高等学校中外合作办学适用教材

A Concise Course to Linear Algebra

线性代数简明教程

刘国庆 赵剑 石玮 编著

化学工业出版社

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·北京·

本书叙述深入浅出，以矩阵为主线，突出矩阵的运算和化简，突出用矩阵方法研究线性方程组、二次型和实际问题模型。本书对于抽象的理论和方法，总是从具体问题入手，再将其推广到一般情形，而略去了许多繁杂的理论推导，并力求将数学与应用相结合。

本书的主要内容包括线性方程组、矩阵代数、行列式、向量空间、矩阵的特征值与特征向量和二次型等。

本书是一本介绍性的线性代数教材，内容简洁，层次清晰，适合高等学校理工科专业线性代数课程双语教学使用。

The matrix is the mainline of the book. With the help of the matrix operation and the matrix simplification, we study the linear equations, the quadratic forms and the real world applications. For the purpose of the insights into the abstract theory and the methods of the linear algebra, we start to discuss the conceptions and the methods with the specific problems, then we directly extend them to the general situation without the complicated theoretical derivation. Furthermore, we try to combine the mathematical methods with the real applications in this book.

The main contents of the book are linear equations, matrix algebra, determinants, vector spaces, eigenvalues and eigenvectors, and quadratic forms, etc.

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Preface

The main goal of the text is to help students master the basic concepts and skills they will use later in their study and careers. The text provides a modern elementary introduction to linear algebra and a broad selection of interesting applications. The material is accessible to students with the maturity that should come from successful completion of two semesters of college-level calculus.

We have attempted to give this book the following distinctive features.

(1) Many fundamental ideas of linear algebra are introduced within the first lectures, then gradually examined from different points of view. A major achievement of the text is that the level of difficulty is fairly even.

(2) Good notation is crucial, and the text reflects the way scientists and engineers actually use linear algebra in practice. The definitions and proofs focus on the columns of a matrix rather than on the matrix entries. This modern approach simplifies many arguments, and it ties vector space ideas into the study of linear systems.

(3) A broad selection of applications illustrates the power of linear algebra to explain fundamental principles and simplify computing in engineering, physics, economics, and statistics.

In this volume, Chapter 1, Chapter 2, Chapter 6 and Chapter 7 are written by Professor Guoqing Liu, Chapter 3 is written by Associated Professor Jian Zhao, and Chapter 4 and Chapter 5 are written by Dr. Wei Shi. All the chapters are checked and revised by Professor Guoqing Liu.

We hope this book can bring readers some help in the studying and teaching of bilingual mathematics. Due to the limit of our ability, it is impossible to avoid some unclear explanations. We would appreciate any constructive criticisms and corrections from readers.

Guoqing Liu, Jian Zhao, Wei Shi
2019. 6

前　言

本书的主要目标是帮助学生们掌握他们在后面的课程学习和未来事业发展中需要的一些线性代数的基本原理和相关技能。本书以现代的视野简明扼要地介绍了线性代数的基本思想和具有广泛背景的有趣应用。这些内容的学习需要学生们具备两个学期的大学微积分课程基础。

我们致力于使本书具备以下特征。

（1）线性代数涉及的许多基本思想和概念在开始的几讲中通过简单的案例引入，再通过随后章节的反复论述，从不同的角度逐渐强化对这些思想和概念的理解。本书的一大成果是通过层层递进的叙述方式降低了对线性代数概念理解的难度。

（2）本书用科学家和工程师们在实际问题中应用线性代数时所熟悉的方法和符号来描绘线性代数。例如，很多的定义和证明都是针对矩阵的列而不是矩阵的元素进行的，使得这些定义和证明看上去更简洁。同时，这样做也将向量空间的思想揉入线性代数方程组的研究中，使得方程组有关解的结论更具应用价值。

（3）书中广泛选择的应用问题彰显了线性代数在解释经济、统计和工程等领域的基本原理方面的重要性，同时也表明利用线性代数可以简化这些领域的科学计算。

本书的第一章、第二章、第六章和第七章由刘国庆教授撰写，第三章由赵剑副教授撰写，第四章和第五章由石玮副教授撰写。全书由刘国庆教授统稿。

我们希望本书的出版对那些用双语学习和教授线性代数的读者有所帮助。由于我们能力的局限，在解释有关概念、理论和方法时，不可避免地会出现疏漏。我们真诚地希望得到来自读者的批评和指正。

本书的编写得到“江苏省第二批中外合作办学高水平示范性建设工程项目培育点：南京工业大学与英国谢菲尔德大学合作举办数学与应用数学（金融数学）专业本科教育项目”（苏教办外【2017】14 号）经费支持。

刘国庆，赵剑，石玮

2019 年 6 月

Contents

Chapter 1
Linear Equations in Linear Algebra

> One has to familiarize the student with actual questions from applications, so that he learns to deal with real world problems. —Lothar Collatz (1910-1990)

Systems of linear equations lie at the heart of linear algebra[1], and this chapter uses them to introduce some of the central concepts of linear algebra in a simple and concrete setting. The central problem about which much of the theory of linear algebra revolves is the problem of finding all solutions to a linear system. As a matter of fact, simple cases of this problem are a part of most high-school algebra backgrounds. We will address the problem of when a linear system has a solution and how to solve such a system for all of its solutions. Examples of linear systems appear in nearly every scientific discipline; we touch on a few in this chapter.

1.1 Systems of Linear Equations

A linear equation in the variables $x_1, x_2, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1+a_2x_2+\cdots+a_nx_n=b$$

where b and the coefficients $a_1, a_2, \cdots, a_n$ are real or complex numbers. The subscript n may be any positive integer. In textbook examples are exercises, n is normally between 2 and 5. In the real-life problems, n might be 100 or 1000, or even

[1] Linear Algebra：线性代数

larger.

【Example 1.1】 The PageRank Algorithm[1]

The PageRank algorithm is a method to assess the "importance" of documents with mutual links, such as web pages, on the basis of the link structure. It was developed by Sergei Brin and Larry Page, the founders of Google Inc., at Stanford University in the late 1990s. The basic idea of the algorithm is the following: Instead of counting links, PageRank essentially interprets a link of page A to page B as a vote of page A for page B. PageRank then assesses the importance of a page by the number of received votes. PageRank also considers the importance of the page that casts the vote, since votes of some pages have a higher value, and thus also assign a higher value to the page they point to. Important pages will be rated higher and thus lead to a higher position in the search results.

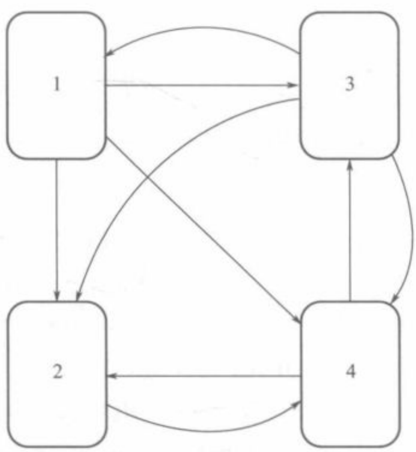

Fig. 1.1

Let us describe this idea mathematically [1]. For a given set of web pages, every page k will be assigned an importance value $x_k \geqslant 0$. A page k is more important than a page j if $x_k > x_j$. If a page k has a link to a page j, we say that page j has a backlink[2] from page k. In the above description these backlinks are the votes. As an example, consider the link structure showed in Fig1.1.

Here the page 1 has links to the pages 2, 3 and 4, and a backlink from page 3.

The easiest approach to define importance of web pages is to count its backlinks; the more votes are cast for a page, the more important the page is. In our example this gives the importance values

$$x_1 = 1, x_2 = 3, x_3 = 2, x_4 = 3.$$

The pages 2 and 4 are thus the most important pages, and they are equally important.

However, backlinks from important pages are more important for the value of a page than those from less important pages. This idea can be modeled by defining x_k as the sum of all importance values of the backlinks of the page k. In our example this results in four equations

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[1] Algorithm：算法

[2] backlink：逆向链接

$$\begin{cases} x_1 = x_3 \\ x_2 = x_1 + x_3 + x_4 \\ x_3 = x_1 + x_4 \\ x_4 = x_1 + x_2 + x_3 \end{cases}$$

A disadvantage of this approach is that it does not consider the number of links of the pages. Thus, it would be possible to significantly increase the importance of a page just by adding links to that page. In order to avoid this, the importance values of the backlinks in the PageRank algorithm are divided by the number of links of the corresponding page. This creates a kind of "internet democracy": Every page can vote for other pages, where in total it can cast one vote. In our example, this gives the equations

$$\begin{cases} x_1 = \dfrac{x_3}{3} \\ x_2 = \dfrac{x_1}{3} + \dfrac{x_3}{3} + \dfrac{x_4}{2} \\ x_3 = \dfrac{x_1}{3} + \dfrac{x_4}{2} \\ x_4 = \dfrac{x_1}{3} + x_2 + \dfrac{x_3}{3} \end{cases}$$

These are four equations for the unknowns, and all equations are linear, i. e. , the unknown occur only in the first power.

Linear systems arise in many applications. Examples in which they occur, in addition to lines and planes, are least-squares fitting of lines, planes, or curves to observed data, methods for obtaining approximate solutions of various differential equations, Kirchhoff's equations relating currents and potentials in electrical circuits, and various economic models. In many applications, the number of equations and unknowns can be quite large, sometimes in the hundreds or thousands. Thus it is very important to understand the structure of such systems and to apply systematic and efficient methods for their solution. Even more important is that, as we shall see, studying such systems leads to several new concepts and theories that are at the heart of linear algebra.

We begin with a simple example.

【Example 1. 2】 Let us solve the following system:

$$\begin{cases} 2x + 3y - z = 8 \\ 4x - 2y + z = 5 \\ x + 5y - 2z = 9 \end{cases} \tag{1.1}$$

We want to proceed as follows: multiply both sides of the first equation by 2 and subtract the result from the second equation to eliminate the $4x$, and subtract 1/2 times the first equation from the third equation to eliminate the x. The system is then changed into the new, equivalent system:

$$\begin{cases} 2x+3y-z=8 \\ -8y+3z=-11 \\ \dfrac{7}{2}y-\dfrac{3}{2}z=5 \end{cases}$$

As our next step we want to get rid of the $\frac{7}{2}y$ term in the last equation. We can achieve this elimination by multiplying the middle equation by $-\frac{7}{16}$ and subtracting the result from the last equation. Then we get

$$\begin{cases} 2x+3y-z=8 \\ -8y+3z=-11 \\ -\dfrac{3}{16}z=\dfrac{3}{16} \end{cases}$$

At this point, we can easily find the solution by starting with the last equation and working our way back up. First, we find $z=-1$, and second, substituting this value into the middle equation we get $y=1$. Last, we enter the values of y and z into the top equation and obtain $x=2$.

The method of solving a linear system used in the example above is called Gaussian elimination.

Notice that in the elimination computations of Example 1.2, the variables x, y, z were not really used. In computer programs there is not even a way (and no need either) to enter the variables. In writing, the coefficients are usually arranged in a rectangular array enclosed in parentheses or brackets, called a matrix (plural: matrices) and designed by a capital letter, as in

$$\boldsymbol{A}=\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 1 \\ 1 & 5 & -2 \end{bmatrix} \tag{1.2}$$

This matrix contains the coefficients on the left of system (1.1) in the same arrangement, and is therefore referred to as the coefficient matrix. We may include the numbers on the right sides of the equations as well:

$$\boldsymbol{B}=\begin{bmatrix} 2 & 3 & -1 & 8 \\ 4 & -2 & 1 & 5 \\ 1 & 5 & -2 & 9 \end{bmatrix} \tag{1.3}$$

This is called the augmented matrix[1] of the system.

The size of a matrix tells how many rows and columns it has. The augmented matrix (1.3) above has 3 rows and 4 columns and is called a 3×4 matrix. If m and n are positive integers, an $m\times n$ matrix is a rectangular array of numbers with m rows and n columns. Matrix notation will simplify the calculations in the following examples.

We write the computations of Example 1.2 as

$$\begin{bmatrix}2 & 3 & -1 & 8\\ 4 & -2 & 1 & 5\\ 1 & 5 & -2 & 9\end{bmatrix}\rightarrow\begin{bmatrix}2 & 3 & -1 & 8\\ 0 & -8 & 3 & -11\\ 0 & 7/2 & -3/2 & 5\end{bmatrix}\rightarrow\begin{bmatrix}2 & 3 & -1 & 8\\ 0 & -8 & 3 & -11\\ 0 & 0 & -3/16 & 3/16\end{bmatrix}$$

The arrows between the matrices above do not designate equality, they just indicate the flow of the computation.

Next, we change from the last augmented matrix to the corresponding system

$$\begin{cases}2x+3y-z=8\\ -8y+3z=-11\\ -\dfrac{3}{16}z=\dfrac{3}{16}\end{cases}$$

Which we solve as in Example 1.2.

Example 1.2 illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix.

The operations we used are called elementary row operations[2].

Definition 1.1 (Elementary Row Operations). We call the following three types of operations on the augmented matrix of a system elementary row operations:

(1) Multiplying a row by a nonzero number.

(2) Exchanging two rows.

(3) Changing a row by subtracting a nonzero multiple of another row from it.

Definition 1.2 (Equivalence of Systems and of Matrices). Two systems of equations are called equivalent if their solution sets are the same. Furthermore, the augmented matrices of equivalent systems are called equivalent to each other as well.

Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system. Two matrices are called row equivalent if

---

[1] augmented matrix：增广矩阵

[2] elementary row operations：初等行运算（变换）

there is a sequence of elementary row operations that transforms one matrix into the other. Hence any two matrices obtainable from each other by a finite number of successive elementary row operations are equivalent, and to indicate that they are related by such row operations, they are said to be row equivalent.

Column operations would also be possible, but they are rarely used, and we shall not discuss them at all.

We have used only the third type of elementary row operation so far. The first kind is not necessary for Gaussian elimination but will be used later in further reductions. The second kind must be used if we encounter a zero where we need a pivot, as in the following example.

【Example 1.3】 Let us solve the following system:

$$\begin{cases} x_1+2x_2=2 \\ 3x_1+6x_2-x_3=8 \\ x_1+2x_2+x_3=0 \\ 2x_1+5x_2-2x_3=9 \end{cases}$$

Solution. We do the computations in matrix form.

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & 6 & -1 & 8 \\ 1 & 2 & 1 & 0 \\ 2 & 5 & -2 & 9 \end{bmatrix} \xrightarrow[r_4-2r_1]{\substack{r_2-3r_1 \\ r_3-r_1}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{bmatrix} \xrightarrow{r_2\leftrightarrow r_4} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{r_4+r_3} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the fourth row just expresses the trivial equation, the back substitution phase should start with the third row of the last matrix. The third row gives $x_3 = -2$, the second row yields $x_2=1$. And the first row results $x_1=0$.

As the example above shows, the number m of equations and the number n of unknowns need not be the same. In this case, the four equations described four planes in three-dimensional space, having a single point of intersection given by the unique solution we have found. Of course, in general, four planes need not have a point of intersection in common or may have an entire line or plane as their intersection (in the latter case the four equations would each describe the same plane). Systems with solutions are called consistent. On the other hand, if there is no intersection, then the system has no solution, and it is said to be inconsistent❶. Inconsistency of the system can happen with just two or three planes as well, for instance if two of them are parallel, and also in two dimensions with parallel lines. So before attacking the general theory, we discuss examples of inconsistent systems and systems with infinitely many solutions. Systems with more equations than unknowns are called overdetermined, and are usually (though not always, see Example 1.3) inconsistent. Systems with fewer equations than unknowns are called underdetermined❷, and they usually (though not always) have infinitely many solutions. For example, two planes in $\mathbf{R}^3$ would usually intersect in a line, but exceptionally they could be parallel and have no intersection. On the other hand, a system with the same number of equations as unknowns is called determined and usually (though not always) has a unique solution. For instance, three planes in $\mathbf{R}^3$ would usually intersect in a point, but by exception they could be parallel and have no intersection or intersect in a line or a plane.

【Example 1.4】 (A 3×3 Inconsistent System). Consider the system given by

$$\begin{cases} x_2-4x_3=8 \\ 2x_1-3x_2+2x_3=1 \\ 5x_1-8x_2+7x_3=1 \end{cases}$$

Determine if it is consistent.

Solution. The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To obtain an x_1 in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

Next, use the x_2 in the second equation to eliminate the $-(1/2)x_2$ term from the third equation.

---

❶ inconsistent：不相容

❷ underdetermined：亚定的

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0x_3 &= 5/2 \end{aligned}$$

This system in triangular form obviously has a build-in contradiction. There are no values of x_1, x_2, x_3 that satisfy the original system because the equation $0x_3 = 5/2$ is never correct. Thus, the original system is inconsistent (i. e., has no solution).

1.2 Row Reduction and Echelon[1] Forms

In this section, we refine the method of Section 1.1 into a row reduction algorithm that will enable us to analyze any system of linear system.

The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system. So the first part of this section concerns an arbitrary rectangular[2] matrix. In the definitions that follow, a nonzero row or column in a matrix means a row or column that contains at least one nonzero entry; a leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

Definition 1.3 A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

(1) All nonzero rows are above any rows of all zeros.

(2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.

(3) All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form[3]):

(4) The leading entry in each nonzero row is 1.

(5) Each leading 1 is the only nonzero entry in its column.

---

[1] echelon：梯形

[2] rectangular：矩形的

[3] reduced row echelon form：最简型行阶梯

【Example 1.5】 The following matrix is in echelon form. The leading entries (■) may have any nonzero value, and the starred entries (*) may have any values (including zero).

$$\begin{bmatrix} ■ & * & * & * \\ 0 & 0 & ■ & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrix is in reduced echelon form because the leading entries are 1's, and there are 0's below and above each leading 1.

$$\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Any nonzero matrix may be row reduced into more than one matrix in echelon form, using different sequences of elementary row operations. However, the reduced echelon form one obtains from a matrix is unique.

Theorem 1.1 (Uniqueness of the Reduced Echelon Form)

Each matrix is row equivalent to one and only one reduced echelon matrix.

The proof of this theorem is temporarily omitted here, because it needs to use the idea from the following Chapter that the columns of row-equivalent matrices have exactly the same linear dependence relations.

If a matrix $\boldsymbol{A}$ is row equivalent to an echelon matrix $\boldsymbol{U}$, we call $\boldsymbol{U}$ an echelon form of $\boldsymbol{A}$; if $\boldsymbol{U}$ is in reduced echelon form we call $\boldsymbol{U}$ the reduced echelon form of $\boldsymbol{A}$.

Pivot Positions

When row operations on a matrix produce an echelon form, further row operations to obtain the reduced echelon form do not change the positions of the leading entries. Since the reduced echelon form is unique, the leading entries are always in the same positions in any echelon form obtained from a given matrix. This leading entries corresponding to leading 1's in the reduced echelon form.

Definition 1.4 A pivot position in a matrix $\boldsymbol{A}$ is a location in $\boldsymbol{A}$ that corresponds to a leading 1 in the reduced echelon form of $\boldsymbol{A}$. A pivot column is a column of $\boldsymbol{A}$ that contains a pivot position.

【Example 1.6】 Row reduce the matrix $\boldsymbol{A}$ below to echelon form and locate the pivot columns of $\boldsymbol{A}$.

$$A=\begin{bmatrix}0&-3&-6&4&9\\-1&-2&-1&3&1\\-2&-3&0&3&-1\\1&4&5&-9&-7\end{bmatrix}$$

Solution. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position. A choice is to interchange rows 1 and 4,

$$A=\begin{bmatrix}1&4&5&-9&-7\\-1&-2&-1&3&1\\-2&-3&0&3&-1\\0&-3&-6&4&9\end{bmatrix}$$

Create zero below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix below

$$\begin{bmatrix}1&4&5&-9&-7\\0&2&4&-6&-6\\0&5&10&-15&-15\\0&-3&-6&4&9\end{bmatrix}$$

The pivot position in the second row must be as far left as possible—namely, in the second column. We choose the 2 in (2, 2) of the matrix as the next pivot. Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

$$\begin{bmatrix}1&4&5&-9&-7\\0&2&4&-6&-6\\0&0&0&0&0\\0&0&0&-5&0\end{bmatrix}$$

Obviously, there is no way to create a leading entry in column 3 of above matrix. We interchange rows 3 and 4, we can produce a leading entry -5 in column 4 as follows.

$$\begin{bmatrix}1&4&5&-9&-7\\0&2&4&-6&-6\\0&0&0&-5&0\\0&0&0&0&0\end{bmatrix}$$

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of $\boldsymbol{A}$ are pivot columns.

With Example 1.6 as a guide, we are ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon matrix.

The Row Reduction Algorithm

The algorithm that follows consists of four steps, and it produces a matrix in