A SHORT COURSE IN
General Relativity

广义相对论简明教程

第 2 版

J.Foster
J.D.Nightingale

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Preface

This book is a short introduction to general relativity, intended primarily as a one-semester course for first-year graduate students (or for seniors) in physics, or in related subjects such as astrophysics. While we expect such students to have been exposed to special relativity in their introductory modern physics courses (most likely in their sophomore year) it is unlikely that they have used the standard 4-vector methods, and so we supply such a review in Appendix A. We strongly advise reading Appendix A first.

Most students approaching general relativity require an introduction to tensors, and these are dealt with in Chapter 1 and the first half of Chapter 2, where geodesics, absolute and covariant differentiation, and parallel transport are discussed. This enables us to discuss the spacetime of general relativity in the latter half of the chapter and takes us on to a discussion of the field equations in Chapter 3. In Chapter 4 the results learned are applied to physics in the vicinity of a massive object, where we have tried to compare general relativistic results with their Newtonian counterparts. Chapters 5 and 6, on gravitational radiation and the elements of cosmology, respectively, give further applications of the theory, but students wanting a more detailed knowledge of these topics (and indeed all topics) would have to turn to the texts referred to in the body of the book.

Over the years, a version of this course has been offered variously (by JDN) at the University of Mississippi (Ole Miss), at Bard College, and at SUNY New Paltz, as well as at the University of Sussex (by JF). It was often found that there was not enough time for Chapters 5 and 6, unless one made judicious cuts elsewhere. A few cuts may be made in the first two chapters, but it would probably be better to omit either Chapter 5, or Chapter 6 (or both) than to omit Appendix A, since a sound knowledge of the 4-vector formalism of special relativity is an essential prerequisite.

Exercises have been provided at the end of most sections and problems at the end of chapters. The former are often quite straightforward (but possibly tedious) verifications needed for a first reading of the book, while the latter are suitable for homework-type problems. Outline solutions can be made available to instructors upon request.

The original version of this book was published in 1979, with translations into Polish in 1985 and Japanese in 1990. That version placed mathematical demands on the reader which were not entirely appropriate for a physics student, requiring him or her to acquire mathematical skills beyond what is needed for a first course in general relativity. For the present edition, the mathematical sections have been completely reorganized and rewritten, so as to make the text more accessible to the physics student with the kind of background gained from following a course in vector calculus, with applications to field theories such as Newtonian gravitation and Maxwell's theory of electromagnetism. Much of the original 1979 material, as well as the new material for this edition, has been taught on both sides of the Atlantic in standard introductory courses.

With gratitude, mention must be made of Bob Marchini, John McNamara, John Ray, Eric Shugart, Richard Halpern, and Peter Skiff, all of whom have been of assistance in one way or another. Particular thanks are due to Tarun Biswas for contributions of examples and insights (taken from a beautifully presented course given at SUNY in the spring of 1992), for his wide-ranging expertise in theoretical physics and for assistance with the technicalities of Kermit and LaTeX. Marc Bensadoun kindly supplied the figure showing the measurements of cosmic background microwave radiation in Chapter 6 and gave us permission to reproduce it here. The original version of the book was completed with the help and encouragement of Arlene Nightingale and the exemplary typing skills of Jill Foster, whose transcription of the original text to computer files served as a foundation for its revision and conversion to LaTeX format. Finally, it is a pleasure to acknowledge the encouragement given by Tom von Foerster of Springer-Verlag and to thank him for his advice and patience while we worked on the text's revision.

David Nightingale, New Paltz
James Foster, Brighton
July 1994

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Introduction

The originator of the general theory of relativity was Einstein, and in 1919 he wrote[1]: *The special theory, on which the general theory rests, applies to all physical phenomena with the exception of gravitation; the general theory provides the law of gravitation and its relation to the other forces of nature.* The claim that the general theory provides the law of gravitation does not mean that H.G. Wells' Mr Cavor could now introduce an antigravity material and glide up to the Moon, nor, for example, that we might produce intense permanent gravitational fields in the laboratory, as we can electric fields. It means only that all the properties of gravity of which we are aware are explicable by the theory, and that gravity is essentially a matter of geometry. Before saying how we get to the general from the special theory, we must first discuss the principle of equivalence.

In electrostatics, when a test particle of charge q and inertial mass m_i is placed in a static field $\mathbf{E}$, it experiences a force $q\mathbf{E}$ and undergoes an acceleration $\mathbf{a}$ given by

$$\mathbf{a} = (q/m_i)\mathbf{E}. \tag{0.1}$$

In contrast, a test particle of gravitational mass m_g and inertial mass m_i placed in a gravitational field $\mathbf{g}$ experiences a force $m_g\mathbf{g}$ and undergoes an acceleration $\mathbf{a}$ given by

$$\mathbf{a} = (m_g/m_i)\mathbf{g}. \tag{0.2}$$

It is an experimental fact (known since Galileo's time) that different particles placed in the same gravitational field acquire the same acceleration (see Fig. 0.1). This implies that the ratio m_g/m_i appearing in equation (0.2) is the same for all particles, and by an appropriate choice of units this ratio may be taken to be unity. This equivalence of gravitational and inertial mass (which allows us to drop the qualification, and simply refer to mass) has been checked experimentally by Eötvös (in 1889 and 1922), and more recently and more accurately (to one part in 10^{11}) by Dicke and his co-workers (in the 1960s). In contrast, the ratio q/m_i occurring in equation (0.1) is not the same for all particles (see Fig. 0.1).

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[1] *The Times*, London, 28 November 1919.

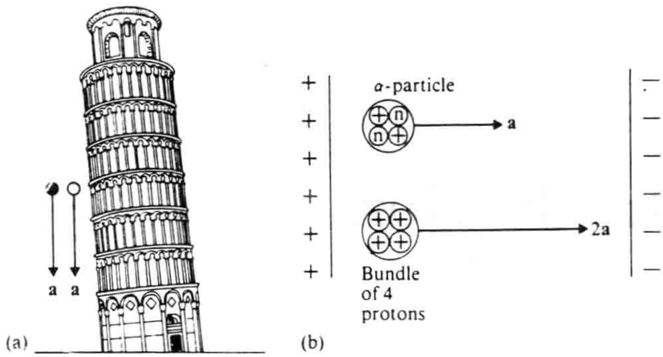

FIGURE 0.1. Test particles in (a) a gravitational field, and (b) an electrostatic field.

Let us now consider the principle of equivalence, which it is instructive to do from the point of view of Einstein's freely falling elevator. If we consider a projectile shot from one side of the elevator cabin to the other, the projectile appears to go in a straight line (the elevator cable being cut) rather than in the usual curved trajectory. Projectiles that are released from rest relative to the cabin remain floating weightless in the cabin. Of course, if the cabin is left to fall for a long time, the particles gradually draw closer together, since they are falling down radial lines towards a common point which is the center of the Earth. However, if we make the proviso that the cabin is in this state for a short time, as well as being spatially small enough for the neglect of tidal forces in general, then the freely falling cabin (which may have X, Y, Z coordinates chalked on its walls, as well as a cabin clock measuring time T) looks remarkably like an inertial frame of reference, and therefore the laws of special relativity hold sway inside the cabin. (The cabin must not only occupy a small region of spacetime, it must also be nonrotating with respect to distant matter in the universe.[2]) All this follows from the fact that the acceleration of any particle relative to the cabin is zero because they both have the same acceleration relative to the Earth, and we see that the equivalence of inertial and gravitational mass is an essential feature of the discussion. We may incorporate these ideas into the *principle of equivalence*, which is this: *In a freely falling (nonrotating) laboratory occupying a small region of spacetime, the laws of physics are the laws of special relativity.*[3]

[2]This statement is related to *Mach's principle*. For a discussion, see Weinberg, 1972, §3, and Anderson, 1967, Chap. 10.

[3]Some authors distinguish between weak and strong equivalence. Our statement is the strong statement; the weak one refers to freely falling particles only, and not to the whole of physics.

As a result of the above discussion, the reader should not believe that we can actually transform gravity away by turning to a freely falling reference frame. It is absolutely impossible to transform away a permanent gravitational field of the type associated with a star (as we shall see in Chap. 3), but it is possible to get closer and closer to an ideal inertial reference frame if we make our laboratory occupy smaller and smaller regions of spacetime.

The way in which Einstein generalized the special theory so as to incorporate gravitation was extremely ingenious, and without precedent in the history of science. Gravity was no longer to be regarded as a force, but as a manifestation of the curvature of spacetime itself. The new theory, known as the *general theory of relativity* (or *general relativity* for short), yields the special theory as an approximation in exactly the way that the principle of equivalence requires. Because of the curvature of spacetime, it cannot be formulated in terms of coordinate systems based on inertial frames, as the special theory can, and we therefore use arbitrary coordinate systems. Indeed, global inertial frames can no longer be defined, the nearest we can get to them being freely falling nonrotating frames valid in limited regions of spacetime only. A full explanation of what is involved is given in Chapter 2, but we can give a limited preview here.

In special relativity, the invariant expression which defines the proper time τ is given by

$$c^2 d\tau^2 = \eta_{\mu\nu} dX^\mu dX^\nu, \tag{0.3}$$

where the four coordinates X^0, X^1, X^2, X^3 are given in terms of the usual coordinates T, X, Y, Z by

$$X^0 \equiv cT, \quad X^1 \equiv X, \quad X^2 \equiv Y, \quad X^3 \equiv Z. \tag{0.4}$$

(See Sec. A.0, but note the change to capital letters. See also Sec. 1.2 for an explanation of the summation convention.) If we change to arbitrary coordinates x^μ, which may be defined in terms of the X^μ in any way whatsoever (they may, e.g., be linked to an accelerating or rotating frame), then expression (0.3) takes the form

$$c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{0.5}$$

where

$$g_{\mu\nu} = \eta_{\rho\sigma} \frac{\partial X^\rho}{\partial x^\mu} \frac{\partial X^\sigma}{\partial x^\nu}.$$

This follows from the fact that $dX^\rho = (\partial X^\rho / \partial x^\mu) dx^\mu$. In terms of the coordinates X^μ, the equation of motion of a free particle is

$$d^2 X^\mu / d\tau^2 = 0, \tag{0.6}$$

which, in terms of the arbitrary coordinates, becomes

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0, \tag{0.7}$$

where

$$\Gamma^{\mu}_{\nu\sigma} = \frac{\partial x^{\mu}}{\partial X^{\rho}} \frac{\partial^2 X^{\rho}}{\partial x^{\nu} \partial x^{\sigma}},$$

as a short calculation (involving the chain rule) shows. Einstein's proposals for the general theory were that in any coordinate system the proper time should be given by an expression of the form (0.5) and that the equation of motion of a free particle (i.e., one moving under the influence of gravity alone, gravity no longer being a force) should be given by an expression of the form (0.7), but that (in contrast to the spacetime of special relativity) *there are no preferred coordinates* X^{μ} which will reduce these to the forms (0.3) and (0.6). This is the essential difference between the spacetimes of special and general relativity. The curvature of spacetime (and therefore gravity) is carried by the $g_{\mu\nu}$, and as we shall see, there is a sense in which these quantities may be regarded as gravitational potentials. We shall also see that the $\Gamma^{\mu}_{\nu\sigma}$ are determined by the $g_{\mu\nu}$, and that it is always possible to introduce *local* inertial coordinate systems of *limited extent* in which $g_{\mu\nu} \approx \eta_{\mu\nu}$ and $\Gamma^{\mu}_{\nu\sigma} \approx 0$, so that equations (0.3) and (0.6) hold as approximations. We thus recover special relativity as an approximation, and in a way which ties in with our discussion of the principle of equivalence.

Because the introduction of curvature forces us to use arbitrary coordinate systems, we need to formulate the theory in a way which is valid in all coordinate systems. This we do by using tensor fields, the mathematics of which is developed in Chapter 1; the way these fit into the theory is explained in Chapter 2. It might be thought that this arbitrariness causes problems, because the coordinates lose the simple physical meanings that the preferred coordinates X^{μ} of special relativity have. However, we still have contact with the special theory at the local level, and in this way problems of physical meaning and the correct formulation of equations may be overcome. The basic idea is contained in the principle of general covariance, which may be stated as follows: *A physical equation of general relativity is generally true in all coordinate systems if (a) the equation is a tensor equation (i.e., it preserves its form under general coordinate transformations), and (b) the equation is true in special relativity.* The way in which this principle works and the reason why it works are explained in Section 2.5.

General relativity should not only reduce to special relativity in the appropriate limit, it should also yield Newtonian gravitation as an approximation. Contacts and comparisons with Newtonian theory are made in Sections 2.6, 2.7, 2.8, and 2.9, and extensively in Chapter 4, where we discuss physics in the vicinity of a massive object. These reveal differences between the two theories which provide possible experimental tests of the general theory, and for convenience we list here the experimental and observational evidence concerning these tests, the so-called five tests of general relativity.

1. *Perihelion advance.* General relativity predicts an anomalous advance

of the perihelion of planetary orbits. The following (and many more) observations exist for the solar system[4]:

Mercury	$43.11 \pm 0.45''$ per century,
Venus	$8.4 \pm 4.8''$ per century,
Earth	$5.0 \pm 1.2''$ per century.

The predicted values are $43.03''$, $8.6''$, and $3.8''$, respectively.

2. *Deflection of light.* General relativity predicts that light deviates from rectilinear motion near massive objects. The following (and many more) observed deflections exist for light passing the Sun at grazing incidence:

1919	Greenwich Observatory	$1.98 \pm 0.16''$,
1922	Lick Observatory	$1.82 \pm 0.20''$,
1947	Yerkes Observatory	$2.01 \pm 0.27''$,
1972	Mullard Radio Observatory, Cambridge (using radio sources and interferometers)	$1.82 \pm 0.14''$.

The predicted value is $1.75''$.

3. *Spectral shift.* General relativity predicts that light emanating from near a massive object is red-shifted, while light falling towards a massive object is blue-shifted. Numerous observations of the spectra of white dwarfs, as well as the remarkable terrestrial experiments carried out at the Jefferson Laboratory[5] verify the general-relativistic prediction.

4. *Time delay in radar sounding.* General relativity predicts a time delay in radar sounding due to the gravitational field of a massive object. Experiments involving the radar sounding of Venus, Mercury, and the spacecrafts *Mariner 6* and *7*, performed in the 1960s and 1970s, have yielded agreement with the predicted values to well within the experimental uncertainties.[6]

5. *Geodesic effect.* General relativity predicts that the axis of a gyroscope which is freely orbiting a massive object should precess. For a gyroscope in a near-Earth orbit this precession amounts to $8''$ per year, and an experiment involving a gyroscope in an orbiting satellite has been planned to verify this.[7]

While all the above effects are small for our solar system, some larger, and presumably general-relativistic, effects have been observed since 1975 for

[4]The figures are taken from Duncombe, 1956.

[5]See Pound and Rebka, 1960.

[6]See Shapiro, 1968; Shapiro et al., 1971; and Anderson et al., 1975.

[7]See Sec. 4.7.

the two mutually orbiting neutron stars PSR 1913+16. However, Einstein's theory of general relativity does not directly treat *two* such massive objects, and Chapter 4 (on physics in the vicinity of a massive object) looks only at the motions of a test particle in the field of *one* massive object.

Finally, let us say something about the notation used in this book. Wherever possible we have chosen it to coincide with that of the more recent and influential texts on general relativity. For working in spacetime, we use Greek suffixes (μ, ν, etc.) and these have the range 0, 1, 2, 3, while for three-dimensional space we use lower-case English suffixes from the middle of the alphabet (i, j, etc.) and these have the range 1, 2, 3. For working on a two-dimensional surface, we use upper-case English suffixes from the beginning of the alphabet (A, B, etc.) and these have the range 1, 2. The signature of the metric tensor is -2, which means that $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$. Rather than use gravitational units in which the gravitational constant G and the speed of light c are unity, we have retained G and c throughout, except in Chapter 6 where $c = 1$. In the sections dealing with general tensor fields and curvature, the underlying space or manifold is of arbitrary dimension, and we have used lower-case English suffixes from the beginning of the alphabet (a, b, etc.) to denote the arbitrary range $1, 2, \ldots, N$. Where an equation defines some quantity or operation, the symbol $\equiv$ is used on its first occurrence, and occasionally thereafter as a reminder. Important equations are displayed in boxes.

1

Vector and tensor fields

1.0 Introduction

In this first chapter we concentrate on the algebra of vector and tensor fields, while postponing ideas that are based on the calculus of fields to Chapter 2. Our starting point is a consideration of vector fields in the familiar setting of three-dimensional Euclidean space and how they can be handled using arbitrary curvilinear coordinate systems. We then go on to extend and generalize these ideas in two different ways, first by admitting tensor fields, and second by allowing the dimension of the space to be arbitrary and its geometry to be non-Euclidean.[1] The eventual goal is to present a model for the spacetime of general relativity as a four-dimensional space that is curved, rather than flat. While some aspects of this model emerge in this chapter, it is more fully developed in Chapters 2 and 3, where we introduce some more mathematical apparatus and relate it to the physics of gravitation.

1.1 Coordinate systems in Euclidean space

In this and the next five sections we shall be working in three-dimensional Euclidean space. We shall take it to be equipped with a Cartesian system of coordinates (x, y, z) and an associated set of unit vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, each pointing in the direction of the corresponding coordinate axis. We shall regard this Cartesian setup as a fixed and permanent feature of our Euclidean space; its purpose is to serve as a basic reference system for the description of other (generally non-Cartesian) coordinate systems.

Suppose then that we have an alternate coordinate system (u, v, w) that is non-Cartesian, such as spherical coordinates (r, θ, ϕ), as in the example below. We can express the Cartesian coordinates x, y, z in terms of u, v, w,

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w), \tag{1.1}$$

and, in principle, invert these to get u, v, w in terms of x, y, z. Through any point P with coordinates (u_0, v_0, w_0) there pass three *coordinate surfaces*,

[1]We use the term *non-Euclidean* simply to mean *not Euclidean*. Mathematicians sometimes restrict the term to describe the geometries that arise as a result of modifying Euclid's parallel postulate.

given by $u = u_0$, $v = v_0$, and $w = w_0$, which meet in *coordinate curves*. The following example serves to illustrate these ideas.

Example 1.1.1
For spherical coordinates we have

$$x = r\sin\theta\,\cos\phi, \quad y = r\sin\theta\,\sin\phi, \quad z = r\cos\theta, \tag{1.2}$$

where the conventional ranges for the coordinates are[2]

$$r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi.$$

The coordinate surface $r = r_0$ is a sphere of radius r_0 (because $x^2+y^2+z^2 = r_0^2$), the coordinate surface $\theta = \theta_0$ is an infinite cone with its vertex at the origin and its axis vertical, and the coordinate surface $\phi = \phi_0$ is a semi-infinite plane with the z axis as its edge. (See Fig. 1.1.)

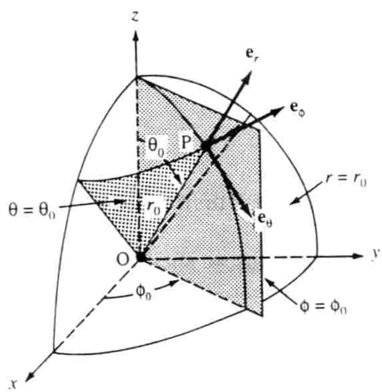

FIGURE 1.1. The coordinate surfaces and coordinate curves of spherical coordinates.

The surfaces $\theta = \theta_0$ and $\phi = \phi_0$ intersect to give a coordinate curve which is a ray (part of a line) that emanates from O and passes through P; the surfaces $\phi = \phi_0$ and $r = r_0$ intersect to give a coordinate curve which is a semicircle having its endpoints on the z axis and passing through P; and the surfaces $r = r_0$ and $\theta = \theta_0$ intersect to give a coordinate curve which is a horizontal circle passing through P with its center on the z axis.

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[2] In practice, one usually lets ϕ wrap around and take all values, so that the ϕ coordinate of a point is unique only up to multiples of 2π.