

Classics in Mathematics

Kôsaku Yosida (吉田耕作)

# Functional Analysis

*Sixth Edition*

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Kôsaku Yosida

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Sixth Edition



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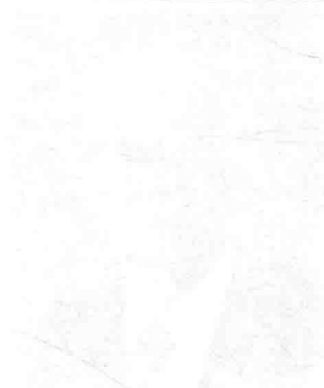
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Classics in Mathematics

Kôzaku Yosida    Functional Analysis

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Kôzaku Yosida (7. 2. 1909–20. 6. 1990) was born in Hiroshima, Japan. After studying mathematics at the University of Tokyo he held posts at Osaka and Nagoya Universities before returning to the University of Tokyo in 1955.

Yosida obtained important and fundamental results in functional analysis and probability. He is best remembered for his joint work with E. Hille which brought forth a theory of semigroups of operators successfully applied to diffusion equations, Markov processes, hyperbolic equations and potential theory. His famous textbook on functional analysis was published in 6 distinct editions between 1965 and 1980.

## Preface to the First Edition

The present book is based on lectures given by the author at the University of Tokyo during the past ten years. It is intended as a textbook to be studied by students on their own or to be used in a course on Functional Analysis, i.e., the general theory of linear operators in function spaces together with salient features of its application to diverse fields of modern and classical analysis.

Necessary prerequisites for the reading of this book are summarized, with or without proof, in Chapter 0 under titles: Set Theory, Topological Spaces, Measure Spaces and Linear Spaces. Then, starting with the chapter on Semi-norms, a general theory of Banach and Hilbert spaces is presented in connection with the theory of generalized functions of S. L. SOBOLEV and L. SCHWARTZ. While the book is primarily addressed to graduate students, it is hoped it might prove useful to research mathematicians, both pure and applied. The reader may pass, e.g., from Chapter IX (Analytical Theory of Semi-groups) directly to Chapter XIII (Ergodic Theory and Diffusion Theory) and to Chapter XIV (Integration of the Equation of Evolution). Such materials as "Weak Topologies and Duality in Locally Convex Spaces" and "Nuclear Spaces" are presented in the form of the appendices to Chapter V and Chapter X, respectively. These might be skipped for the first reading by those who are interested rather in the application of linear operators.

In the preparation of the present book, the author has received valuable advice and criticism from many friends. Especially, Mrs. K. HILLE has kindly read through the manuscript as well as the galley and page proofs. Without her painstaking help, this book could not have been printed in the present style in the language which was not spoken to the author in the cradle. The author owes very much to his old friends, Professor E. HILLE and Professor S. KAKUTANI of Yale University and Professor R. S. PHILLIPS of Stanford University for the chance to stay in their universities in 1962, which enabled him to polish the greater part of the manuscript of this book, availing himself of their valuable advice. Professor S. ITO and Dr. H. KOMATSU of the University of Tokyo kindly assisted the author in reading various parts

of the galley proof, correcting errors and improving the presentation. To all of them, the author expresses his warmest gratitude.

Thanks are also due to Professor F. K. SCHMIDT of Heidelberg University and to Professor T. KATO of the University of California at Berkeley who constantly encouraged the author to write up the present book. Finally, the author wishes to express his appreciation to Springer-Verlag for their most efficient handling of the publication of this book.

Tokyo, September 1964

KÔSAKU YOSIDA

### **Preface to the Second Edition**

In the preparation of this edition, the author is indebted to Mr. FLORET of Heidelberg who kindly did the task of enlarging the Index to make the book more useful. The errors in the second printing are corrected thanks to the remarks of many friends. In order to make the book more up-to-date, Section 4 of Chapter XIV has been rewritten entirely for this new edition.

Tokyo, September 1967

KÔSAKU YOSIDA

### **Preface to the Third Edition**

A new Section (9. Abstract Potential Operators and Semi-groups) pertaining to G. HUNT's theory of potentials is inserted in Chapter XIII of this edition. The errors in the second edition are corrected thanks to kind remarks of many friends, especially of Mr. KLAUS-DIETER BIERSTEDT.

Kyoto, April 1971

KÔSAKU YOSIDA

### **Preface to the Fourth Edition**

Two new Sections "6. Non-linear Evolution Equations 1 (The Kōmura-Kato Approach)" and "7. Non-linear Evolution Equations 2 (The Approach Through The Crandall-Liggett Convergence Theorem)" are added to the last Chapter XIV of this edition. The author is grateful to Professor Y. Kōmura for his careful reading of the manuscript.

Tokyo, April 1974

KÔSAKU YOSIDA

### **Preface to the Fifth Edition**

Taking advantage of this opportunity, supplementary notes are added at the end of this new edition and additional references to books have been entered in the bibliography. The notes are divided into two categories. The first category comprises two topics: the one is concerned with the time reversibility of Markov processes with invariant measures, and the other is concerned with the uniqueness of the solution of time dependent linear evolution equations. The second category comprises those lists of recently published books dealing respectively with Sobolev Spaces, Trace Operators or Generalized Boundary Values, Distributions and Hyperfunctions, Contraction Operators in Hilbert Spaces, Choquet's Refinement of the Krein-Milman Theorem and Linear as well as Non-linear Evolution Equations.

A number of minor errors and a serious one on page 459 in the fourth edition have been corrected. The author wishes to thank many friends who kindly brought these errors to his attention.

Kamakura, August 1977

KÔSAKU YOSIDA

### **Preface to the Sixth Edition**

Two major changes are made to this edition. The first is the rewriting of the Chapter VI,6 to give a simplified presentation of Mikusiński's Operational Calculus in such a way that this presentation does not appeal to Titchmarsh's theorem. The second is the rewriting of the Lemma together with its Proof in the Chapter XII,5 concerning the Representation of Vector Lattices. This rewriting is motivated by a letter of Professor E. Coimbra of Universidad Nova de Lisboa kindly suggesting the author's careless phrasing in the above Lemma of the preceding edition.

A number of misprints in the fifth edition have been corrected thanks to kind remarks of many friends.

Kamakura, June 1980

KÔSAKU YOSIDA

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by Kôzaku Yosida

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## 0. Preliminaries

It is the purpose of this chapter to explain certain notions and theorems used throughout the present book. These are related to *Set Theory*, *Topological Spaces*, *Measure Spaces* and *Linear Spaces*.

### 1. Set Theory

**Sets.**  $x \in X$  means that  $x$  is a *member* or *element* of the set  $X$ ;  $x \notin X$  means that  $x$  is not a member of the set  $X$ . We denote the set consisting of all  $x$  possessing the property  $P$  by  $\{x; P\}$ . Thus  $\{y; y = x\}$  is the set  $\{x\}$  consisting of a single element  $x$ . The *void set* is the set with no members, and will be denoted by  $\emptyset$ . If every element of a set  $X$  is also an element of a set  $Y$ , then  $X$  is said to be a *subset* of  $Y$  and this fact will be denoted by  $X \subseteq Y$ , or  $Y \supseteq X$ . If  $\mathfrak{X}$  is a set whose elements are sets  $X$ , then the set of all  $x$  such that  $x \in X$  for some  $X \in \mathfrak{X}$  is called the *union* of sets  $X$  in  $\mathfrak{X}$ ; this union will be denoted by  $\bigcup_{X \in \mathfrak{X}} X$ . The *intersection* of the sets  $X$  in  $\mathfrak{X}$  is the set of all  $x$  which are elements of every  $X \in \mathfrak{X}$ ; this intersection will be denoted by  $\bigcap_{X \in \mathfrak{X}} X$ . Two sets are *disjoint* if their intersection is void. A family of sets is disjoint if every pair of distinct sets in the family is disjoint. If a sequence  $\{X_n\}_{n=1,2,\dots}$  of sets is a disjoint family, then the union  $\bigcup_{n=1}^{\infty} X_n$  may be written in the form of a sum  $\sum_{n=1}^{\infty} X_n$ .

**Mappings.** The term *mapping*, *function* and *transformation* will be used synonymously. The symbol  $f: X \rightarrow Y$  will mean that  $f$  is a single-valued function whose *domain* is  $X$  and whose *range* is contained in  $Y$ ; for every  $x \in X$ , the function  $f$  assigns a uniquely determined element  $f(x) = y \in Y$ . For two mappings  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ , we can define their *composite mapping*  $gf: X \rightarrow Z$  by  $(gf)(x) = g(f(x))$ . The symbol  $f(M)$  denotes the set  $\{f(x); x \in M\}$  and  $f(M)$  is called the *image* of  $M$  under the mapping  $f$ . The symbol  $f^{-1}(N)$  denotes the set  $\{x; f(x) \in N\}$  and  $f^{-1}(N)$  is called the *inverse image* of  $N$  under the mapping  $f$ . It is clear that

$$Y_1 = f(f^{-1}(Y_1)) \text{ for all } Y_1 \subseteq f(X), \text{ and } X_1 \subseteq f^{-1}(f(X_1)) \text{ for all } X_1 \subseteq X.$$

If  $f: X \rightarrow Y$ , and for each  $y \in f(X)$  there is only one  $x \in X$  with  $f(x) = y$ , then  $f$  is said to have an *inverse* (mapping) or to be *one-to-one*. The inverse mapping then has the domain  $f(X)$  and range  $X$ ; it is defined by the equation  $x = f^{-1}(y) = f^{-1}(\{y\})$ .

The domain and the range of a mapping  $f$  will be denoted by  $D(f)$  and  $R(f)$ , respectively. Thus, if  $f$  has an inverse then

$$f^{-1}(f(x)) = x \text{ for all } x \in D(f), \text{ and } f(f^{-1}(y)) = y \text{ for all } y \in R(f).$$

The function  $f$  is said to map  $X$  *onto*  $Y$  if  $f(X) = Y$  and *into*  $Y$  if  $f(X) \subseteq Y$ . The function  $f$  is said to be an *extension* of the function  $g$  and  $g$  a *restriction* of  $f$  if  $D(f)$  contains  $D(g)$ , and  $f(x) = g(x)$  for all  $x$  in  $D(g)$ .

### Zorn's Lemma

**Definition.** Let  $P$  be a set of elements  $a, b, \dots$ . Suppose there is a binary relation defined between certain pairs  $(a, b)$  of elements of  $P$ , expressed by  $a < b$ , with the properties:

$$\begin{cases} a < a, \\ \text{if } a < b \text{ and } b < a, \text{ then } a = b, \\ \text{if } a < b \text{ and } b < c, \text{ then } a < c \text{ (transitivity)}. \end{cases}$$

Then  $P$  is said to be *partially ordered* (or *semi-ordered*) by the relation  $<$ .

**Examples.** If  $P$  is the set of all subsets of a given set  $X$ , then the set inclusion relation ( $A \subseteq B$ ) gives a partial ordering of  $P$ . The set of all complex numbers  $z = x + iy$ ,  $w = u + iv, \dots$  is partially ordered by defining  $z < w$  to mean  $x \leq u$  and  $y \leq v$ .

**Definition.** Let  $P$  be a partially ordered set with elements  $a, b, \dots$ . If  $a < c$  and  $b < c$ , we call  $c$  an *upper bound* for  $a$  and  $b$ . If furthermore  $c < d$  whenever  $d$  is an upper bound for  $a$  and  $b$ , we call  $c$  the *least upper bound* or the *supremum* of  $a$  and  $b$ , and write  $c = \sup(a, b)$  or  $a \vee b$ . This element of  $P$  is unique if it exists. In a similar way we define the *greatest lower bound* or the *infimum* of  $a$  and  $b$ , and denote it by  $\inf(a, b)$  or  $a \wedge b$ . If  $a \vee b$  and  $a \wedge b$  exist for every pair  $(a, b)$  in a partially ordered set  $P$ ,  $P$  is called a *lattice*.

**Example.** The totality of subsets  $M$  of a fixed set  $B$  is a lattice by the partial ordering  $M_1 < M_2$  defined by the set inclusion relation  $M_1 \subseteq M_2$ .

**Definition.** A partially ordered set  $P$  is said to be *linearly ordered* (or *totally ordered*) if for every pair  $(a, b)$  in  $P$ , either  $a < b$  or  $b < a$  holds. A subset of a partially ordered set is itself partially ordered by the relation which partially orders  $P$ ; the subset might turn out to be linearly ordered by this relation. If  $P$  is partially ordered and  $S$  is a subset of  $P$ , an  $m \in P$  is called an *upper bound* of  $S$  if  $s < m$  for every  $s \in S$ . An  $m \in P$  is said to be *maximal* if  $p \in P$  and  $m < p$  together imply  $m = p$ .

**Zorn's Lemma.** Let  $P$  be a non-empty partially ordered set with the property that every linearly ordered subset of  $P$  has an upper bound in  $P$ . Then  $P$  contains at least one maximal element.

It is known that Zorn's lemma is equivalent to Zermelo's axiom of choice in set theory.

## 2. Topological Spaces

### Open Sets and Closed Sets

**Definition.** A system  $\tau$  of subsets of a set  $X$  defines a *topology* in  $X$  if  $\tau$  contains the void set, the set  $X$  itself, the union of every one of its subsystems, and the intersection of every one of its finite subsystems. The sets in  $\tau$  are called the *open sets* of the *topological space*  $(X, \tau)$ ; we shall often omit  $\tau$  and refer to  $X$  as a topological space. Unless otherwise stated, we shall assume that a topological space  $X$  satisfies *Hausdorff's axiom of separation*:

For every pair  $(x_1, x_2)$  of distinct points  $x_1, x_2$  of  $X$ , there exist disjoint open sets  $G_1, G_2$  such that  $x_1 \in G_1, x_2 \in G_2$ .

A *neighbourhood* of the point  $x$  of  $X$  is a set containing an open set which contains  $x$ . A neighbourhood of the subset  $M$  of  $X$  is a set which is a neighbourhood of every point of  $M$ . A point  $x$  of  $X$  is an *accumulation point* or *limit point* of a subset  $M$  of  $X$  if every neighbourhood of  $x$  contains at least one point  $m \in M$  different from  $x$ .

**Definition.** Any subset  $M$  of a topological space  $X$  becomes a topological space by calling "open" the subsets of  $M$  which are of the form  $M \cap G$  where  $G$ 's are open sets of  $X$ . The induced topology of  $M$  is called the *relative topology* of  $M$  as a subset of the topological space  $X$ .

**Definition.** A set  $M$  of a topological space  $X$  is *closed* if it contains all its accumulation points. It is easy to see that  $M$  is closed iff<sup>1</sup> its *complement*  $M^c = X - M$  is open. Here  $A - B$  denotes the totality of points  $x \in A$  not contained in  $B$ . If  $M \subseteq X$ , the intersection of all closed subsets of  $X$  which contain  $M$  is called the *closure* of  $M$  and will be denoted by  $M^a$  (the superscript "a" stands for the first letter of the German: abgeschlossene Hülle).

Clearly  $M^a$  is closed and  $M \subseteq M^a$ ; it is easy to see that  $M = M^a$  iff  $M$  is closed.

### Metric Spaces

**Definition.** If  $X, Y$  are sets, we denote by  $X \times Y$  the set of all ordered pairs  $(x, y)$  where  $x \in X$  and  $y \in Y$ ;  $X \times Y$  will be called the *Cartesian product* of  $X$  and  $Y$ .  $X$  is called a *metric space* if there is defined a func-

<sup>1</sup> iff is the abbreviation for "if and only if".