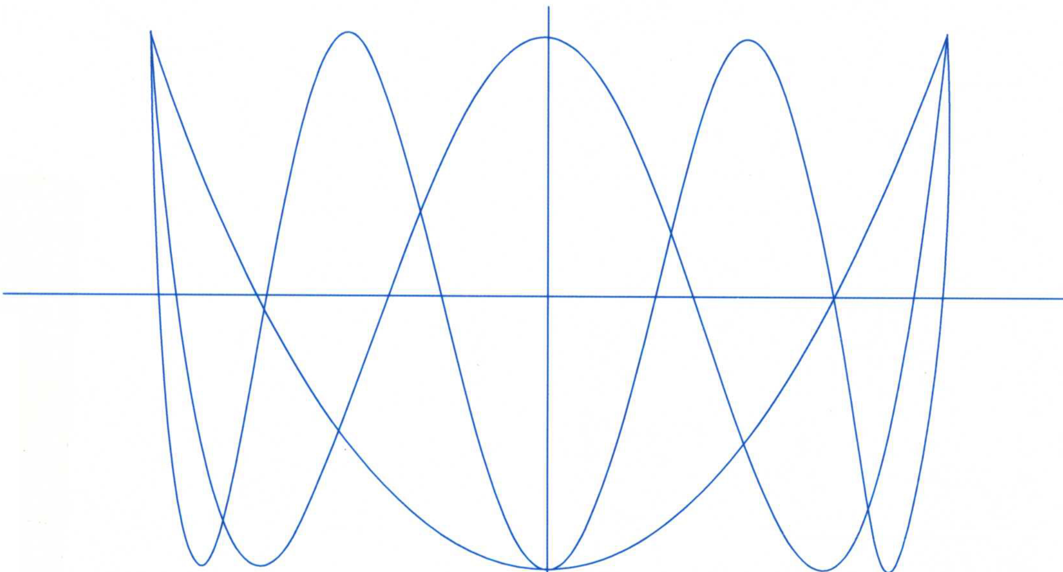
# Complex Analysis

Li Na    Ma Lixin



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# Preface

Complex analysis came into being in the 18th century. In 1774, Euler considered two equations derived from integrals of complex analysis in one of his papers. French mathematician D' Alembert had already obtained them in his paper on fluid mechanics earlier than him. Therefore, later people mentioned these two equations and called them "Darambel-Euler equations". By the 19th century, these two equations had been studied in more details when Cauchy and Riemann studied fluid mechanics, so they were also called "Cauchy-Riemann conditions".

The comprehensive development of complex analysis was in the 19th century, just as the direct expansion of calculus dominated mathematics in the 18th century. The new branch of complex analysis dominated mathematics in the 19th century. Mathematicians at that time recognized that complex analysis was the most abundant branch of mathematics. It was called the enjoyment of mathematics in this century. Some people praised it as one of the most harmonious theories in abstract science. The earliest works for the establishment of complex analysis were Euler's and Dallanbell's. Laplace of France also studied the integrals of complex analysis. They were pioneers in the creation of this discipline. Later, it was Cauchy, Riemann and German mathematician Wilstrass who laid a foundation for the development of this subject. At the beginning of the 20th century, great progress had been made in complex analysis. The students of Wilstrass, the Swedish mathematician Levler, the French mathematician Poincare and Adama did a lot of research work. A broader research field of complex analysis was opened up. They made contributions to the development of this subject.

Complex analysis involves a wide range of applications. Many complex calculations make use of it. Complex analysis has been widely used not only in other disciplines, but also in many branches of mathematics. It has penetrated into the disciplines of differential equation, integral equation, probability theory and number theory. Complex analysis has great influence on their development.

This book quotes *Complex Variables and Applications* by James Ward Brown, which has a far-reaching impact on the teaching of complex functions. It has been adopted by many famous universities such as California Institute of Technology, University of California, Berkeley, Georgia Institute of Technology, Purdue University, Dartmouth College and University of Southern California. It is easy to understand and learn the theory by citing the original works. This book has a beautiful style and strong practicability. The examples listed in this book are concise and wonderful. Basically, all the propositions given are demonstrated. This book can be used as a reference for senior students, graduate students and teachers of mathematics in colleges and universities.

# Contents

# Chapter 1 Complex Numbers and Functions

The complex function is a function of complex variables. The complex function is a branch of analytics, and it is also called complex analysis.

One of the advantages of dealing with the real numbers instead of the rational numbers is that certain equations which do not have any solutions in the rational numbers have a solution in the real numbers. For instance, $x^2 = 2$ or $x^2 = 3$ are such equations. However, we also know some equations having no solution in the real numbers, for instance, $x^2 = -1$ or $x^2 = -2$. In this chapter, we define a new kind of numbers where such equations have solutions. We will survey the algebraic and geometric structures of the complex number system.

## 1 Complex Numbers

### 1.1 Complex Number Field

**Definition 1.1.1** We call the numbers form $z = x + iy$ complex numbers, in which $x$ and $y$ are all real numbers, $i$ is a number that satisfies $i^2 = -1$ and $i$ is called imaginary unit. We call $x$ and $y$ the real and imaginary parts of $z$ and denote this by

$$\mathrm{Re}z = x, \mathrm{Im}z = y. \tag{1.1.1}$$

We notice that $z = x$ is a real number if $y = 0$, and $z = iy$ is called pure imaginary number if $x = 0$.

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if they have the same real part and the same imaginary part.

The ordinary laws of arithmetic operations are defined as:

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2),$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1),$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \quad (1.1.2)$$

As a special case the reciprocal of a complex number $z = x + iy \neq 0$ is given by

$$\frac{1}{x + iy} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

From the discussion above, we conclude that the set $C$ of all complex numbers becomes a field, called the field of complex numbers, or the complex field. We may consider $R$ as a subset of $C$.

## 1.2 Complex Plane

For mapping: $C \to R^2 : z = x + iy\ a(x, y)$ then built a one-to-one correspondence between the set of complex numbers and the plane $R^2$.

With respect to a given rectangular coordinate system in a plane, the complex number $z = x + iy$ can be represented by the point with coordinates $(x, y)$. The first coordinate axis ( $x$ -axis) takes the name of real axis, and the second coordinate axis( $y$ -axis) is called the imaginary axis. The plane itself is referred to as the complex plane.

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point $(x, y)$ that represents $z$ in the complex plane. In fact, we often refer to $z$ as the point $z$ or the vector $z$. The number, the point, and the vector will be denoted by the same letter $z$.

According to the definition of the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the number $z_1 + z_2$ corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components(Fig. 1. 1. 1). Hence

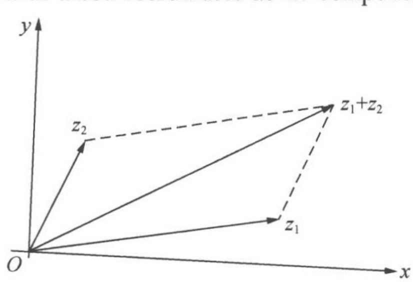


**Fig. 1. 1. 1**

$z_1 + z_2$ may be obtained as shown in Fig. 1. 1. 1. The difference $z_1 - z_2 = z_1 + (-z_2)$ *corresponds to the sum of the vectors for* $z_1$ and $-z_2$ (Fig. 1. 1. 2).

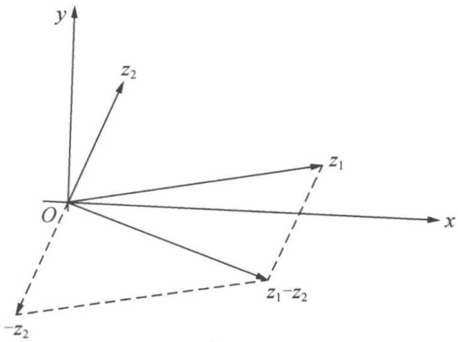


**Fig. 1. 1. 2**

## 1. 3 Modulus, Conjugation, Argument, and Polar Representation

**Definition 1. 1. 2** If $z = x + iy$ then we define

$$|z| = \sqrt{x^2 + y^2} \tag{1.1.3}$$

to be the absolute value of $z$.

If we think of $z$ as a point in the plane $(x, y)$, then $|z|$ is the length of the line segment from the origin to $z$. It reduces to the usual absolute value in the real number system when $y = 0$.

**Theorem 1. 1. 1** The absolute value of a complex number satisfies the following properties. If $z_1, z_2$, $z$ are complex numbers, then

$$|-z| = |z|, \tag{1.1.4}$$

$$|-z| = |z|, \ |z_1 z_2| = |z_1||z_2|, \ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{1.1.5}$$

$$|z_1 \pm z_2| \leqslant |z_1| + |z_2|, \ \big||z_1| - |z_2|\big| \leqslant |z_1 \pm z_2|. \tag{1.1.6}$$

Equation (1. 1. 6) is called the triangle inequality because, if we represent $z_1$ and $z_2$ in the plane, equation (1. 1. 6) says that the length of one side of the triangle is less than the sum of the lengths of the other two sides. Or, the shortest distance between two points is a straight line (Fig. 1. 1. 3).

By the mathematical induction we also get:

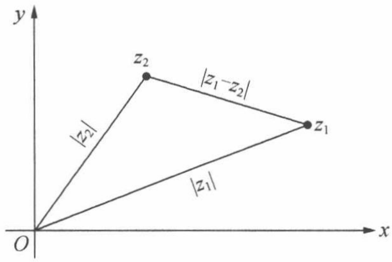


**Fig. 1. 1. 3**

**Theorem 1. 1. 2** If $z_1, z_2, \cdots, z_n$ are complex numbers then we have

$$|z_1 + z_2 + \cdots + z_n| \leqslant |z_1| + |z_2| + \cdots + |z_n|. \tag{1.1.7}$$

**Definition 1. 1. 3** The complex conjugate, or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\bar{z}$; that is $\bar{z} = x - iy$ (Fig. 1. 1. 4).

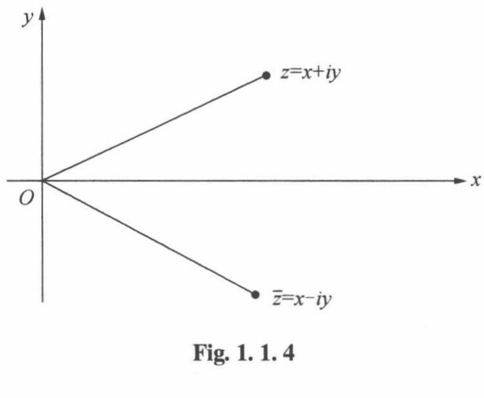


**Fig. 1. 1. 4**

The point $z$ and its conjugate $\bar{z}$ lie symmetrically with respect to the real axis. This is also easy; in fact, it is the point obtained by reflecting $z$ across the $x$-axis (i. e., the real axis). A number is real if and only if it is equal to its conjugate.

**Theorem 1. 1. 3** The complex conjugate of a complex number satisfies the following properties. If $z_1, z_2$, $z$ are complex numbers, then

$$\mathrm{Re}z = \frac{z + \bar{z}}{2}, \ \mathrm{Im}z = \frac{z - \bar{z}}{2i}; \tag{1.1.8}$$

$$z\bar{z} = |z|^2, \ |z| = |\bar{z}|, \ \bar{\bar{z}} = z; \tag{1.1.9}$$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \ \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}; \tag{1.1.10}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}\quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}(z_2 \neq 0). \tag{1.1.11}$$

Let $(x,y) = x + iy$ be a complex number. We know that any point in the plane can be represented by polar coordinates $(r,\theta)$ :

$$x = r\cos\theta, y = r\sin\theta. \tag{1.1.12}$$

Hence we can write $z = (x,y) = x + iy = r(\cos\theta + i\sin\theta)$ . In this trigonometric form of a complex number $r$ is always $\geqslant 0$ and equal to the modulus $|z|$ (Fig. 1.1.5).

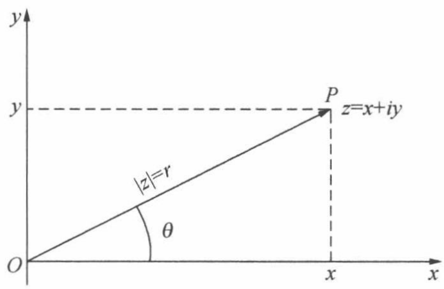


**Fig. 1.1.5**

**Definition 1.1.4** The polar angle $\theta$ is called the argument of the complex number, and we denote it by Arg$z$. The principal value of Arg$z$, denoted by arg$z$, is that unique value $\theta$ such that $-\pi < \theta \leqslant \pi$. Note that

$$\mathrm{Arg}z = \{\arg z + 2n\pi : n = 0, \pm 1, \pm 2, \cdots\}.$$

Simply, we write

$$\mathrm{Arg}z = \arg z + 2n\pi(n = 0, \pm 1, \pm 2, \cdots). \tag{1.1.13}$$

Also, when $z$ is a negative real number, arg$z$ has value $\pi$, not $-\pi$.

$$\underset{(z\neq 0)}{\arg z} = \begin{cases} \arctan\dfrac{y}{x} & (x > 0) \\ \dfrac{\pi}{2} & (x = 0, y > 0) \\ \arctan\dfrac{y}{x} + \pi & (x < 0, y \geqslant 0) \\ \arctan\dfrac{y}{x} - \pi & (x < 0, y < 0) \\ -\dfrac{\pi}{2} & (x = 0, y < 0) \end{cases} \tag{1.1.14}$$

where $-\frac{\pi}{2} < \arctan\frac{y}{x} < \frac{\pi}{2}$.

We list some important identity involving arguments:

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}z_1 + \operatorname{Arg}z_2 . \tag{1.1.15}$$

$$\operatorname{Arg}(z_2^{-1}) = -\operatorname{Arg}z_2 . \tag{1.1.16}$$

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}z_1 - \operatorname{Arg}z_2 . \tag{1.1.17}$$

**Example 1.1.1** Compute $\operatorname{Arg}(2-2i)$ and $\operatorname{Arg}(-3+4i)$.

$$\operatorname{Arg}(2-2i) = \arg(2-2i) + 2n\pi = \arctan\frac{-2}{2} + 2n\pi = -\frac{\pi}{4} + 2n\pi.$$

$$\begin{aligned}\operatorname{Arg}(-3+4i) &= \arg(-3+4i) + 2n\pi = \arctan\frac{4}{-3} + \pi + 2n\pi \\ &= (2n+1)\pi - \arctan\frac{4}{3}.\end{aligned}$$

**Example 1.1.2** Find the principal argument $\arg z$ when

$$z = \frac{-2}{1+\sqrt{3i}}.$$

Observe that

$$\operatorname{Arg}z = \operatorname{Arg}(-2) - \operatorname{Arg}(1+\sqrt{3i}) .$$

Since

$$\arg(-2) = \pi , \arg(1+\sqrt{3i}) = \frac{\pi}{3},$$

one value of $\operatorname{Arg}z$ is $\frac{2\pi}{3}$; and, because $\frac{2\pi}{3}$ is between $-\pi$ and $\pi$, we find that $\arg z = \frac{2\pi}{3}$.

## 1.4 Powers and Roots of Complex Numbers

(1) Powers

**Definition 1.1.5** We define the expression $e^{i\theta}$ to be

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ (Euler's formula)}, \tag{1.1.18}$$

where $\theta$ is to be measured in radians. Thus $e^{i\theta}$ is a complex number(Fig. 1.1.6).

It enables us to write the polar form of a complex number in exponential form as

$$z = re^{i\theta} .$$

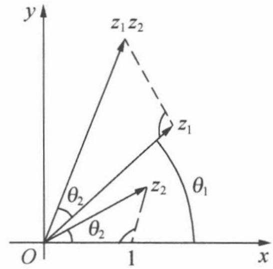


**Fig. 1.1.6**

**Theorem 1.1.4**
$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}. \tag{1.1.19}$$
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}. \tag{1.1.20}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}. \tag{1.1.21}$$
$$z^{-1} = \frac{1}{r}e^{-i\theta}. \tag{1.1.22}$$

Expressions (1.1.19), (1.1.20), (1.1.21), and (1.1.22) are, of course, easily remembered by applying the usual algebraic rules for real numbers and $e^x$.

**Definition 1.1.6** We define the powers of $z = re^{i\theta}$ as
$$z^n = r^n e^{in\theta} \quad (n = 0, \pm 1, \pm 2, \cdots). \tag{1.1.23}$$

For $r = 1$ we obtain de Moivre's Formula
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n = 0, \pm 1, \pm 2, \cdots), \tag{1.1.24}$$
which provides an extremely simple way to express $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$.

**Example 1.1.3** The number $-1-i$ has exponential form
$$-1-i = \sqrt{2}e^{i(-\frac{3}{4}\pi)}. \tag{1.1.25}$$

Expression (1.1.25) is only one of an infinite number of possibilities for the exponential form of $-1-i$:
$$-1-i = \sqrt{2}e^{i(-\frac{3}{4}\pi+2n\pi)} \ (n = 0, \pm 1, \pm 2, \cdots).$$

**Example 1.1.4** Put $(\sqrt{3}+i)^7$ in rectangular form.

We write
$$(\sqrt{3}+i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6} = (2^6 e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3}+i).$$

(2) Square Roots

To find the $n$th root of a complex number $z$ we have to solve the equation

$$w^n = z. \tag{1.1.26}$$

Suppose that $z \neq 0, z = re^{i\theta}, w = \rho e^{i\phi}$. Then Formula(1.1.26) takes the form

$$\rho^n e^{in\phi} = re^{i\theta}.$$

This equation is certainly fulfilled if $\rho^n = r$ and $n\phi = \theta + 2k\pi$. Hence we obtain the root

$$w = \sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}}, k = 0, \pm 1, \pm 2, \cdots$$

However, only the values $k = 0, 1, 2, \cdots, n-1$ give different value of $z$.

**Definition 1.1.7** We define the $n$th root of a complex number $z$ as

$$w = \sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}}, k = 0, 1, 2, \cdots, n-1. \tag{1.1.27}$$

There are $n$th roots of any complex number $z \neq 0$. They have the same modulus, and their arguments are equally spaced.

Geometrically, the $n$th roots are the vertices of a regular polygon with $n$ sides inscribed in that circle(Fig. 1.1.7).

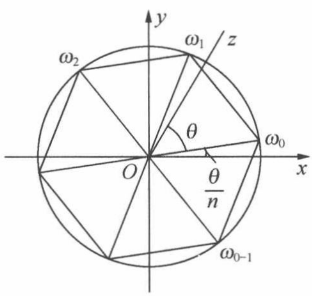


**Fig. 1.1.7**

**Example 1.1.5** Determine the $n$th roots of unity(Fig. 1.1.8).

We write

$$1 = 1e^{i0},$$

and find that

$$1^{\frac{1}{n}} = \sqrt[n]{1}e^{i\frac{0+2k\pi}{n}} = e^{i\frac{2k\pi}{n}} \quad (k = 0, 1, 2, \cdots, n-1).$$

**Example 1.1.6** Find the value of $\sqrt[4]{1+i}$ (Fig. 1.1.9).

Because $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$, then $\sqrt[4]{1+i} = \sqrt[8]{2}e^{i\frac{\frac{\pi}{4}+2k\pi}{4}}$ $(k = 0, 1, 2, 3)$.

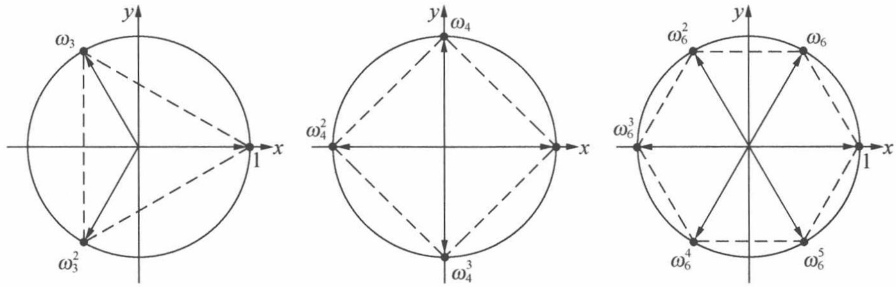


**Fig. 1. 1. 8**

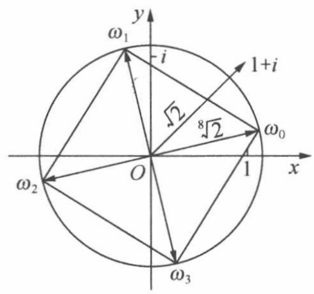


**Fig. 1. 1. 9**

# 2 Regions in the Complex Plane

## 2. 1 Some Basic Concepts

**Definition 1. 2. 1** An $\varepsilon$ -neighborhood

$$N(z_0,\varepsilon) = \{z: |z - z_0| < \varepsilon\} \tag{1.2.1}$$

of a given pint $z_0$, consists of all points $z$ lying inside but not on a circle centered at $z_0$ and with a specified positive radius $\varepsilon$.

In other words, a neighborhood of $z_0$ is a set which contains all points sufficiently near to $z_0$.

**Definition 1. 2. 2** A deleted neighborhood, or centerless neighborhood,

$$N^0(z_0,\varepsilon) = \{z:0 < |z - z_0| < \varepsilon\}, \tag{1.2.2}$$

consists of all points $z$ in an $\varepsilon$ neighborhood of $z_0$ except for the point $z_0$ itself