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Equivalence, Invariants, and Symmetry



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国外优秀数学著作
原 版 系 列

等值、不变量和对称性

[美] 皮特·奥立弗（Peter J. Olver） 著

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To my family — Chehrzad, Parizad, Sheehan, and Noreen

Preface

The volume you are about to read will, I hope, prove to be a stimulating, unusual, and provocative blend of mathematical flavors. As its title indicates, the book revolves around three interconnected and particularly fertile themes, each arising in a wide variety of mathematical disciplines, and each having a wealth of significant and substantial applications. Equivalence deals with the determination of when two mathematical objects are the same under a change of variables. The symmetries of a given object can be interpreted as the group of self-equivalences. Conditions guaranteeing equivalence are most effectively expressed in terms of invariants, whose values are unaffected by the changes of variables. Issues of this generality naturally arise in all fields of mathematics, and, particularly in geometry, often lie at the heart of the subject. Although each of these concepts has a discrete counterpart, our primary focus will be on the continuous. The areas of immediate concern are analytical — differential equations, variational problems, vector fields, and differential forms — although algebraic objects, such as polynomials, matrices, and quadratic forms, also play an important role. This book will explore the available methods for systematically and algorithmically solving the problems of symmetry, equivalence, the classification of invariants, and the determination of canonical forms, thereby elucidating the many interconnections, some surprising, between the particular manifestations of these problems in seemingly unrelated situations.

The book naturally divides into four interconnected parts. The first, comprising Chapters 1–3, constitutes the algebro-geometric foundation of our subject. The first chapter provides a rapid survey of the basic facts from differential geometry, including manifolds, vector fields, and differential forms. Chapters 2 and 3 could, with some more fleshing out, form a basic course on Lie groups and representation theory. The primary omissions are the detailed structure theory of Lie groups and algebras, and the classification theory for irreducible representations, neither of which play a significant role in our applications. The second

part, comprising Chapters 4–7, provides an in depth study of applications of symmetry methods to differential equations. We begin with a discussion of jet spaces, on which systems of differential equations are naturally realized as geometrical subsets, and conclude with a discussion of contact transformations. The next three chapters deal with the construction and classification of differential invariants, followed by applications to differential equations and variational problems. My earlier text, [**186**], can be used to supplement the material in this part with many additional applications. In the third part, Chapters 8–12, the focus shifts to equivalence problems, and the Cartan approach to their solution. The different possible branches that arise in Cartan's equivalence method are discussed in detail, and illustrated by a wide variety of applications. The final three chapters survey the required results from the theory of partial differential equations and differential systems. In particular, we describe the proofs of the fundamental existence theorems of Frobenius, and of Cartan and Kähler, that underlie all the methods.

I should mention that the arrangement of the chapters is slightly unnatural from a developmental standpoint. The most foundational material — the existence theorems for systems of partial differential equations — appears in the last three chapters. The geometry of coframes is used in part in the symmetry analysis of differential equations, particularly in the construction of differential invariants, which are used to classify equations admitting prescribed symmetry groups. However, I decided on a more pedagogical arrangement so as not to frighten off the prospective reader with overly technical topics at the outset. Thus, the book begins with the more readily digested methods from Lie group theory, then moves on to the more sophisticated approach of Cartan based on differential forms, and completes the repast with the hard analysis for dessert. I hope the more logically minded reader will not be too inconvenienced by this choice of presentation.

There is a wealth of related material in the mathematical literature, and one of the hardest tasks was choosing which results and applications to include, and which, for reasons of space, to omit. For example, at one time I envisioned a much more substantial section on classical invariant theory; however, this unfortunately had to be considerably shortened for the final version. (I hope, in the near future, to use this additional material as the basis for an introductory text on classical invariant theory and its applications.) Experts in any of the subjects touched on in the

book will, no doubt, find many of their favorite theoretical developments or applications omitted or tersely commented upon. In many cases, this was made necessary by the space limitations and the need to choose results that were particularly representative, interesting, and/or interdisciplinary. I hope the omissions will not prevent anyone from enjoying what I did finally decide to include.

For clarity, I have adopted a fairly informal, discursive style to present the methods, with the hope that this will lead to new understanding of their utility and effectiveness. Too often in the recent literature, these powerful and constructive techniques have been obscured by elaborate, theoretical machinery that many have chosen to "rigorize" them with. In my opinion, in the practical realm, this only serves to obscure the fundamental issues, thereby interfering with a student's direct understanding. My own presentation relies ultimately on the original sources, particularly Lie and Cartan, which I wholeheartedly recommend to the genuine scholar. As a consequence, results are not always stated in complete generality, and occasionally some minor points of rigor are left for the reader to properly sort out. As long as one exercises the proper amount of caution, such technical details will rarely lead one astray in the practical applications.

Although a substantial fraction of the book covers results that are well known to followers of Lie and Cartan, the modern applicability of these methods is illustrated by a surprising variety of new applications. Much of Chapter 5 on the theory of differential invariants is new, and, apart from being surveyed in conference proceedings, has not appeared in the literature before. Some of the classification results for differential equations and variational problems based on symmetry are also new, and others have only appeared scattered in the literature. The applications of differential invariants to computer vision are the result of a recent collaboration with Allen Tannenbaum and Guillermo Sapiro. The integration of the Chazy equation using prolongation and symmetry appears in joint work with Peter Clarkson. Many of the applications of the Cartan equivalence method are based on results from my long-standing collaboration with Niky Kamran. The emphasis on the classifying manifolds in the solution to the equivalence problem for coframes is new, and offers a substantial advantage over the more traditional classifying function approach. The results on global equivalence and non-associative Lie groups are stated here for the first time. The applications of classical

invariant theory appear in various contexts, and have been reformulated in a more consistent form; in particular, the connection with the Cartan solution to an equivalence problem from the calculus of variations is based on an earlier paper, [**183**].

The basic prerequisites for the book are multi-variable calculus — specifically the implicit and inverse function theorems and the divergence theorem — basic tensor and exterior algebra, and a smattering of group theory. Results from elementary linear algebra and complex analysis, and basic existence theorems for ordinary differential equations are used without comment. Many standard results are, for lack of space, left unproved, although ample references are supplied. It is hoped that the book will form the basis of an advanced graduate course in symmetry and equivalence problems. I have included a large number of exercises, which range in difficulty from the fairly obvious to quite substantial. References to the solutions of the more difficult exercises are provided.

It is my hope that this book will serve as a catalyst for the further development, both in theory and in applications, of this fascinating and fertile mathematic field. I am certain that there are many fundamental contributions yet to be made, and that the devoted student cannot help but play a role in its accelerating mathematical development, or in its ever-broadening range of applications.

Acknowledgments

The book has its origins in a series of ten London Mathematical Society Invited Lectures, delivered at the University of Bath, under the auspices of the London Mathematical Society, in April 1992. I should like to express my sincere appreciation to Geoffrey Burton and John Toland who played an indispensable role in the planning and organization. I should also like to thank the London Mathematical Society for their patience and understanding as the original lecture notes slowly metamorphosed into a full-fledged book, thereby delaying its appearance considerably beyond the original schedule.

Later versions of these notes were used as a basis for a course at the University of Maryland during spring semester 1993. I must thank the Maryland Mathematics Department, the participants in the course, and particularly John Maddocks, for the opportunity to further refine my ideas during the course.

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This book was typeset in plain TEX using a system of macros for automatic equation and reference numbering that I developed. I would like to thank Ian Anderson for sharing his own TEX macros, which formed the foundation of the OTEX system. The figures were drawn using the computer algebra system MATHEMATICA.

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Introduction

The most basic of the themes to be discussed is the concept of equivalence. The fundamental equivalence problem is to determine whether two geometric objects can be transformed into each other by a suitable change of variables. For example, one might be interested in whether two given differential equations are, in fact, the same equation rewritten in terms of different independent and dependent variables. As a particular instance, the linearization problem is to determine whether a differential equation can be mapped to a linear differential equation via a change of variables. In this manner, a solution to the general equivalence problem for differential equations will include, as a very particular case, the characterization of all linearizable differential equations. As a second example, in the calculus of variations, the basic equivalence problem is to determine when two variational problems can be identified under a change of variables. This problem forms a subcase of the differential equation equivalence problem, since the change of variables will also map their Euler-Lagrange equations to each other. In algebra, the problem of equivalence of (homogeneous) polynomials under linear transformations also falls under our broad purview. Geometric equivalence problems lie at the foundations of Riemannian geometry, where one wants to know whether two given Riemannian manifolds can be identified under a (local) diffeomorphism, and, similarly, in symplectic geometry, where one asks the same question for two symplectic manifolds.

The determination of the symmetry group of a geometric object can be regarded as a special case of the general equivalence problem. Indeed, provided it lies in the admissible class of changes of variables, a symmetry is merely a self-equivalence of the object. Thus, for instance, the solution to the equivalence problem for differential equations will include a determination of all symmetries of a given differential equation. Similarly, the isometries of a Riemannian manifold are merely the set of self-equivalences of the underlying metric. Besides addressing the general equivalence problem, we shall also develop methods to directly

calculate the symmetry group of a given object. Two equivalent objects have isomorphic symmetry groups — indeed, conjugating any symmetry of the first object by the equivalence transformation produces a symmetry of the second. Thus, one means of recognizing equivalent objects is by inspecting their symmetry groups: If the two symmetry groups are not equivalent, e.g., they have different dimensions, or different structures, then the two objects cannot be equivalent. Of course, having isomorphic symmetry groups is no guarantee that the two objects are equivalent; nevertheless, in many highly symmetric cases, including linearization problems, the existence of a suitable symmetry group is both necessary and sufficient for the equivalence of the two objects.

The concept of an invariant proves to be of crucial importance in understanding an equivalence problem. By definition, an invariant is a quantity which is unaffected by the changes of variables. Consequently, two equivalent objects must necessarily have the same invariants. Conversely, in the regular case, if we know enough invariant functions, we can use them to completely characterize the equivalent objects, and thereby completely solve the equivalence problem. Thus, the construction of invariants and their characterization lies at the heart of most approaches to equivalence problems. In addition, every regular invariant system of equations (algebraic, differential, variational, etc.) can be characterized by functional relationships among the invariant functions, so the invariants form the fundamental building blocks which can be used to construct suitably symmetric objects, a process of immense utility in modern physics. Whereas invariant functions are the most important invariant quantities associated with the equivalence, many other invariant objects — vector fields, differential forms, differential operators, etc. — arise naturally and will be seen to play important roles.

Last, but not least, we shall consider the problem of determining canonical forms for the objects of interest. By definition, a "canonical form" is provided by a particularly simple equivalent object. Familiar examples include the Jordan canonical form for matrices, and the well-known canonical forms for quadratic and cubic polynomials. The determination of the canonical forms requires a solution to the equivalence problem; indeed, one way of characterizing two equivalent objects is that they have the same canonical form. The particular canonical form of a prescribed object is most readily characterized by its invariants; vice versa, the invariants themselves can often be found once a

canonical form is known. For example, the eigenvalues and the Jordan block structure provide the proper invariants that will characterize those matrices equivalent to a given Jordan canonical form. In our discussion of equivalence problems, we shall, when feasible, attempt to describe a list of canonical forms.

It is these four issues — equivalence, symmetry, invariant, and canonical form — that form the core of the book. Our goal is to develop a variety of techniques, some very general, some quite specific, that will allow one to handle some or all of these problems in a wide variety of mathematical contexts. Particular attention will be paid to problems arising in geometry, differential equations, and the calculus of variations, although this will by no means restrict our discussion. These topics arise in many contexts, both in mathematics and in physical applications, and our survey will include problems from quantum mechanics, elasticity, computer vision, gas dynamics, relativity, optics, Hamiltonian mechanics, solitons, representation theory, and differential and projective geometry, to name a few. In a more theoretical direction, the theory of equivalence and invariants of ordinary polynomials, known as "classical invariant theory", has a long and distinguished history of its own, which, especially in view of recently discovered connections with the calculus of variations, is worth presenting in some detail.

In our study, the emphasis will be on the analytical, the continuous, and the local. Our geometric objects will be defined on smooth, finite-dimensional manifolds (most commonly Euclidean spaces), and depend smoothly or analytically on local coordinates. We shall exclusively deal with *local* equivalence problems, in which one asks for equivalence of the objects when restricted to sufficiently small open subsets. Moreover, our treatment will primarily deal with the regular case, away from singularities. This is not to say that the global and singular problems are less interesting (indeed, one can successfully argue that the opposite is true), but rather an indication of our necessarily restricted scope, imposed by space limitations. In consequence, we shall emphasize continuous symmetry groups, which, in the finite dimensional manifestation, appear as Lie groups of transformations, or, in the infinite-dimensional version, as "Lie pseudo-groups". Again, this is not because discrete groups are unimportant, but just to keep the length manageable. Even with our chosen restrictions, the proposed subject area is still vast, and we shall repeatedly be forced to choose particular subtopics.