine books on mathematics.

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I will greatly appreciate receiving suggestions, corrections, or comments from readers, teachers and students.

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July 9, 2018

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Chapter 1

Complex Number Field

In this chapter, we assume various corresponding properties of real numbers to be known and survey the algebraic and geometric structure of the complex number system.

1.1 Addition and Multiplication

Definition 1.1.1 A ***complex number*** is an ordered pair $z = (x, y)$ of two real numbers x and y; the real numbers x and y here are called the ***real*** and ***imaginary parts*** of z, respectively; and denoted by

$$\text{Re } z = x, \quad \text{Im } z = y. \tag{1.1.1}$$

The set of all complex numbers is always denoted by $\mathbb{C}$ and called the ***complex number system***.

$$\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}. \tag{1.1.2}$$

By definition, complex numbers can be interpreted as points on the real plane $\mathbb{R}^2$. Conversely, points on the plane $\mathbb{R}^2$ can be regarded as complex numbers. So, the plane $\mathbb{R}^2$ can be viewed as a plane consisting of all complex numbers and then is called the ***complex plane***. When real numbers x are displayed as points $(x, 0)$ on the ***real axis***, it is clear that the set of complex numbers includes the real numbers as a subset. Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called ***pure imaginary numbers***. The y axis is, then, referred to as the ***imaginary axis***.

Definition 1.1.2 Two complex numbers z_1 and z_2 are ***equal***, denoted by $z_1 = z_2$ if they have the same real parts and the same imaginary parts.

Thus,

$$z_1 = z_2 \text{ if and only if } \mathrm{Re}\ z_1 = \mathrm{Re}\ z_2, \quad \mathrm{Im}\ z_1 = \mathrm{Im}\ z_2.$$

Definition 1.1.3 For two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, the ***sum*** $z_1 + z_2$ and the ***product*** $z_1 z_2$ of them are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{1.1.3}$$

and

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2). \tag{1.1.4}$$

These two operations are called the ***addition*** and ***multiplication*** of complex numbers.

Note that the operations defined by equations (1.1.3) and (1.1.4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0), \quad (x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

If we identify a real x with the complex number $(x, 0)$, i.e., $x \equiv (x, 0)$, then the complex number system is a natural extension of the real number system (Figure 1-1). Thus, $\mathbb{C} \supset \mathbb{R}$, i.e., $\mathbb{R} \subset \mathbb{C}$.

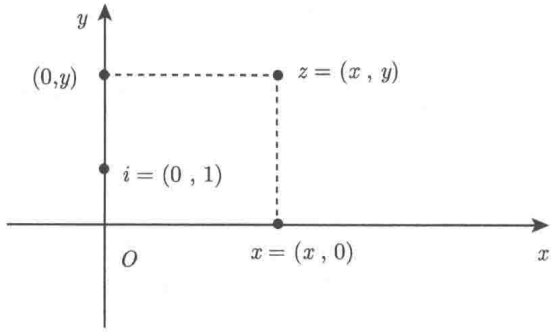

Figure 1-1

Hence, any complex number $z = (x, y)$ can be written as

$$z = (x, 0) + (0, y),$$

according to the definition of addition. According to the definition of multiplication, we have $(0, 1)(y, 0) = (0, y)$ and then by using the definition of addition again,

$$z = (x, 0) + (0, 1)(y, 0).$$

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If we put $i = (0, 1)$, called the ***imaginary unit*** (Figure 1-1), then the identity above can be written as

$$z = x + iy, \tag{1.1.5}$$

which is called the ***rectangular form*** of the number z. Thus, the complex number system can be written as

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

Also, with the convention $z^2 = zz, z^3 = zz^2$, etc., we find that

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1. \tag{1.1.6}$$

Thus, the equation $z^2 + 1 = 0$ has two roots $z = i = (0, 1)$ and $z = -i = (0, -1)$ in $\mathbb{C}$.

In view of expression (1.1.5), definitions (1.1.3) and (1.1.4) become

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \tag{1.1.7}$$

and

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2). \tag{1.1.8}$$

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left by replacing i^2 with -1 when it occurs.

1.2 Basic Algebraic Properties

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here some algebraic properties and verify some of them. Most of the others are verified in the exercises.

The ***commutative laws***:

$$z_1 + z_2 = z_2 + z_1, \quad z_1z_2 = z_2z_1 \tag{1.2.1}$$

and the ***associative laws***:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1z_2)z_3 = z_1(z_2z_3) \tag{1.2.2}$$

follow easily from the definitions (1.1.3) and (1.1.4) and the corresponding laws of real numbers.

For example, if $z_1 = (x_1, y_1)$, and $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1.$$

Verification of the rest of the above laws, as well as the ***distributive law***:

$$z(z_1 + z_2) = zz_1 + zz_2, \tag{1.2.3}$$

are similar.

According to the commutative law for multiplication, $iy = yi$. Hence, one can write $z = x + yi$ instead of $z = x + iy$. Also, because of the associative laws, a sum $z_1 + z_2 + z_3$ and a product $z_1 z_2 z_3$ are well defined without parentheses, as is the case with real numbers. Especially, for every positive integer n and every complex number z, we make the following conventions

$$nz = \overbrace{z + z + \cdots + z}^{n} \quad (n \text{ multiple of } z)$$

and

$$z^n = \overbrace{zz \cdots z}^{n} \text{ (the } n \text{ power of } z).$$

Due to the facts that

$$z + 0 = z, \quad z \cdot 1 = z, \tag{1.2.4}$$

hold for every complex number z, we call $0 = (0, 0)$ the ***additive identity*** and $1 = (1, 0)$ the ***multiplicative identity*** for complex numbers, respectively.

For each complex number $z = (x, y)$, there is an ***additive inverse:***

$$-z = (-x, -y), \tag{1.2.5}$$

satisfying the equation $z + (-z) = 0$. Additive inverses are used to define ***subtraction***:

$$z_1 - z_2 = z_1 + (-z_2), \quad \forall z_1, z_2 \in \mathbb{C}. \tag{1.2.6}$$

Thus, when $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, we have

$$z_1 - z_2 = (x_1 - x_2, y_1 - y_2) = (x_1 - x_2) + i(y_1 - y_2). \tag{1.2.7}$$

Hence, expression (1.2.5) can also be written as $-z = -x - iy$ since (Exercise 1.2.8)

$$-(iy) = (-i)y = i(-y).$$

For any *nonzero* complex number $z = (x, y) = x + iy$, there is a number z^{-1} such that $zz^{-1} = 1$, called the ***multiplicative inverse*** of z. To find it, we observe that

$$(x, y)(u, v) = (1, 0) \text{ if and only if } \begin{cases} xu - yv = 1, \\ yu + xv = 0. \end{cases}$$

Obviously, the last system of ***linear simultaneous equations*** has a unique solution

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

So the multiplicative inverse of $z = (x, y) = x + iy$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} \ (z \neq 0). \tag{1.2.8}$$

Note that the inverse z^{-1} is not defined when $z = 0$.

From the discussion above, we conclude that the set $\mathbb{C}$ of all complex numbers becomes a ***field***, called the ***field of complex numbers***, or the ***complex number field***.

Exercises 1.2

1. Verify that

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$;

(b) $(2, -3)(-2, 1) = (-1, 8)$;

(c) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

2. Show that

(a) $\text{Re}(iz) = -\text{Im } z$;

(b) $\text{Im}(iz) = \text{Re } z$.

3. Show that $(1 + z)^2 = 1 + 2z + z^2 (\forall z \in \mathbb{C})$. Generally,

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (\forall a, b \in \mathbb{C}).$$

4. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation

$$z^2 - 2z + 2 = 0.$$

5. Verify that the associative law for addition and the distributive law hold.

6. Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

7. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Ans. $z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{3}\right)$.

8. For any nonempty sets $E, F \subset \mathbb{C}$, define the following sets

$E \pm F = \{z \pm w : z \in E, w \in F\}$, $EF = \{zw : z \in E, w \in F\}$,

$E/F = \{z/w : z \in E, w \in F\}$(if $0 \notin F$), $wE = \{wz : z \in E\}(w \in \mathbb{C})$,

$w \pm E = \{w \pm z : z \in E\}(w \in \mathbb{C})$, $E^n = \{z^n : z \in E\}(n \in \mathbb{N})$.

Find four examples of E that satisfy the following conditions, respectively.

(a) $E + E \neq 2E$;

(b) $E - E \neq \{0\}$;

(c) $EE \neq E^2$;

(d) $E/E \neq \{1\}$.

9. Show that $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, $\mathbb{Z} = \mathbb{N} - \mathbb{N}$ and $\mathbb{Q} = \mathbb{Z}/\mathbb{N}$.

1.3 Further Properties

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described in Section 1.2.

We begin with the observation that the existence of multiplicative inverses enables us to show that

$$\textit{If } z_1 z_2 = 0, \textit{ then either } z_1 = 0 \textit{ or } z_2 = 0.$$

To show this, we let $z_1 z_2 = 0$ and $z_1 \neq 0$. Then inverse z_1^{-1} exists and then

$$z_2 = 1 \cdot z_2 = (z_1^{-1} z_1) z_2 = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0.$$

Equivalently, we have

$$\textit{If } z_1 \textit{ and } z_2 \textit{ are nonzero, then so is } z_1 z_2.$$

Division by a nonzero complex number is defined as follows:

$$\frac{z_1}{z_2} = z_1 z_2^{-1} \ (z_2 \neq 0). \tag{1.3.1}$$

If $z_1 = (x_1, y_1) = x_1 + iy_1$ and $z_2 = (x_2, y_2) = x_2 + iy_2$, then equation (1.3.1) and expression (1.2.8) tell us that

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \ (z_2 \neq 0). \tag{1.3.2}$$

Although expression (1.3.2) is not easy to remember, it can be obtained by writing (see Exercise 1.3.4)

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}, \tag{1.3.3}$$

multiplying out the products in the numerator and denominator on the right, and then using the property

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2)z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \ (z_3 \neq 0). \tag{1.3.4}$$

There are some expected identities, involving quotients, that follow from the relation

$$\frac{1}{z_2} = z_2^{-1} \ (z_2 \neq 0), \tag{1.3.5}$$

which is equation (1.3.1) when $z_1 = 1$. Relation (1.3.5) enables us, for example, to write equation (1.3.1) in the form

$$\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2}\right) \quad (z_2 \neq 0). \tag{1.3.6}$$

Also, by observing that (see Exercise 1.3.3)

$$(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = 1 \quad (z_2 \neq 0),$$

and hence that $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$, one can use relation (1.3.5) to show that

$$\frac{1}{z_1 z_2} = (z_1 z_2)^{-1} = z_1^{-1} z_2^{-1} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) (z_1 \neq 0, z_2 \neq 0). \tag{1.3.7}$$

Another useful identity, to be derived in the exercises, is

$$\frac{z_1 z_2}{z_3 z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) (z_3 \neq 0, z_4 \neq 0). \tag{1.3.8}$$

Exercises 1.3

1. Reduce each of these quantities to a real number:

(a) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$; (b) $\dfrac{5i}{(1-i)(2-i)(3-i)}$; (c) $(1-i)^4$.

2. Show that

(a) $(-1)z=-z$; (b) $\dfrac{1}{1/z}=z \quad (z\neq 0)$.

3. Use the associative and commutative laws for multiplication to show that

$$(z_1z_2)(z_3z_4)=(z_1z_3)(z_2z_4).$$

4. Use identity (1.3.8) in Section 1.3 to derive the ***cancellation law***:

$$\frac{z_1z}{z_2z}=\frac{z_1}{z_2} \quad (z_2\neq 0, z\neq 0).$$

5. Use mathematical induction to verify the binomial formula

$$(z_1+z_2)^n=\sum_{k=0}^{n} C_n^k z_1^{n-k}z_2^k \quad (n=1,2,\cdots)$$

is valid for any two complex numbers z_1 and z_2, where

$$C_n^k=\frac{n!}{k!(n-k)!} \quad (k=0,1,2,\cdots,n).$$

1.4 Moduli of Complex Numbers

It is natural to associate any nonzero complex number $z=x+iy$ with the directed line segment, or ***vector***, from the origin to the point (x,y) that represents z (Section 1.1) in the complex plane. In fact, we often refer to z as the point z or the vector z. In Figure 1-2, the number $z=x+iy$ and $-2+i$ are displayed graphically as both points and radius vectors.

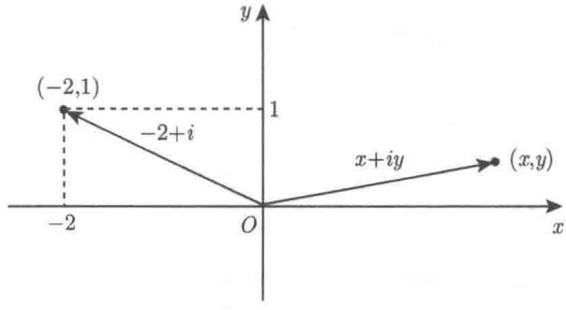

Figure 1-2

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