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C^* -Algebras and Finite-Dimensional Approximations

C^* -代数和有限维逼近

Nathanial P. Brown

Narutaka Ozawa



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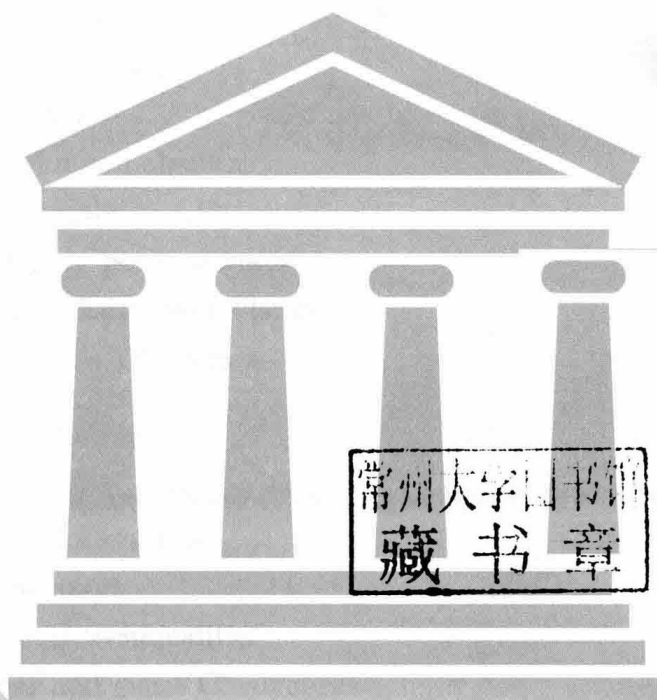
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出版者的话

近年来,我国的科学技术取得了长足进步,特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时,国内的科研队伍与国外的交流合作也越来越密切,越来越多的科研工作者可以熟练地阅读英文文献,并在国际顶级期刊发表英文学术文章,在国外出版社出版英文学术著作。

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我们希望这套书的出版,能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用,也希望今后能有更多的海外优秀英文著作被介绍到中国。

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Preface

This is a book about C^* -algebras, various types of approximation, and a few of the surprising applications that have been recently discovered. In short, we will study *approximation theory in the context of operator algebras*.

Approximation is ubiquitous in mathematics; when the object of interest cannot be studied directly, we approximate by tractable relatives and pass to a limit. In our context this is particularly important because C^* -algebras are (almost always) infinite dimensional and we can say precious little without the help of approximation theory. Moreover, most concrete examples enjoy some sort of finite-dimensional localization; hence it is very important to exploit these features to the fullest. Indeed, over the years approximation theory has been at the heart of many of the deepest, most important results: Murray and von Neumann's uniqueness theorem for the hyperfinite II_1 -factor and Connes's remarkable extension to the injective realm; Haagerup's discovery that reduced free group C^* -algebras have the metric approximation property; Higson and Kasparov's resolution of the Baum-Connes conjecture for Haagerup's groups; Popa's work on subfactors and Cartan subalgebras; Voiculescu's whole free entropy industry, which is defined via approximation; Elliott's classification program, which collapses without approximate intertwining arguments; and one can't forget the influential work of Choi, Effros, and Kirchberg on nuclear and exact C^* -algebras.

Approximation is everywhere; it is powerful, important, the backbone of countless breakthroughs. We intend to celebrate it. This subject is a functional analyst's delight, a beautiful mixture of hard and soft analysis, pure joy for the technically inclined. Our wheat may be other texts' chaff, but we see no reason to hide our infatuation with the grace and power which is approximation theory. We don't mean to suggest that mastering technicalities

is the point of operator algebras – it isn't. We simply hope to elevate them from a necessary ally to a revered friend. Also, one shouldn't think these pages are a one-stop shopping place for all aspects of approximation theory – they aren't. The main focus is nuclearity and exactness, with several related concepts and a few applications thrown in for good measure.

From the outset of this project, we were torn between writing user-friendly notes which students would appreciate – many papers in this subject are notoriously difficult to read – or sticking to an expert-oriented, research-monograph level of exposition. In the end, we decided to split the difference. Part 1 of these notes is written with the beginner in mind, someone who just finished a first course in operator algebras (C^* - and W^* -algebras). We wanted the basic theory to be accessible to students working on their own; hence Chapters 2 - 10 have a lot of detail and proceed at a rather slow pace.¹ Chapters 11 - 17 and all of the appendices are written at a higher level, something closer to that found in the literature.

Here is a synopsis of the contents.

Part 1: Basic Theory

The primary objective here is an almost-comprehensive treatment of nuclearity and exactness.² Playing the revisionist-historian role, we define these classes in terms of finite-dimensional approximation properties and later demonstrate the tensor product characterizations. We also study several related ideas which contribute to, and benefit from, nuclearity and exactness.

The first chapter is just a collection of results that we need for later purposes. We often utilize the interplay between C^* -algebras and von Neumann algebras; hence this chapter reviews a number of “basic” facts on both sides. (Some are not so basic and others are so classical that many students never learn them.)

Chapter 2 contains definitions, simple exercises designed to get the reader warmed up, and a few basic examples (AF algebras, C^* -algebras of amenable groups, type I algebras).

¹Except for a few sections in Chapters 4 and 5, where much more is demanded of the reader. This was necessary to keep the book to a reasonable length.

²The most egregious omission is probably Kirchberg's $\mathcal{O}_2$ -embedding theorem for separable exact C^* -algebras. We felt there were not enough general (i.e., outside of the classification program) applications to warrant including the difficult proof. The paper [107] is readily available and has a self-contained, well-written proof. Rørdam's book [168] has a nearly complete proof and a forthcoming book of Kirchberg and Wassermann will certainly contain all the details. Another significant omission is a discussion of general locally compact groups; we stick to the discrete case. The ideas are adequately exposed in this setting and we don't think beginners benefit from more generality.

In Chapter 3 we give a long introduction to the theory of C^* -tensor products. Most of the chapter is devoted to definitions and a thorough discussion of the subtleties which make C^* -tensor products both interesting and hazardous. However, the last two sections contain important theorems, taking us back to the original definitions of nuclearity and exactness.

In the next two chapters we show that many natural examples of C^* -algebras admit some sort of finite-dimensional approximants. In Chapter 4 we discuss a number of general constructions which one finds in the literature (crossed products by amenable actions, free products, etc.). Chapter 5 is an introduction to exact discrete groups and some related topics which are relevant to noncommutative geometry. Both of these chapters contain redundancies in the sense that we start with special cases and gradually tack on generality. The Bourbakians may protest, but we feel this approach is pedagogically superior.

Someone who works through Chapters 2 - 5 will have a pretty good feel for most aspects of nuclearity and exactness. There is, however, one important permanence property which requires much more work: Both nuclearity and exactness pass to quotients. In some sense, the next four chapters are required to prove these fundamental facts. This doesn't mean we've taken the most direct route, however. On the contrary, we take our sweet time and present a number of related approximation properties which are of independent interest and play crucial roles in the quotient results.

Chapter 6 contains the basics of amenable tracial states. These "invariant means" on C^* -algebras can be characterized in terms of approximation or tensor products. They also yield a simple proof of the deep fact that every finite injective von Neumann algebra is semidiscrete.

In Chapter 7 we study quasidiagonal C^* -algebras. They are also defined via approximation, but the flavor is quite different from nuclearity or exactness. Most of the basic theory is presented, including Voiculescu's homotopy invariance theorem, though much of it isn't necessary for applications to exactness. (For this we only need Dadarlat's approximation theorem for exact quasidiagonal C^* -algebras; see Section 7.5.)

This leads naturally to Chapter 8: AF Embeddability. For applications, the most important fact is that every exact C^* -algebra is a subquotient of an AF C^* -algebra. We give the proof in the beginning of the chapter so those only interested in exactness can quickly proceed forward. For others, we have included the homotopy invariance theorem for AF embeddability and a short survey of related results.

In Chapter 9 we put all the pieces together, completing the basic-theory portion of the book. The main result gives two more tensor product characterizations of exactness, from which corollaries flow: Exact C^* -algebras are

locally reflexive (another important finite-dimensional approximation property), nuclearity and exactness pass to quotients, and a few others.

Finally, we conclude Part 1 with a chapter summarizing permanence properties. This is just for ease of reference, in case one forgets whether or not extensions of exact C^* -algebras are exact.

Part 2: Special Topics

The next four chapters are a disjoint collection of related concepts. They are logically independent and meant to spark the reader's interest – much more could be written about any one of them.

Chapter 11 is primarily about simple quasidiagonal C^* -algebras. Motivated by Elliott's classification program, we spend time discussing the generalized inductive limit approach (of Blackadar and Kirchberg) to nuclear quasidiagonal C^* -algebras. We also prove a theorem of Popa, showing that quasidiagonality is often detectable internally. Finally, we present Connes's amazing uniqueness theorem for the injective II_1 -factor, exploiting Popa's techniques.

Chapter 12 introduces some properties of discrete groups that have been extremely important over the years. First, we discuss Kazhdan's property (T), prove that $SL(3, \mathbb{Z})$ has this property, and demonstrate Connes's result that II_1 -factors with property (T) have few outer automorphisms. Next, we define Haagerup's approximation property – the antithesis of property (T) – and prove that a group which acts properly on a tree (e.g., a free group) enjoys this property. The latter sections of this chapter discuss related approximation properties and their interrelations.

Chapter 13 – on Lance's weak expectation property and the local lifting property for C^* -algebras – gives a streamlined approach to some of Kirchberg's influential work around these ideas. We also reproduce Junge and Pisier's theorem on the tensor product of $\mathbb{B}(\ell^2)$ with itself.

Part 2 concludes with Chapter 14: Weakly Exact von Neumann Algebras. This concept was first suggested by Kirchberg; the theorems and proofs are similar to C^* -results found in Part 1 of the book. It is not yet clear if this theory will bear fruit like its C^* -predecessor, but it seemed like a natural topic to include.

Part 3: Applications

The last three chapters, comprising Part 3, are devoted to applications. We hope to convince you that approximation properties are useful; seemingly unrelated problems will crack wide open when pried with the right technical tool.

Chapter 15 contains solidity and prime factorization results for certain group von Neumann algebras. The solidity results generalize one of the celebrated achievements of free probability theory, while the prime factorization results are natural analogues of some spectacular recent work in dynamical systems. Both depend in a crucial way on some of the C^* -ideas and tensor product techniques contained in Part 1.

Chapter 16 resolves a problem in single operator theory which, at present, appears to require exact quasidiagonal C^* -algebras. We need the fact that exactness implies local reflexivity – one of the deepest, most difficult theorems in C^* -algebras – and it is hard to imagine an operator-theoretic proof which could circumvent this fact.

The final chapter is based on some work of Simon Wassermann. He observed that property (T) groups together with quasidiagonal ideas lead to natural examples for which the Brown-Douglas-Fillmore semigroup is not a group. Approximation properties, or the lack thereof, are at the heart of the argument.

So, that's what you'll find in this book. To the student: We hope these notes are reasonably accessible and a helpful introduction to an area of active research. To the veteran: We hope this will be a useful reference for C^* -approximation theory. As mentioned earlier, Kirchberg and Wassermann are working on an exact C^* -algebra text and there will certainly be overlap between these notes and theirs. However, the emphasis and selection of topics will likely differ; with any luck, the union of our books will satisfy the needs of most. We should also mention that a web page correcting this book's inevitable errors can be found at www.ams.org/bookpages/gsm-88.

This project would have been impossible without the support of our academic institutions and the encouragement of many friends and colleagues. Thank you. The first author gratefully acknowledges NSF support; the second author thanks JSPS, NSF, and the Sloan Foundation. Finally, we are deeply indebted to Reiji Tomatsu for his meticulous note-taking during the second author's lectures on some of the material covered in this book; Tyrone Crisp, Steve Hair, and Naokazu Mizuta for their fastidious proofreading; and Kenley Jung and Takeshi Katsura for some helpful questions, remarks, and suggestions.

Nate and Taka, 2007

Contents

Preface	xi
Chapter 1. Fundamental Facts	1
§1.1. Notation	1
§1.2. C^* -algebras	2
§1.3. Von Neumann algebras	3
§1.4. Double duals	5
§1.5. Completely positive maps	9
§1.6. Arveson's Extension Theorem	17
§1.7. Voiculescu's Theorem	18
Part 1. Basic Theory	
Chapter 2. Nuclear and Exact C^* -Algebras: Definitions, Basic Facts and Examples	25
§2.1. Nuclear maps	25
§2.2. Nonunital technicalities	28
§2.3. Nuclear and exact C^* -algebras	32
§2.4. First examples	38
§2.5. C^* -algebras associated to discrete groups	42
§2.6. Amenable groups	48
§2.7. Type I C^* -algebras	55
§2.8. References	58
Chapter 3. Tensor Products	59

§3.1.	Algebraic tensor products	59
§3.2.	Analytic preliminaries	66
§3.3.	The spatial and maximal C^* -norms	72
§3.4.	Takesaki's Theorem	77
§3.5.	Continuity of tensor product maps	82
§3.6.	Inclusions and The Trick	85
§3.7.	Exact sequences	92
§3.8.	Nuclearity and tensor products	99
§3.9.	Exactness and tensor products	105
§3.10.	References	112
Chapter 4.	Constructions	115
§4.1.	Crossed products	115
§4.2.	Integer actions	121
§4.3.	Amenable actions	124
§4.4.	$X \rtimes \Gamma$ -algebras	129
§4.5.	Compact group actions and graph C^* -algebras	133
§4.6.	Cuntz-Pimsner algebras	136
§4.7.	Reduced amalgamated free products	154
§4.8.	Maps on reduced amalgamated free products	157
§4.9.	References	165
Chapter 5.	Exact Groups and Related Topics	167
§5.1.	Exact groups	167
§5.2.	Groups acting on trees	176
§5.3.	Hyperbolic groups	182
§5.4.	Subgroups of Lie groups	193
§5.5.	Coarse metric spaces	194
§5.6.	Groupoids	200
§5.7.	References	209
Chapter 6.	Amenable Traces and Kirchberg's Factorization Property	211
§6.1.	Traces and the right regular representation	211
§6.2.	Amenable traces	214
§6.3.	Some motivation and examples	223
§6.4.	The factorization property and Kazhdan's property (T)	227
§6.5.	References	235

Chapter 7. Quasidiagonal C^* -Algebras	237
§7.1. The definition, easy examples and obstructions	237
§7.2. The representation theorem	243
§7.3. Homotopy invariance	247
§7.4. Two more examples	252
§7.5. External approximation	255
§7.6. References	260
Chapter 8. AF Embeddability	261
§8.1. Stable uniqueness and asymptotically commuting diagrams	261
§8.2. Cones over exact RFD algebras	267
§8.3. Cones over general exact algebras	268
§8.4. Homotopy invariance	274
§8.5. A survey	279
§8.6. References	282
Chapter 9. Local Reflexivity and Other Tensor Product Conditions	283
§9.1. Local reflexivity	284
§9.2. Tensor product properties	285
§9.3. Equivalence of exactness and property C	293
§9.4. Corollaries	297
§9.5. References	299
Chapter 10. Summary and Open Problems	301
§10.1. Nuclear C^* -algebras	301
§10.2. Exact C^* -algebras	303
§10.3. Quasidiagonal C^* -algebras	306
§10.4. Open problems	309
Part 2. Special Topics	
Chapter 11. Simple C^* -Algebras	313
§11.1. Generalized inductive limits	313
§11.2. NF and strong NF algebras	317
§11.3. Inner quasidiagonality	323
§11.4. Excision and Popa's technique	328
§11.5. Connes's uniqueness theorem	335
§11.6. References	337

Chapter 12. Approximation Properties for Groups	339
§12.1. Kazhdan's property (T)	339
§12.2. The Haagerup property	354
§12.3. Weak amenability	361
§12.4. Another approximation property	369
§12.5. References	374
Chapter 13. Weak Expectation Property and Local Lifting Property	375
§13.1. The local lifting property	375
§13.2. Tensorial characterizations of the LLP and WEP	378
§13.3. The QWEP conjecture	380
§13.4. Nonsemisplit extensions	385
§13.5. Norms on $\mathbb{B}(\ell^2) \odot \mathbb{B}(\ell^2)$	388
§13.6. References	391
Chapter 14. Weakly Exact von Neumann Algebras	393
§14.1. Definition and examples	393
§14.2. Characterization of weak exactness	397
§14.3. References	403
Part 3. Applications	
Chapter 15. Classification of Group von Neumann Algebras	407
§15.1. Subalgebras with noninjective relative commutants	407
§15.2. On bi-exactness	411
§15.3. Examples	414
§15.4. References	420
Chapter 16. Herrero's Approximation Problem	421
§16.1. Description of the problem	421
§16.2. C^* -preliminaries	423
§16.3. Resolution of Herrero's problem	425
§16.4. Counterexamples	426
§16.5. References	429
Chapter 17. Counterexamples in K-Homology and K-Theory	431
§17.1. BDF preliminaries	431
§17.2. Property (T) and Kazhdan projections	435
§17.3. Ext need not be a group	438

§17.4. Topology on Ext	439
§17.5. References	441

Part 4. Appendices

Appendix A. Ultrafilters and Ultraproducts	445
Appendix B. Operator Spaces, Completely Bounded Maps and Duality	449
Appendix C. Lifting Theorems	459
Appendix D. Positive Definite Functions, Cocycles and Schoenberg's Theorem	463
Appendix E. Groups and Graphs	471
Appendix F. Bimodules over von Neumann Algebras	479
Bibliography	493
Notation Index	503
Subject Index	505

Fundamental Facts

We stated in the preface that these notes should be accessible to anyone with a “first course” in operator algebras under their belt. An excellent first course would consist of the material contained in [127], for example, and we assume familiarity with that book. However, we’ll need numerous other facts that may or may not have made it into *your* first course; the purpose of this chapter is to summarize the requisite results.

It goes without saying that advanced students and seasoned researchers should skip this chapter, referring back if necessary. Indeed, the only things required before starting Chapter 2 are basic properties of completely positive maps and Arveson’s Extension Theorem (Sections 1.5 and 1.6). We advise the novice to nail down this material and then to jump ahead to Chapter 2 – mathematics books need not be read linearly.¹

1.1. Notation

We use $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ to denote generic complex Hilbert spaces. The n -dimensional Hilbert space is usually denoted ℓ_n^2 , while ℓ^2 is the separable, infinite-dimensional Hilbert space. Here $\mathbb{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, $\mathbb{K}(\mathcal{H})$ denotes the compacts and $Q(\mathcal{H}) = \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ is the Calkin algebra. (An abstract copy of the compacts will be denoted by $\mathbb{K}$.) Also, Tr is the canonical (typically unbounded, densely defined) trace on $\mathbb{B}(\mathcal{H})$. We let $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) denote the trace class (resp. Hilbert-Schmidt) operators, with canonical norm $\|T\|_1 = \text{Tr}(|T|)$ (resp. $\|T\|_2 = \sqrt{\text{Tr}(|T|^2)}$).

¹If you feel obliged to go carefully through this chapter, be advised that tensor products are required in a few places. Thus, first reading the beginning of Chapter 3 might be necessary. How’s that for nonlinear?

Here $M_n(\mathbb{C})$ is the $n \times n$ complex matrices and tr is its unique tracial state. A collection of $n \times n$ matrices $\{e_{i,j}\}_{1 \leq i,j \leq n}$ is called a *system of matrix units* if $e_{i,j}e_{s,t} = \delta_{j,s}e_{i,t}$. It is often convenient to index our matrices over a finite set F , in which case we write $M_F(\mathbb{C})$ ($= \mathbb{B}(\ell^2(F))$) and let $\{e_{p,q}\}_{p,q \in F}$ denote the canonical matrix units.

We reserve A, B, C and D for C^* -algebras while M and N will typically denote von Neumann algebras. We let A_{sa} be the self-adjoint elements, A_1 the closed unit ball and A_+ the positive elements in A . The symbols E and F will denote operator systems (or operator spaces). We usually use I and J for ideals in C^* -algebras (e.g. $I \triangleleft A$), though they are occasionally index sets too. All ideals are assumed closed and two-sided.

The set of states on A – positive linear functionals of norm one – will be denoted $S(A)$. If $\varphi \in S(A)$, we let $L^2(A, \varphi)$ be the GNS (Gelfand-Naimark-Segal) Hilbert space and $\pi_\varphi: A \rightarrow \mathbb{B}(L^2(A, \varphi))$ be the GNS representation. For an element $a \in A$, we let $\hat{a} \in L^2(A, \varphi)$ denote its natural image.

1.2. C^* -algebras

Quasical central approximate units. Quasical central approximate units are an indispensable tool. See [53, Theorem I.9.16] for a proof of the following fact.

Theorem 1.2.1. *Let $I \triangleleft A$ be an ideal. Then I has an approximate unit $\{e_i\} \subset I$ such that $\|e_i a - a e_i\| \rightarrow 0$, as $i \rightarrow \infty$, for all $a \in A$. In fact, if $\{f_k\} \subset I$ is any approximate unit for I , then a quasical central approximate unit can always be extracted from its convex hull.*

We don't need it too many times, but it is worth mentioning that quasical central approximate units allow a particular type of approximate decomposition.

Proposition 1.2.2. *Let A be unital and $\{e_i\} \subset I \triangleleft A$ be a quasical central approximate unit. For every pair $a, b \in A$ such that $a - b \in I$ we have*

$$\|a - \left((1 - e_i)^{\frac{1}{2}} b (1 - e_i)^{\frac{1}{2}} + e_i^{\frac{1}{2}} a e_i^{\frac{1}{2}} \right)\| \rightarrow 0.$$

Proof. First notice that for every $x \in A$ and polynomial p we have $\|p(e_i)x - xp(e_i)\| \rightarrow 0$ (by some standard estimates). Since continuous functions can be approximated by polynomials, it follows that $\|e_i^{\frac{1}{2}}x - xe_i^{\frac{1}{2}}\| \rightarrow 0$ and $\|(1 - e_i)^{\frac{1}{2}}x - x(1 - e_i)^{\frac{1}{2}}\| \rightarrow 0$.

Next, observe that

$$\|(1 - e_i)^{\frac{1}{2}}a(1 - e_i)^{\frac{1}{2}} - (1 - e_i)^{\frac{1}{2}}b(1 - e_i)^{\frac{1}{2}}\| = \|(1 - e_i)^{\frac{1}{2}}(a - b)(1 - e_i)^{\frac{1}{2}}\| \rightarrow 0$$