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英文授课精编教材



高等数学 下

**ADVANCED
MATHEMATICS II**

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主编



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Chapter 8 Infinite series

From the elementary mathematics, We should know that finite real numbers u_1, u_2, \dots, u_n , can be added together and the sum is also a real number. Can infinitely many real numbers be added together? What is the sum? To be specific, let's look at an example first.

Ancient Chinese philosopher Chuang Tzu said that by cutting one wooden stick in half day by day, the stick will never run out. It describes the change of the remaining length after each cutting. What is the total length cut down at last? Suppose the length cutted down after i days is denoted by s_i , we have

the first day $s_1 = \frac{1}{2}$;

the second day $s_2 = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}$;

.....

the tenth day $s_{10} = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{10}} = \frac{1023}{1024}$;

.....

the n th day $s_n = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$; (1)

If cutting without end, the sum of the length is

$$s = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots \quad (2)$$

In fact,

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^n}\right) = 1. \quad (3)$$

The formula (2) denoting the sum of infinite real numbers is called the infinite series(or series). The formula (1) which is the sum of first n terms of the infinite series(2) is called partial sum. From the formula (3), we see that the sum of the infinite real numbers is equal to the limit of the partial sum as n approaches to infinity and the limit 1 is called the sum of the infinite series. Note that the sum is

not calculated by adding total numbers together but by limit.

The sum of infinite terms occurs widely in practical problems, and it is actually the infinite series. Infinite series is a useful tool to express functions, study characters of functions, evaluate functions and solve differential equations. It also plays an important role in the theoretical researches of mathematic and practical application.

In this chapter, we will first discuss infinite series and introduce some basic contents about infinite series. Then we will also talk about power series and discuss how to write a function into a power series.

8.1 The concepts and characters of infinite series

8.1.1 The concepts of constant term progression

An infinite sequence

$$u_1, u_2, u_3, \dots, u_n, \dots,$$

the sum of total terms is

$$u_1 + u_2 + u_3 + \dots + u_n + \dots,$$

which is called an infinite series (or just series) and is denoted by the symbol $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$ for short, that is,

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \dots + u_n + \dots, \quad (1)$$

where u_n is called the n th term (or general term). Now, we give some examples.

① $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$ is called a harmonic series with $u_n = \frac{1}{n}$.

② $\sum_{n=1}^{\infty} aq^{n-1} = a + aq + aq^2 + \dots + aq^{n-1} + \dots$ (both a and q are constants) is called a geometric series with $u_n = aq^{n-1}$, where q is the common ratio.

③ $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$ ($p > 0$) is called a p -series with $u_n = \frac{1}{n^p}$.

④ $\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \dots + (-1)^{n-1} + \dots$ is a series with $u_n = (-1)^{n-1}$.

⑤ $\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + \dots + 2n + \dots$ is a series with $u_n = 2n$.

8.1.2 The convergence and divergence of infinite series

For series(1), how to add the infinite terms? From the example in the introduction of this chapter, we add up the first n terms and observe its characteristic, and then, take the limit to get the sum of the infinite terms. In general, we denote the sum of the first n terms as

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^n u_k, \quad (2)$$

which is called partial sum of the first n terms. If $n=1, 2, 3, \dots$, we will get a new sequence $s_1, s_2, s_3, \dots, s_n, \dots$, which is called the sequence of partial sums denoted by $\{s_n\}$.

Since the partial sums s_n will get closer and closer to the sum of the series as n approaches to infinity, we define the sum of the series by the limit of the partial sums as $n \rightarrow \infty$.

Definition If the sequence $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, the series $\sum_{n=1}^{\infty} u_n$ is called convergent, the number s is called the sum of series $\sum_{n=1}^{\infty} u_n$ and we write

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots = s.$$

It is also called that the series $\sum_{n=1}^{\infty} u_n$ converges to s . If the sequence $\{s_n\}$ is divergent, the series $\sum_{n=1}^{\infty} u_n$ is called divergent and the divergent series has no sum. In particular, if the sequence $\{s_n\}$ approaches to infinity, we call that the sum is infinity. However, the series is actually divergent.

Therefore, if the series(1) is convergent, the partial sum s_n will be the approximate value of the sums, that is, $s \approx s_n$. We define the difference $r_n = s - s_n = u_{n+1} + u_{n+2} + \cdots$ to be the remainder term of the series(1). The absolute value of remainder term

$$|r_n| = |u_{n+1} + u_{n+2} + \cdots|$$

is the error produced by substituting s_n for s .

Example 1 Is the geometric series

$$\sum_{n=0}^{\infty} aq^n = a + aq + aq^2 + \cdots + aq^n + \cdots \quad (3)$$

convergent or divergent, where q is the common ratio and $a \neq 0$? (If $a=0$, each term is 0. It is convergent and the sum is 0.)

Solution If $q \neq 1$, the partial sum is

$$s_n = a + aq + \cdots + aq^{n-1} = \frac{a - aq^n}{1 - q} = \frac{a}{1 - q} - \frac{aq^n}{1 - q}.$$

If $|q| < 1$, we have $\lim_{n \rightarrow \infty} s_n = \frac{a}{1 - q}$, the series is convergent and its sum is $\frac{a}{1 - q}$.

If $|q| > 1$, we have $\lim_{n \rightarrow \infty} s_n = \infty$, the series is divergent.

If $q = 1$, we have $s_n = na \rightarrow \infty$, the series is divergent.

If $q = -1$, the series is $a - a + a - a + \cdots$, we have s_n is a if n is odd and 0 if n is even,

that is, the limit of s_n does not exist, the series is divergent.

Above all, the geometric series (3) is convergent and its sum is $\frac{a}{1-q}$ if $|q| < 1$ and divergent if $|q| \geq 1$, that is,

$$\sum_{n=0}^{\infty} aq^n = \begin{cases} \frac{a}{1-q}, & |q| < 1, \\ \text{divergent}, & |q| \geq 1. \end{cases}$$

For example, series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots + (-1)^{n-1} \frac{1}{2^{n-1}} + \dots$ is a geometric series and the common ratio is $q = -\frac{1}{2}$, so the series is convergent and its sum is

$$s = \frac{a}{1-q} = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}.$$

Example 2 Show that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots \quad (4)$$

is divergent.

Solution The partial sum is

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

By the inequation $x > \ln(1+x)$ where $x > 0$ ($f(x) = x - \ln(1+x)$ is continuous if $x \geq 0$. If $x > 0$, $f'(x) > 0$ and $f(0) = 0$. Thus if $x > 0$, $f(x) > 0$), we get that

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \\ &> \ln(1+1) + \ln\left(1 + \frac{1}{2}\right) + \dots + \ln\left(1 + \frac{1}{n}\right) \\ &= \ln 2 + \ln \frac{3}{2} + \dots + \ln \frac{n+1}{n} \\ &= \ln 2 + (\ln 3 - \ln 2) + \dots + [\ln(n+1) - \ln n] \\ &= \ln(n+1). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \ln(n+1) = +\infty$, we have $\lim_{n \rightarrow \infty} s_n = +\infty$. Therefore the harmonic progression is divergent.

Note The character that harmonic progression is divergent plays an important role in the convergence tests of series and the discussion of conditions that series converges.

Example 3 Show that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} + \dots = 1$.

Solution Because

$$\begin{aligned} s_n &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \end{aligned}$$

$$= 1 - \frac{1}{n+1},$$

we have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1.$$

Thus the series is convergent and its sum is 1.

Then give the necessary condition where the progression is convergent.

Theorem If $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n \rightarrow \infty} u_n = 0$.

Proof Because $\sum_{n=1}^{\infty} u_n$ is convergent, we have $\lim_{n \rightarrow \infty} s_n = s$ and $\lim_{n \rightarrow \infty} s_{n-1} = s$. It is obvious that $u_n = s_n - s_{n-1}$, so

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = s - s = 0.$$

Note 1 $\lim_{n \rightarrow \infty} u_n = 0$ is just the necessary condition of the convergence of $\sum_{n=1}^{\infty} u_n$ but not the sufficient condition, which means that if $\lim_{n \rightarrow \infty} u_n = 0$ the series $\sum_{n=1}^{\infty} u_n$ might be convergent or divergent. For example, for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, we have $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ but the harmonic series is actually divergent.

Note 2 From Theorem 1, we have that if $\lim_{n \rightarrow \infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$ must be divergent. For example,

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \dots,$$

its general term $(-1)^{n-1}$ does not tend to 0, so the series is divergent.

To determine whether a series is convergent or not, we usually first evaluate the limit of the general term. If u_n doesn't tend to 0, the series is divergent definitely.

8.1.3 The characters of convergent series

Because the convergence or divergence of a series is corresponding to the convergence or divergence of the sequence $\{s_n\}$, so we can get the properties of convergent series from the properties of limit of sequence.

Property 1 If $\sum_{n=1}^{\infty} u_n$ is convergent and k is a constant, then the series $\sum_{n=1}^{\infty} ku_n$ is also

convergent and $\sum_{n=1}^{\infty} ku_n = k \sum_{n=1}^{\infty} u_n$.

In a word, after multiplying each term of a convergent series by a constant, the new series is still convergent.

Proof For given series $\sum_{n=1}^{\infty} u_n$, $s_n = u_1 + u_2 + \dots + u_n$, then for the series $\sum_{n=1}^{\infty} ku_n$, we have the partial sums

$$\sigma_n = ku_1 + ku_2 + \dots + ku_n = k(u_1 + u_2 + \dots + u_n) = ks_n.$$

Since $\lim_{n \rightarrow \infty} s_n = s$, we have $\lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} ks_n = ks$. Thus the series $\sum_{n=1}^{\infty} ku_n$ is also convergent and its sum is ks .

For example, if $|q| < 1$, the geometric series $\sum_{n=1}^{\infty} q^n$ is convergent and its sum is $s = \frac{q}{1-q}$. Thus if $|q| < 1$, $\sum_{n=1}^{\infty} aq^n$ is also convergent and $\sum_{n=1}^{\infty} aq^n = a \sum_{n=1}^{\infty} q^n = \frac{aq}{1-q}$.

Property 2 If series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are both convergent and their sum are s and σ respectively, then $\sum_{n=1}^{\infty} (u_n \pm v_n)$ is also convergent and its sum is $s \pm \sigma$, that is

$$\sum_{n=1}^{\infty} (u_n \pm v_n) = \sum_{n=1}^{\infty} u_n \pm \sum_{n=1}^{\infty} v_n.$$

Proof Assume that the partial sums of series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are respectively

$$\begin{aligned} s_n &= u_1 + u_2 + \dots + u_n, \\ \sigma_n &= v_1 + v_2 + \dots + v_n, \end{aligned}$$

then the partial sums of $\sum_{n=1}^{\infty} (u_n \pm v_n)$ is

$$\begin{aligned} \lambda_n &= (u_1 \pm v_1) + (u_2 \pm v_2) + \dots + (u_n \pm v_n) \\ &= (u_1 + u_2 + \dots + u_n) \pm (v_1 + v_2 + \dots + v_n) \\ &= s_n \pm \sigma_n. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} s_n = s$, $\lim_{n \rightarrow \infty} \sigma_n = \sigma$, then $\lim_{n \rightarrow \infty} \lambda_n = \lim_{n \rightarrow \infty} (s_n \pm \sigma_n) = s \pm \sigma$. Thus $\sum_{n=1}^{\infty} (u_n \pm v_n)$ is also convergent and its sum is $s \pm \sigma$.

Example 4 Show that series $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} + \frac{5}{3^n} \right)$ is convergent and find the sum.

Solution The series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a geometric series with $a=1$ and $q=\frac{1}{2}$, so $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent. The series $\sum_{n=1}^{\infty} \frac{5}{3^n}$ is a geometric series with $a=\frac{5}{3}$ and $q=\frac{1}{3}$, so $\sum_{n=1}^{\infty} \frac{5}{3^n}$ is also convergent. Then we have $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} + \frac{5}{3^n} \right)$ is convergent and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} + \frac{5}{3^n} \right) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} + \sum_{n=1}^{\infty} \frac{5}{3^n}$$

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} + \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} \\
 &= \frac{1}{1 - \frac{1}{2}} + \frac{5}{3} \cdot \frac{1}{1 - \frac{1}{3}} = 2 + \frac{5}{2} = \frac{9}{2}.
 \end{aligned}$$

Property 3 Remove, add or change finite number of terms, the new series is still convergent and its sum may differ from the original one.

Proof First, we will show that the new series generated by removing finite number of terms is still convergent.

We should prove that the series which removes one term is still convergent. Actually, the process of removing finite number of terms is same to removing one term for finite times. Assumed that we remove the term u_k from the convergent series $\sum_{n=1}^{\infty} u_n$, then the new series is

$$\sum_{n=1}^{\infty} u'_n = u_1 + u_2 + \cdots + u_{k-1} + u_{k+1} + \cdots. \quad (5)$$

Let s_n be the partial sums of the original series and σ_n be the partial sums of the new series (5). If $n > k$, it is obvious that

$$s_n = \sigma_{n-1} + u_k,$$

where u_k is a constant irrelevant to n . Thus we have that the sequence $\{\sigma_n\}$ is convergent if the sequence $\{s_n\}$ is convergent as $n \rightarrow \infty$. Therefore, the new series $\sum_{n=1}^{\infty} u'_n$ is convergent.

In the similar way, we can prove that adding or changing finite number of terms does not influence the convergence of series.

Property 4 If a series is convergent, the new series obtained by adding brackets to some terms is also convergent and has the same sum of the original series.

Proof Assumed that $u_1 + u_2 + u_3 + u_4 + \cdots$ is convergent, that is, the limit of the partial sums s_1, s_2, s_3, \cdots exists. The new series obtained by adding brackets to the original one is

$$(u_1 + \cdots + u_{n_1}) + (u_{n_1+1} + \cdots + u_{n_2}) + \cdots + (u_{n_{k-1}+1} + \cdots + u_{n_k}) + \cdots = \sum_{k=1}^{\infty} v_k,$$

where $v_k = u_{n_{k-1}+1} + u_{n_{k-1}+2} + \cdots + u_{n_k}$, which means that every bracket is one term of the new series. Obviously, there are the following relations between the partial sums of the new series and the original series:

$$A_1 = u_1 + \cdots + u_{n_1} = s_{n_1},$$

$$A_2 = (u_1 + \cdots + u_{n_1}) + (u_{n_1+1} + \cdots + u_{n_2}) = s_{n_2},$$

.....

$$A_k = (u_1 + \dots + u_{n_1}) + (u_{n_1+1} + \dots + u_{n_2}) + \dots + (u_{n_{k-1}+1} + \dots + u_{n_k}) = s_{n_k},$$

.....

It is obvious that the sequence $\{A_k\}$ is a subsequence of the sequence $\{s_n\}$. From the relationship between convergent sequence and its subsequences, since the sequence $\{s_n\}$ is convergence, we have the sequence $\{A_k\}$ is convergent and

$$\lim_{k \rightarrow \infty} A_k = \lim_{n \rightarrow \infty} s_n.$$

So the new series obtained by adding brackets converges to the same sum of the original series.

Note Convergent series cannot remove brackets optionally. In fact, the new series generated by removing brackets may not be convergent. For example, the series

$$(1-1) + (1-1) + \dots + (1-1) + \dots = 0 + 0 + \dots + 0 + \dots = 0$$

is convergent, but the series which is removed brackets

$$1 - 1 + 1 - 1 + \dots + (-1)^{n-1} + \dots$$

is divergent.

Exercises 8-1

1. Find the general terms of the following series and rewrite the series as the form \sum :

(1) $\frac{1!}{2} + \frac{2!}{5} + \frac{3!}{10} + \frac{4!}{17} + \dots;$

(2) $-\frac{1}{2} + \frac{2}{2^2} - \frac{3}{2^3} + \frac{4}{2^4} - \dots;$

(3) $\frac{a^2}{3} - \frac{a^3}{5} + \frac{a^4}{7} - \frac{a^5}{9} + \dots;$

(4) $\frac{\sqrt{a}}{2} + \frac{a}{2 \cdot 4} + \frac{a\sqrt{a}}{2 \cdot 4 \cdot 6} + \frac{a^2}{2 \cdot 4 \cdot 6 \cdot 8} + \dots.$

2. Given that the partial sum of $\sum_{n=1}^{\infty} u_n$ is $s_n = \frac{2n}{n+1}$.

(1) Find the general term.

(2) Find the first five terms.

(3) Find the sum s .

3. For the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$, u_n can be rewritten as $u_n = \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$.

(1) Find the partial sum s_n .

(2) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

4. For the convergent series $\sum_{n=1}^{\infty} u_n$, discuss the convergence or divergence of the following series:

(1) if $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} (u_n + 0.01)$, $\sum_{n=1}^{\infty} v_n$ is _____;

(2) if $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} u_{n+100}$, $\sum_{n=1}^{\infty} v_n$ is _____;

(3) if $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{u_n} (u_n \neq 0)$, $\sum_{n=1}^{\infty} v_n$ is _____.

5. By the definition of series' convergence or divergence, determine whether the following series is convergent or divergent.

(1) For series $\sum_{n=1}^{\infty} \frac{n}{3^n}$, show that $\frac{2}{3}s_n = s_n - \frac{1}{3}s_n = \frac{1}{2}\left(1 - \frac{1}{3^n}\right) - \frac{n}{3^{n+1}}$ and find the sum.

(2) For series $\sum_{n=1}^{\infty} \arctan \frac{1}{2n^2}$, show that by the mathematical induction $s_n = \arctan \frac{n}{n+1}$ and find the sum.

(3) For series $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$, find the partial sum s_n and show that the series is divergent.

6. By the convergence and divergence of geometric series and harmonic series as well as the properties of series, determine the convergence or divergence of the following series:

(1) $\frac{\ln 2}{2} + \frac{\ln^2 2}{2^2} + \frac{\ln^3 2}{2^3} + \dots + \frac{\ln^n 2}{2^n} + \dots$; (2) $\frac{1}{1001} + \frac{1}{2001} + \frac{1}{3001} + \dots + \frac{1}{1000n+1} + \dots$;

(3) $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{10n}\right)$; (4) $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right)$;

(5) $\sum_{n=1}^{\infty} 10^{10} \frac{1}{a^n}$ ($a > 0$); (6) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{5}}$.

8.2 Positive series and its convergence test

One of the core problems is how to judge the convergence or divergence of a series because we can just do operations, find sum and approximate value of convergent series. The method to determine whether a positive series is convergent or divergent is the basic method for other series.

8.2.1 The basic characters of positive series

If each term of the series $\sum_{k=1}^{\infty} u_k$ is nonnegative real number, that is, $u_n \geq 0 (n=1, 2, \dots)$, then the series is called positive series. Positive series is not only the most basic, but also very important. Later we will see that many series' problems of convergence or divergence come down to the problems of positive series.

Assume that $\sum_{k=1}^{\infty} u_k$ is a positive series and s_n is the partial sum, then $s_{n+1} - s_n = u_{n+1} \geq 0$ ($n=1, 2, \dots$), and the partial sums $\{s_n\}$ is strictly increasing. If the sequence $\{s_n\}$ has upper bound, then $\{s_n\}$ is convergent since a monotonic bounded sequence has a limit, and the positive series is convergent. On the other hand, if the positive series is convergent, the partial sums $\{s_n\}$ is also convergent. Therefore we have the following theorem of the convergent positive series.

Theorem 1 The positive series $\sum_{k=1}^{\infty} u_k$ is convergent if and only if its partial sums $\{s_n\}$ has upper bound.