CTM 5
Classical Topics in Mathematics

Felix Klein

Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree

With a New Introduction and Commentary

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Classical Topics in Mathematics

Mathematics is the queen of sciences. She is pure, noble and attractive, and also has a distinct character in comparison with subjects in sciences such as physics: its permanent relevance and eternal validness of its theories and theorems. Whatever was once proved will stay true forever.

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A Note From the Editor of the Series

The regular solids, or Platonic solids, are well-known and have attracted people's attention by their rich symmetry and elegant simplicity since the ancient time. It is probably less known that they represent the highest achievement of the Greek mathematics, and their classification was discussed in the last book (Book 13) in Euclid's famous book "Elements".

The symmetry of the regular solids is also related to a lot of work in the invariant theory and singularity theory in the contemporary mathematics, which is related to the ubiquity of ADE classification and the modular miracle.

The crucial bridge between the ancient and contemporary mathematics for the regular solids is this book by Klein *The icosahedron and quintic equations*. Indeed, the most complicated regular solid is the icosahedron. Up to and through the 19th century, the most important problem was to solve algebraic equations. Abel showed that a general equation of the fifth degree cannot be solved by radicals, and Galois' theory of groups clarified the reason of failure. In this classic book, Klein showed how to relate these two seemingly unrelated topics: the symmetry group of the icosahedron and solution to quintic equations, and he also tied them together with hypergeometric functions and monodromy groups, which are crucial notions in the contemporary mathematics. This clearly shows Klein's vision of the unity of mathematics.

This book has had a huge impact on the development of mathematics. As Dieudonné wrote in 1986: "Klein's Lectures on the icosahedron, published in 1884, soon became one of the most famous books on mathematics for a half-century; no other book so forcefully put in the limelight the fundamental unity of mathematics. Klein showed that three apparently disjoint theories: the symmetries of the icosahedron ("geometry"), the resolution of fifth degree equations ("algebra") and the differential equation of hypergeometric functions ("analysis") were in fact dominated by the structure of a single object, the simple group A_5 of 60 elements."

This book was also the starting point of the four other books written by Klein and Fricke on elliptic modular functions and automorphic functions, whose English translation has been published for the first time in this series.

On the other hand, mathematics has made much progress since Klein's time, and it is not easy read such a classic. Commentaries and introductions by experts are needed to understand better their connection with problems under current study. Peter Slodowy was the leading expert on the ADE classification and singularity theory, and he has written extensively about them. Both the new material in this expanded edition of Klein's classic by him and his another expository paper will be essential to the modern reader to appreciate more this book of Klein.

Unfortunately, these writings of Slodowy are in German. This new English edition of Klein's classic together with the material by Slodowy will be very valuable to everyone who is interested in the ADE classification, and their many unexpected connections and applications.

The translation and editing of this book have been a nontrivial project. I am grateful to Lei Yang for his effort to make accurate translation both in terms of language and mathematics, and Pat Bolland for his help in polishing a part of the translation of the writings of Slodowy included in this book. Especially, I would like to thank the senior editor of HEP, Liping Wang, for initiating this series CTM and her unfailing continual support in producing every volume in this series. It is sad that when this book is published, Liping Wang will have left HEP. With many more volumes she has planned already, hope that CTM can be continued and has a good future. The publication of this book by Klein on discrete subgroups of Lie groups and automorphic forms on the icosahedron completes the grand book project by Klein. Hopefully this is a service to the English speaking mathematics community.

Lizhen Ji
March 2019.

Preface and Introduction of the Republication

by

Peter Slodowy

Translated by

Lei Yang

Preface of the Republication

The republication of Felix Klein's "Lectures on the icosahedron and the solution of equations of the fifth degree" corresponds to the constantly growing demand for this work, which was published in Leipzig more than a hundred years ago. A good deal of interest in Klein's book might be certainly due to the continuous relevance of the "icosahedral mathematics", i.e. the mathematics, in which the geometry and symmetry of the icosahedron, as well as the other Platonic solids and the regular polygons, play an essential role. In this regard, the following developments in the last twenty years are mentioned: the study of the so-called Klein singularities, also known as Du Val singularities, rational double points or simple singularities (see e.g. Du Val [1934], M. Artin [1966], Brieskorn [1968], [1970], Arnol'd [1972] or the survey articles of Arnol'd [1974], Brieskorn [1976], Durfee [1979], Slowdowy [1983]), the investigation of certain elliptic and Hilbert-Blumenthal modular surfaces (see Hirzebruch [1976], [1977], Naruki [1978], Burns [1983]), the construction of an indecomposable vector bundle of rank 2 on $\mathbf{P}^4$ (Horrocks-Mumford [1973]) and the analysis of its properties (see Barth-Hulek-Moore [1984], [1987], Barth-Hulek [1985], Decker-Schreyer [1986], Hulek [1986], [1987], Hulek-Lange [1988] and the survey article of Hulek [1989]). A particularly remarkable fact, especially with regard to Klein's thanks to Sophus Lie in the preface of his book, is the relationship between the Platonic solids, or more precisely the finite subgroups of $SU(2,\mathbf{C})$, and the complex simple Lie groups of types A_r, D_r, E_r, which was discovered by Grothendieck and Brieskorn (see Brieskorn [1970]). While this discovery built on deep studies on the theory of resolution and deformation of the singularities of surfaces mentioned above and the geometry of the conjugation classes of simple algebraic groups, a more direct, although more formal derivation of this relationship was given by J. McKay, who showed how the irreducible characters of finite binary groups can be parameterized in a natural way by the vertices of the extended Coxeter-Witt-Dynkin diagrams of the corresponding Lie groups (see McKay [1980], Ford-McKay [1979]).

Most of the developments and results mentioned above have given impetus to further fruitful researches, which are either connected with these results or have been generalized in more complicated situations (thus, McKay's observation was further developed by Happel-Preiser-Ringel [1980], Iwahori-Yokonuma [1982], Steinberg [1985], Kostant [1985], Springer [1987]; the relationship to the theory of singularities was established by Gonzalez-Sprinberg and Verdier [1981], [1983], Knörrer [1985a], then set by Artin-Verdier [1985], Esnault-Knörrer [1985], Auslander [1986] in the general context, see also the survey articles of Knörrer [1985b], Schreyer [1987]). However, it has always given discoveries again, and it will probably continue in the future, in which the icosahedron or the regular polyhedron has, in a surprising manner, thrown new light on initially more distant issues. The connection to the previous results has partly remained a problem (see e.g. Sherbak [1983], [1988] and the surveys of Arnol'd [1983] and Bennequin [1984]; while preparing for this introduction, the works of Kronheimer [1986], [1987] and Capelli-Itzykson-Zuber [1987], Ginsparg [1988] appeared, those were based on theoretical physical questions). In this case, Klein's "icosahedral book" has served as a popular reference for the "icosahedral mathematics", and it will continue to play this role.

This aspect of the book may be significant. However, he presented it only as a quarry where one can find mathematical treasures on occasion. If one wants to have a complete picture of the book, one has to consider the second part of its title. Namely, Klein's main goal was to give an original synthesis of the theories on the equations of the fifth degree, that had been created on independent paths of Hermite, Brioschi and Kronecker in 1858 with the construction of transcendental solutions. For Klein the icosahedron and its geometry should be in the foreground.

At this point, it may suffice to give a rough description of Klein's main result in this regard. Let G be the icosahedral group, i.e. the group of rotational symmetries of a regular icosahedron. This group operates on the sphere circumscribed on the icosahedron, which we identify with the Riemann sphere, i.e. the complex projective line $\mathbf{P}^1$. The quotient of $\mathbf{P}^1$ by G can be identified again with $\mathbf{P}^1$ and the quotient map $\mathbf{P}^1 \to \mathbf{P}^1/G$ is a branched covering of degree 60, the order of G. The problem to calculate a point of preimage under this map can be viewed as the solution of an equation of degree 60. Klein called such an equation an icosahedral equation. The aim of the second part of the icosahedral book is the constructive proof that the solution of any equation of the fifth degree (with complex coefficients) can be reduced to the solution of an icosahedral equation by means of essentially rational manipulation. The solution of the latter can be accomplished by means of hypergeometric series or by means of elliptic integrals and modular functions.

In his book, Klein did not have a direct approach to this goal. His intention was not to present a sophisticated theory with definitions defined clearly, theorems and formal proofs. Instead, he wanted to familiarize a broad circle of readers with numerous mathematical ideas and developments that played a central role in his own work and in his thinking. Yet his style is more narrative and descriptive than systematic and deductive. A relatively short proof of Klein's main results can be given from a modern perspective (see the representations of Weber [1899] and Dickson [1930]). Given the length and deviousness of the icosahedral book, this may be a disappointment for some readers. On the other hand, it may be, however, precisely the fullness of the icosahedral book as well as Klein's art to weave different mathematical areas together, which have kept alive the interest in this text till today. For some eulogies in this regard, references are made as D. Hilbert's famous lecture at the International Congress of Mathematicians in Paris (Hilbert [1900]) and H. Weyl's article about Klein's work and personality (Weyl [1930]).

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As prominent authors confirm (see e.g., Dickson [1930], Serre [1978]), that Klein's style presents considerable difficulties to the reader, especially when he is educated by modern textbooks. Moreover, the text of the icosahedral book is a testimony of the stage of mathematics at the time of its completion. So Klein presented the theory of Galois in a formulation, as it still went back to Galois himself substantially and was elaborated in the textbooks of Serret [1866] and Jordan [1870]. Although there have been the basics already in works of Dedekind and Kronecker, nowadays the most common interpretation of the theory of Galois in terms of field extensions and their automorphisms needed several decades to win full influence (under Weber, Hilbert, Steinitz, Artin, Noether, van der Waerden). Klein did not yet work in the categories into which we now place geometric investigations. In this sense, it very often changes its viewpoint. While some sections of the book, including naturally those which deal with the transcendental solution of the icosahedral equation, fall into the area of complex analysis, they can be attributed to other category of algebraic varieties and regular morphisms over a base field of characteristic zero. Although they are not mentioned as such, Klein gave attention to the field of definition by making occasional remarks about the coefficients occurring in explicit formulas. Klein often changed enough even to a birational viewpoint, i.e. on the level of the field of functions of the involved varieties. As is known, this means no attenuation, as long as one is concerned with smooth curves. Some statements of Klein about higher-dimensional varieties are, however, classified with caution based on the explicit formulas in the correct area of validity. In another direction, Klein could still make use of the concepts and results of the representation theory, as developed by Frobenius, Burnside and Schur from 1896. Numerous representation theoretical problems, which one would tackle with character theoretical methods nowadays, were solved by Klein with geometrical or invariant theoretical arguments. Klein was aware of the possibility of an algebraic treatment of many questions, but he emphasized again and again his preference for geometrical arguments, which he attributed to the "privilege of the invention" (see Klein's introduction of [1879a], as well as Section II, Chap. 5, §8 of the icosahedral book; Klein's view got a certain support in the introductory remarks of Horrocks and Mumford [1973], where although these authors stated the effectiveness of character theoretical calculations, there is still a lack of geometrical insight).

In order to make it easier for the modern reader to access Klein's icosahedral book, we have added a mathematical introduction to the text, which explains Klein's main results in a more modern formulation. We borrow something from our essay published previously (Slodowy [1986]). We also give a survey on the structure of the book and the role of each chapter. Separately, we have written numerous comments, which—in the form of margin—refer to individual chapters or passages, and which are mainly intended for the reader, who is already busy with the reading of the book or he has finished. The existence of a comment to a passage is marked by an asterisk on the edge. At the end of the book we give a sketch of the further developments of the Klein's theory.

For a normally non-historically oriented mathematician, the attempt to understand a historical text as the icosahedral book is a quite unusual job. Certainly there is a mathematical core which can be worked out clearly. But there are also manifold ramifications and shades, whose importance can be opened up only in a historical context. We have taken from a uniform presentation of this environment distance, since they would become too extensive. There are also other sources available, as Wussing [1969], Scholz [1980], and particularly Gray [1986], in which the significance of Klein's explicit transformation group theoretical thinking is worked out and the emergence of his theory of equations is described in connection with the development of the theories of linear differential equations, the elliptic modular functions and the automorphic functions. In detail, Klein's own comments in his collected mathematical papers [1921], [1922], [1923] and his lectures on the development of mathematics in the 19th century [1926] are proved to be very helpful.

Finally, we can send the reader only on his own journey of discovery through "that tract of beautiful country, seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower" (Cayley [1883], see also the introduction to the first English edition of the icosahedral book, 1888).

Bonn, January 1990 Peter Slodowy

Introduction to the Subject of the Icosahedral Book

§1. Resolutions of Algebraic Equations from the Viewpoint of Galois Theory.

Formulas for the resolution of linear and quadratic equations were known since ancient times. At the beginning of the 16th century, Italian mathematicians (Scipio del Ferro, Ferrari, in the schools of Tartaglia and Cardano, 1515—1545) succeeded in solving cubic and quartic algebraic equations. For the solutions of cubic equations one has the so-called formula of Cardano. By means of the substitution $x = y - a/3$, each equation

$$x^3 + ax^2 + bx + c = 0, \quad a, b, c \in \mathbf{C}$$

can be reduced to such a form

$$y^3 + py + q = 0.$$

If $d = q^2/4 + p^3/27$, then the three solutions are

$$y_{1,2,3} = \sqrt[3]{-q/2 + \sqrt{d}} + \sqrt[3]{-q/2 - \sqrt{d}}.$$

(The ambiguity of the nine values caused by the two cubic roots can be reduced to three, considering the fact that the product of the two roots equals $-p/3$.)

Until the early 19th century many mathematicians, including Tschirnhaus, Euler, Bézout, Malfatti, Vandermonde and also Lagrange tried to solve equations of higher degrees by iteration and rational combination of roots (i.e. radicals). Success always appeared only in equations of a very special form, and finally Ruffini (1799) and Abel (1824/26) showed that the solution of general equations of the fifth degree cannot be achieved with the help of radicals. The criticism on Ruffini's and Abel's proofs fell silent at the public reception of the work of Galois (1831, published 1846) whose result was unquestionable.

From the present viewpoint, Galois' treatment of the resolution problems for an algebraic equation

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$$

with complex or transcendent coefficients (these are the classic cases) is as follows. Let k be the field generated by the coefficients $a_1, \ldots, a_n$ over $\mathbf{Q}$ and $K = k(x_1,\ldots,x_n)$ the splitting field generated by the solutions $x_1, \ldots, x_n$ of $P(x) = 0$. Then the Galois group $\mathrm{Gal}(P,k)$ of the equation $P(x) = 0$ over k is equal to the Galois group $\mathrm{Gal}(K,k)$ of the field extension $k \subset K$, i.e. equal to the group of all field automorphisms of K, which leave the basis element of k fixed. In the classical context $\mathrm{Gal}(P,k)$ is a group of permutations of the n roots $x_1,\ldots,x_n$ of P. Now this interpretation can be given by considering the action of $\mathrm{Gal}(K,k)$ on the roots $x_1,\ldots,x_n$. These roots are permuted namely by $\mathrm{Gal}(K,k)$. Moreover, every automorphism of K over k is also uniquely determined by this permutation.

The Galois group $\mathrm{Gal}(P,k)$ is a qualitative measure for the complexity of the resolution process of the equation $P(x) = 0$ or, in other words, for the complexity of the algebraic field extension $k \subset K$.

Every finite group G has a composition series, i.e. a sequence of subgroups G_i

$$G = G_1 \supset G_2 \supset \cdots \supset G_m = \{1\},$$

such that G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is simple. (We also consider the cyclic groups of prime order as simple.) Although the sequence of G_i need not be uniquely determined, the simple quotients G_i/G_{i+1}, are, up to rearrangement. Now, a composition series of the Galois group $\mathrm{Gal}(P,k) = \mathrm{Gal}(K,k)$ corresponds to an ascending sequence

$$k = k_1 \subset k_2 \subset \cdots \subset k_m = K$$

of the fixed-field $k_i = K^{G_i}$, where k_{i+1} is a Galois extension of k_i with the group G_i/G_{i+1}. The problem of the construction of the roots $x_1,\ldots,x_n$ of the equation $P(x) = 0$ or (more abstractly) the field K is thus decomposed into a series of simpler steps, namely the construction of the solutions of suitable auxiliary equations $P_i(y) = 0$, $P_i \in k_i[y]$, which generate the field k_{i+1} over k_i. This is particularly clear in the case that the group G is solvable, i.e. that all simple quotients of a composition series are abelian, thus, are cyclic of prime order. Just in this case, the roots $x_1,\ldots,x_n$ can be represented namely by iterated radicals. Let us assume that k contains all the roots of unity of order $|G|$, which can be achieved by radicals, the representability of the roots by radicals follows from the following theorem of normal form for cyclic extensions, which is applied to the intermediate extensions $k_i \subset k_{i+1}$.

Theorem: *Let k be a field and $q \in \mathbf{N}$, $q \geqslant 2$, a number not divisible by* $\mathrm{char}(k)$. *The field k contains the group μ_q of the q-th roots of unity. Let $K \supset k$ be a Galois extension with cyclic group* $\mathrm{Gal}(K,k) = \mathbf{Z}/(q)$. *Then there is a $u \in k$ such that K contains all solutions of the equation $z^q - u = 0$ and is generated by each such solution, i.e.*

$$K = k(z) \ for\ all\ z\ with\ z^q - u = 0.$$

Moreover, there is an isomorphism

$$\varrho : \mathrm{Gal}(K,k) \to \mu_q$$

such that $\sigma(z) = \varrho(\sigma^{-1}) \cdot z$ for all $\sigma \in$ $\mathrm{Gal}(K,k)$ *and z with $z^q - u = 0$.*

Equations of the form $z^q - u = 0$ were formerly known as “pure equations” and z was called the “Lagrange resolvent”.

As a subgroup of the solvable symmetric groups S_2, S_3 or S_4, the Galois group for a polynomial P of degree 2, 3 or 4 is also solvable. Thus the equation $P(x) = 0$ is solvable by radicals, and we encounter the simple groups A_n and the non-solvable top groups S_n (with composition series $S_n \supset A_n \supset \{1\}$) in the case of degree $n \geqslant 5$. So S_n occurs as the Galois group $\mathrm{Gal}(P, \mathbf{Q}(a_1, \ldots, a_n))$ of the general equation of the n-th degree

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

with algebraically independent coefficients $a_1, \ldots, a_n$. Moreover, $\mathrm{Gal}(P, \mathbf{Q}) = S_n$ also applies to most polynomial P of the n-th degree with rational coefficients. Examples of rational polynomial of the fifth degree with Galois group S_5 are those which are irreducible over $\mathbf{Q}$ and have the exactly three real zeros, such as the

$$x^5 - 10x - 2 \quad \text{(see Artin [1968 Theorem 46])}.$$

The existence of equations, which can not be solved with the help of radicals, raises the following natural question: by means of which additional functions can the roots of these equations be represented in the coefficients of equations, or–more generally–in elements of the base fields? Naturally one wants to deal with as few (well-understood) functions as possible. In view of the reduction process described above, it suffices to put this question for equations with simpler Galois group. Since an answer is given in the cyclic case by the theorem of normal form, therefore arises, in more abstract form, the question of a normal form for Galois extensions $k \subset K$ with a given (simpler, non-abelian) Galois group $G = \mathrm{Gal}(K, k)$.

In his "icosahedral book", Klein dealt with this question in the case that G is isomorphic to the simple group A_5 and thus is isomorphic to the icosahedral group (see I 1, §8). In this case, we can interpret K namely as the splitting field of an equation of the fifth degree over k. It suffices to find an element $x \in K \setminus k$, which is fixed by a subgroup of G isomorphic to A_4. Then this element satisfies an irreducible equation of the fifth degree

$$x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 = 0$$

with coefficients $a_i \in k$, and the conjugates $x_1 = x, x_2, \ldots, x_5$ of x generate K over k. The role of the pure equations solvable by radicals is taken in Klein's theory by the so-called "icosahedral equation", whose definition we will explain in more detail.

§2. The Icosahedral Equation.

Let us rewrite a regular icosahedron inscribed in a sphere S^2, which we interpret as the Riemann sphere $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ by means of the stereographic projection, so the rotations of the icosahedral group G are realized by fractional linear transformations

$$z \mapsto \frac{az + b}{cz + d}, \qquad z \in \mathbf{C} \cup \{\infty\}.$$

Klein fixed the icosahedron such that its vertices obtain the coordinates

$$z = 0, \infty, \varepsilon^\nu(\varepsilon + \varepsilon^4), \varepsilon^\nu(\varepsilon^2 + \varepsilon^3), \quad \nu = 0, 1, 2, 3, 4, \quad \varepsilon = \exp(2\pi i/5).$$

Then the coefficients a, b, c, d can be chosen as elements of the field $\mathbf{Q}(\varepsilon)$. Thus we obtain an embedding of G into the projective linear group $\mathrm{PGL}_2(\mathbf{Q}(\varepsilon))$ over $\mathbf{Q}(\varepsilon)$ (see I 2, §6).

Now we consider the space of the orbits $\mathbf{P}^1/G$ of G on $\mathbf{P}^1$. This can be identified again with a complex projective line $\mathbf{P}^1$, and the natural quotient map $q:\mathbf{P}^1 \to \mathbf{P}^1/G$ is a morphism of algebraic varieties defined over $\mathbf{Q}(\varepsilon)$. The map q is realized as a branched covering of degree 60. Branches occur precisely at the points of the Riemann sphere, which correspond to the face centers, the edge midpoints and the vertices of the icosahedron. Klein fixed an inhomogeneous coordinate u (referred by him with Z) on the space of orbits $\mathbf{P}^1/G \cong \mathbf{P}^1$, such that the images of the singular orbits are in more appropriate order 0, 1, ∞. Because of these fixations, the form of the map

$$\begin{aligned} q:\mathbf{P}^1 &\to \mathbf{P}^1/G \\ z &\mapsto u \end{aligned}$$

can be determined explicitly. With respect to homogeneous coordinates z_1, z_2, $z = z_1/z_2$, q can be written as a quotient

$$u = q(z) = \frac{P(z_1, z_2)}{Q(z_1, z_2)}$$

of homogeneous polynomials P, Q of degree 60, which are invariant under the binary icosahedral group $\widehat{G}$, the preimage of $G \subset \mathrm{PGL}_2(\mathbf{Q}(\varepsilon))$ in $\mathrm{SL}_2(\mathbf{Q}(\varepsilon))$. Moreover, P (or Q, or $Q-P$) must be proportional to the third (or fifth, or second) power of a form H (or f, or T), which simply vanishes exactly at the face centers (or vertices, or edge midpoints) of the icosahedron. Such forms can be easily determined beginning from f (see I 2, §13 and §14):

$$\begin{aligned} f &= z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10}), \\ H &= -(z_1^{20} + z_2^{20}) + 228(z_1^{15} z_2^5 - z_1^5 z_2^{15}) - 494 z_1^{10} z_2^{10}, \\ T &= (z_1^{30} + z_2^{30}) + 522(z_1^{25} z_2^5 - z_1^5 z_2^{25}) - 10005(z_1^{20} z_2^{10} + z_1^{10} z_2^{20}). \end{aligned}$$

Finally, we obtain

$$q(z) = \frac{H(z_1, z_2)^3}{1728 f(z_1, z_2)^5} = \frac{H(z,1)^3}{1728 f(z,1)^5}.$$

Now, the condition $q(z) = u$ corresponds to an equation of degree 60, the so-called icosahedral equation

$$((z^{20}+1) - 228(z^{15} - z^5) + 494 z^{10})^3 + 1728 u z^5 (z^{10} + 11 z^5 - 1)^5 = 0.$$

§3. The Resolution of the Icosahedral Equation.

Just as the resolution of a "pure" equation $z^n - u = 0$, i.e. the calculation of an n-th root of a complex number u is a more well-understood and simpler analytical process, one will require something similar to the resolution of the icosahedral equation. For Klein there stood two methods for disposal. In connection to a work of H. A. Schwarz

[1873], the (local) inverse function to $q: \mathbf{P}^1 \to \mathbf{P}^1/G$ can be written as a quotient of two solutions of a hypergeometric differential equation

$$z'' + \frac{(\alpha+\alpha'-1)+(\beta+\beta'+1)u}{u(u-1)} z' + \frac{\alpha\alpha' - (\alpha\alpha'+\beta\beta'-\gamma\gamma')u + \beta\beta' u^2}{u^2(u-1)^2} z = 0$$

with the exponent data $\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix} = \begin{pmatrix} 1/6 & 1/10 & 1/4 \\ -1/6 & -1/10 & 3/4 \end{pmatrix}$. Hence, the same is represented as a quotient of hypergeometric series

$$F(a,b,c;u) = 1 + \frac{a \cdot b}{1 \cdot c} u + \frac{a(a+1)b(b+1)}{1(1+1)c(c+1)} u^2 + \cdots$$

with suitable $a, b, c \in \mathbf{Q}$.

A more indirect, but historically more significant solution of the icosahedral equation is the means of elliptic integrals and modular functions. We will describe this method more precisely, since Klein gave it only little space in the "icosahedral book".

Let $H = \{\tau \in \mathbf{C} | \mathrm{Im}(\tau) > 0\}$ be the upper half-plane of the complex numbers and $\Gamma = \mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})/\langle \pm 1 \rangle$ the "full modular group". This operates on H properly discontinuously by fractional linear transformations

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}, \quad \tau \in H, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),$$

and the natural quotient map

$$H \to H/\Gamma$$

is realized by the Dedekind-Klein's J-function

$$J: H \to \mathbf{C}$$
$$J(\tau) = \tfrac{1}{1728}\left(\tfrac{1}{q} + 744 + 196884q + \cdots\right), \quad q = e^{2\pi i \tau}$$

(see this also in the following works of Klein and Fricke mentioned later as well as newer reference of Serre [1970 Chap. VII]). Let

$$\Gamma(5) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma | a-1 \equiv b \equiv c \equiv d-1 \equiv 0(5) \right\}$$

be the principal congruence subgroup of the fifth level. Then the quotient group $\Gamma/\Gamma(5) \cong \mathrm{PSL}_2(\mathbf{F}_5)$ is isomorphic to the alternating group A_5 and thus to the icosahedral group G (the former had already been recognized by Galois in 1831). Moreover, the natural action of this group on the quotient space $H/\Gamma(5)$ can be identified with the action of the icosahedral group on the Riemann sphere, from which the twelve vertices of the icosahedron are removed. Thus we obtain a commutative diagram

$$\begin{array}{ccccc} H & \xrightarrow{J_5} & H/\Gamma(5) & \hookrightarrow & \mathbf{P}^1 \\ & J \searrow & \downarrow q' & & q \downarrow \\ & & H/\Gamma & \hookrightarrow & \mathbf{P}^1/G, \end{array}$$

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