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齐性流、模空间及算术

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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Introduction

These are the proceedings of the 2007 Clay Summer School on Homogeneous Flows, Moduli Spaces and Arithmetic, which took place at the Centro di Ricerca Matematica Ennio De Giorgi in Pisa between June 11th and July 6th, 2007. More than 100 young researchers and graduate students attended this intensive four week school, as well as 18 lecturers and other established researchers.

As suggested by the name, the topic of this summer school consisted of two connected but distinct areas of active current research: flows on homogeneous spaces of algebraic groups (or Lie groups), and dynamics on moduli spaces of abelian or quadratic differentials on surfaces. These two subjects have common roots and have several important features in common; most importantly, they give concrete examples of dynamical systems with highly interesting behavior and a rich and powerful theory. Moreover, both have applications whose scope lies well outside that of the theory of dynamical systems.

The first three weeks of the summer school were devoted to the basic theory, and consisted mostly of three long lecture series. Based on these lecture series, the following four sets of notes were written:

[1] *Interval exchange maps and translation surfaces* by J. C. Yoccoz
[2] *Unipotent flows and applications* by A. Eskin
[3] *Quantitative nondivergence and its Diophantine applications* by D. Kleinbock
[4] *Diagonal actions on locally homogeneous spaces* by M. Einsiedler and E. Lindenstrauss

Furthermore, there was a shorter lecture series

[5] *Fuchsian groups, geodesic flows on surfaces of constant negative curvature and symbolic coding of geodesics* by S. Katok.

Extensive notes for all the lecture series given in the first three weeks of the school are included in this proceedings volume (the content of the course by Eskin and Kleinbock has been separated into two different sets of notes). These papers were written to be read independently, and any of the five papers [1]–[5] could serve as a good starting point for the interested reader. More advanced topics were covered by several lecture series and individual lectures mostly given in the last week of the summer school; it was left to the discretion of the lecturers in these shorter courses whether to provide notes for these proceedings (though they were strongly encouraged to contribute). A list of these lecture notes with some additional details is given below.

The common root of both main topics of the summer school mentioned above lie (at least in part) in the theory of flows on surfaces of constant negative curvature, particularly the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$, where pioneering work was done in the early 20th century by mathematicians such as Artin, Hedlund, Morse and others, and this theory has been developed much further in the times since. One highlight was the discovery that the geodesic flow on the modular surface is intimately connected to the continued fraction expansion of real numbers; indeed, when things are properly set up, one can view the continued fraction expansion as a symbolic coding of trajectories of the geodesic flow. These flows and their symbolic codings are carefully explained in Katok's notes; in later sections of that work, recent extensions of this classical result are also discussed.

One can view the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ in two ways: firstly, it can be viewed as the locally homogeneous space $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$, in which case the geodesic flow as well as another important geometric flow — the horocycle flow — can be viewed as in the projection of trajectories of the one parameter groups

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \quad \text{and} \quad u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tag{1}$$

on the quotient space $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$. Another way to view $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is as a moduli space of flat structure (up to rotations) on a two-dimensional torus. These two different points of view generalize to the two main themes of this Clay Summer School: flows on homogeneous spaces, and flows on moduli spaces of abelian or quadratic differentials (which are essentially fancy names for flat structures in two related but slightly different senses).

Flows on moduli spaces of flat structures. The torus is the only surface admitting a flat structure with no singularities. When one considers flat structures for surfaces of higher genus, one is forced to admit singularities: points where the total angles add up to more than 2π. It turns out that *interval exchange maps* play an important role in studying the analogue of the geodesic flow (sometimes called the Teichmüller geodesic flow) on these moduli spaces of flat structures. We recall that interval exchange maps are the following simple yet intriguing dynamical system: divide the unit interval $[0,1]$ into finitely many intervals $I_1, I_2, \dots, I_d$ and then permute these intervals according to a permutation $\pi \in S_d$. Yoccoz' contribution to this proceedings provides an introduction to this theory, and provides full proofs of the most fundamental theorems (by Keane, Masur, Veech, Zorich) in the first ten sections and an introduction to some more advanced topics (Kontsevich-Zorich cocycle, cohomological equation, connected components of the moduli space, exponential mixing of the Teichmüller flow) in the last four sections. Further advanced topics are provided by notes based on the shorter lecture series

[6] *Chaoticity of the Teichmüller flow* by A. Avila

given in the last week of school; in these notes the interested reader can find surveys of the proof of two recent theorems: the simplicity of the Lyapunov spectrum for the Kontsevich-Zorich cocycle and that a typical interval exchange map with three or more intervals is weak mixing.

Flows on homogeneous spaces and applications to arithmetic. Flows on homogeneous spaces concerns the dynamics of group actions on quotient spaces $\Gamma\backslash G$, where G is usually taken to be either a **(i)** Lie group, or **(ii)** an algebraic group over $\mathbb{R}$, or **(iii)** an algebraic group over the p-adic numbers $\mathbb{Q}_p$, or **(iv)** a product of algebraic groups as in (ii) and (iii) above, involving several different fields (sometimes called an S-algebraic group, where S refers to the set of "primes" p that are used[1].)

A simple case is the case of $G = \mathrm{SL}(2, \mathbb{R})$ and Γ a lattice in G, for instance $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. In this case we have discussed (e.g. (1)) the action of two one-parameter subgroups of $\mathrm{SL}(2, \mathbb{R})$: the group g_t corresponding to the geodesic flow on the unit tangent bundle on $\Gamma\backslash\mathbb{H}$ and u_t, which corresponds to the horocycle flow on the same space. These two flows behave very differently: the u_t-flow is very rigid, and one can algebraically classify orbit closures, invariant measures, measurable factors, self joinings, and even the asymptotic distribution of individual orbits. The g_t-flow is very flexible: it is certainly ergodic, but individual orbits can behave very badly. Moreover, the g_t-flow is measure-theoretically equivalent to a Bernoulli shift which has a wealth of measurable factors and self joinings.

The group u_t is an example of a *unipotent group*. In a fundamental series of papers published in 1990; 91, M. Ratner proved that the above mentioned rigidity properties of u_t-flow are shared by all unipotent group actions on homogeneous spaces, in particular establishing in complete generality Raghunathan's conjecture about orbit closures for such actions (some cases of which were known previously, notably in the context of the Oppenheim conjecture discussed below). For the g_t-flow the situation is rather different: while a diagonalizable one-parameter group in general behaves very much like g_t, higher-dimensional diagonalizable groups seem to behave much more rigidly (though not as rigidly as unipotent group actions).

The notes by Eskin discuss in detail unipotent flows, with an emphasis on applications, particularly regarding values attained by indefinite quadratic forms and Oppenheim's Conjecture. This long-standing conjecture was proved by Margulis in the mid-80s using homogeneous dynamics, and in particular unipotent dynamics. In dynamical terms, what Margulis has shown is that any bounded orbit of $\mathrm{SO}(2,1)$ on $\mathrm{SL}(3, \mathbb{Z})\backslash \mathrm{SL}(3, \mathbb{R})$ is closed. The notes also give a detailed exposition of a more delicate result giving precise asymptotics to the distribution of these values by Eskin, Margulis and Mozes (under certain assumptions on the signature of a quadratic form). Some of the ideas and methods used in the theory of unipotent flows, and in particular some of the ideas used by Ratner in her proof of the Measure Classification Theorem are also described in these notes. Eskin's notes also contain other interesting applications of unipotent rigidity as well as connections to dynamics of rational billiards.

Kleinbock's notes focus on a method originally introduced by Margulis and developed significantly since, to show that orbits of unipotent group actions do not diverge to infinity. In particular, a quantitative version of the non-divergence statement due to S. G. Dani is an important ingredient in the proof of various versions of orbit closure and equidistribution theorems, including Ratner's Orbit Closure Theorem. However these techniques are more widely applicable and, in particular, were used by Kleinbock and Margulis to prove a conjecture of Sprindžuk on Diophantine approximations; this connection is also carefully discussed.

The notes by Einsiedler and Lindenstrauss discuss diagonalizable group actions, based mostly on work by the authors and by A. Katok in various combinations. A crucial role in current analysis of these actions is played by the concept of entropy.

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[1] For this purpose ∞ is a prime and $\mathbb{Q}_\infty = \mathbb{R}$.

These notes give a detailed and self-contained account of the theory of the entropy in the locally homogeneous context, the construction of leafwise measures on foliations, and the connection between the two. Subsequently an account is given of two rather different and complementary methods to study measures invariant under multidimensional diagonalizability actions under suitable entropy assumptions, which go under the names of the *high entropy method* and the *low entropy method*. Two applications of this theory are also discussed: a partial result toward a conjecture of Littlewood on simultaneous Diophantine approximations, and how these techniques can be used to establish Arithmetic Quantum Unique Ergodicity on compact surfaces.

The material given in these three basic papers about homogeneous dynamics is complemented by the following two more advanced notes:

[7] *Counting and equidistribution on homogeneous spaces, via mixing and unipotent flows* by H. Oh

[8] *Equidistribution of Heegner points and L-functions* by G. Harcos

In Oh's notes, the use of equidistribution of unipotent flows (and the closely related but more quantitative mixing properties of diagonalizable flows) to count integer and rational points on certain varieties, a theme touched upon in Eskin's note, is developed further, and several state-of-the-art applications are explained. The notes by Harcos give some brief background in the theory of L-functions and how it relates to equidistribution of periodic orbits of the diagonal group in SL(2).

Semiclassical analysis and dynamics. One of the applications of the theory of diagonalizable actions discussed in the Einsiedler-Lindenstrauss note is establishing Arithmetic Quantum Unique Ergodicity for compact (arithmetic) surfaces. The Quantum Unique Ergodicity conjecture deals with the asymptotic distribution of eigenfunctions of the Laplacian; the arithmetic case is a very special case where the surface is arithmetic and eigenfunctions of the Laplacian are chosen so as to respect the rich set of symmetries of such surfaces. This question is considered from a completely different point of view in the notes

[9] *Eigenfunctions of the Laplacian on negatively curved manifolds: a semiclassical approach* by N. Anantharaman.

In these notes the basics of semiclassical analysis are reviewed, the connections between eigenvalues of the Laplacian and the geodesic flow, which have been discussed to some extent in the Einsiedler-Lindenstrauss notes, are developed in a much more systematic way, and very recent work relating entropy and limiting distributions of eigenfunctions of the Laplacian in general compact negatively curved manifolds (including the variable curvature case) is exposed.

Acknowledgement. This Clay Summer School was hosted by the Centro di Ricerca Matematica Ennio De Giorgi in Pisa; we are grateful to its director, Mariano Giaquinta, for accepting to host the school in this inspiring venue. The hospitality of this institute was outstanding, and the local staff, particularly Antonella Gregorace, Ilaria Gabbani, and Valentina Giuffra, went out of their way to help this school be a success. The summer school would not come to being without the vision and generosity of the Clay Mathematics Institute, and the hard work put into the school by its president, Jim Carlson, and its program manager, Christa Carter. We would especially like to thank CMI's publication manager Vida Salahi for all her work and dedication in bringing this volume to completion.

In addition to the authors of the notes listed above, the following mathematicians gave one or more lectures during this school: G. Forni, A. Gamburd, Y. Manin, G. Margulis, J. Marklof, M. Mirzakhani, S. Mozes, N. Templier, C. Ulcigrai, and A. Venkatesh. All lecturers and participants contributed to the enthusiastic and stimulating atmosphere at this school, and we thank them warmly for this.

For a variety of reasons, these lecture notes appear almost 3 years after the summer school. Quite a bit of work went into them, and indeed this is one of the reasons for the delay. They contain a substantial amount of material which cannot be found in any textbook, and we hope you, the reader, would find them useful!

Manfred Einsiedler, David Ellwood, Alex Eskin, Dmitry Kleinbock, Elon Lindenstrauss, Gregory Margulis, Stefano Marmi and Jean-Christophe Yoccoz

May 2010

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Interval exchange maps and translation surfaces

Jean-Christophe Yoccoz

Introduction

Let T be a 2-dimensional torus equipped with a flat Riemannian metric and a vector field which is unitary and parallel for that metric. Then there exists a unique lattice $\Lambda \subset \mathbb{R}^2$ such that T is isometric to $\mathbb{R}^2/\Lambda$ and the vector field on T corresponds to the vertical vector field $\frac{\partial}{\partial y}$ on $\mathbb{R}^2/\Lambda$. The corresponding "Teichmüller space" (classification modulo diffeomorphisms isotopic to the identity) is thus $GL(2,\mathbb{R})$, viewed as the space of lattices equipped with a basis; the "moduli space" (classification modulo the full diffeomorphism group) is the homogeneous space $GL(2,\mathbb{R})/GL(2,\mathbb{Z})$, viewed as the space of lattices in $\mathbb{R}^2$.

The dynamics of the vertical vector field on $\mathbb{R}^2/\Lambda$ can be analyzed through the return map to a non vertical closed oriented geodesic S on $\mathbb{R}^2/\Lambda$; in the natural parameter on S which identifies S with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (after scaling time), the return map is a rotation $x \mapsto x + \alpha$ on $\mathbb{T}$ for some $\alpha \in \mathbb{T}$. When $\alpha \notin \mathbb{Q}/\mathbb{Z}$, all orbits are dense and equidistributed on $\mathbb{R}^2/\Lambda$: the rotation and the vectorfield are uniquely ergodic (which means that they have a unique invariant probability measure, in this case the respective normalized Lebesgue measures on S and $\mathbb{R}^2/\Lambda$).

In the irrational case, an efficient way to analyze the recurrence of orbits is to use the continuous fraction of the angle α. It is well-known that the continuous fraction algorithm is strongly related to the action of the 1-parameter diagonal subgroup in $SL(2,\mathbb{R})$ on the moduli space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of "normalized" lattices in $\mathbb{R}^2$. It is also important in this context that the discrete subgroup $SL(2,\mathbb{Z})$ of $SL(2,\mathbb{R})$ is itself a lattice, i.e. has finite covolume, but is not cocompact.

Our aim is to explain how every feature discussed so far can be generalized to higher genus surfaces. In the first ten sections, we give complete proofs of the basic facts of the theory, which owes a lot to the pionneering work of W. Veech [**Ve1**]-[**Ve5**], with significant contributions by M. Keane [**Kea1, Kea2**], H. Masur [**Ma**], G. Rauzy [**Rau**], A. Zorich [**Zo2**]-[**Zo4**], A. Eskin, G.Forni [**For1**]-[**For3**] and many others. In the last four sections, we present without proofs some more advanced results in different directions.

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The reader is advised to consult [**Zo1**] for an excellent and very complete survey on translation surfaces. See also [**Y1**] for a first and shorter version of these notes.

In Section 1 we give the definition of a translation surface, and introduce the many geometric structures attached to it. Section 2 explains how translation surfaces occur naturally in connection with billiards in rational polygonal tables. In Section 3, we introduce interval exchange maps, which occur as return maps of the vertical flow of a translation surface. We explain in Section 4 Veech's fundamental zippered rectangle construction which allow to obtain a translation surface from an interval exchange map and appropriate suspension data. The relation between interval exchange maps and translation surfaces is further investigated in Section 5, which concludes with Keane's theorem on the minimality of interval exchange maps with no connection. Section 6 introduces the Teichmüller spaces and the moduli spaces; the fundamental theorem of Masur and Veech on the finiteness of the canonical Lebesgue measure in normalized moduli space is stated. In Section 7, we introduce the Rauzy-Veech algorithm for interval exchange maps with no connection, which is a substitute for the continuous fraction algorithm. The basic properties of this algorithm are established. Invariant measures for interval exchange maps with no connection are considered in Section 8. In Section 9, the dynamics in parameter space are introduced, whose study lead ultimately to a proof of the Masur-Veech theorem. Almost sure unique ergodicity of interval exchange maps, a related fundamental result of Masur and Veech, is proven in Section 10.

In Section 11, we introduce the Kontsevich-Zorich cocycle, and present the related results of Forni and Avila-Viana. In section 12, we consider the cohomological equation for an interval exchange map and present the result of Marmi, Moussa and myself, which extend previous fundamental work of Forni. In Section 13, we present the classification of the connected components of the moduli space by Kontsevich and Zorich. In the last section, we discuss the exponential mixing of the Teichmüller flow proved by Avila, Gouezel and myself.

1. Definition of a translation surface

1.1. We start from the following combinational data :

- a compact orientable topological surface M of genus $g \geqslant 1$;
- a non-empty finite subset $\Sigma = \{A_1, \dots, A_s\}$ of M ;
- an associated family $\kappa = (\kappa_1, \dots, \kappa_s)$ of positive integers which should be seen as **ramification indices**.

Moreover we require (for reasons that will be apparent soon) that κ and g are related through

$$2g - 2 = \sum_{i=1}^{s} (\kappa_i - 1) . \tag{1.1}$$

The classical setting considered in the introduction corresponds to $g = 1, s = 1, \kappa_1 = 1$.

DEFINITION 1.1. A structure of translation surface on (M, Σ, K) is a maximal atlas ζ for $M - \Sigma$ of charts by open sets of $\mathbb{C} \simeq \mathbb{R}^2$ which satisfies the two following properties:
(i) any coordinate change between two charts of the atlas is locally a translation of $\mathbb{R}^2$;

(ii) for every $1 \leqslant i \leqslant s$, there exists a neighbourhood V_i of A_i, a neighbourhood W_i of 0 in $\mathbb{R}^2$ and a ramified covering $\pi : (V_i, A_i) \to (W_i, 0)$ of degree κ_i such that every injective restriction of π is a chart of ζ.

1.2. Because many structures on $\mathbb{R}^2$ are translation-invariant, a translation surface $(M, \Sigma, \kappa, \zeta)$ is canonically equipped with several auxiliary structures:

- a preferred orientation ; actually, one frequently starts with an **oriented** (rather than orientable) surface M and only considers those translation surface structures which are compatible with the preferred orientation ;
- a structure of Riemann surface ; this is only defined initially by the atlas ζ on $M - \Sigma$, but is easily seen to extend to M in a unique way : if V_i is a small disk around $A_i \in \Sigma$, $V_i - \{A_i\}$ is the κ_i- fold covering of $W_i - \{0\}$, with W_i a small disk around $0 \in \mathbb{C}$, hence is biholomorphic to $\mathbb{D}^*$;
- a flat metric on $M - \Sigma$; the metric exhibits a true singularity at each A_i such that $\kappa_i > 1$; the total angle around each $A_i \in \Sigma$ is $2\pi\kappa_i$;
- an area form on $M - \Sigma$, extending smoothly to M ; in the neighbourhood of $A_i \in \Sigma$, it takes the form $\kappa_i^2(x^2 + y^2)^{\kappa_i - 1} dx \wedge dy$ in a natural system of coordinates ;
- the geodesic flow of the flat metric on $M - \Sigma$ gives rise to a 1-parameter family of constant unitary directional flows on $M - \Sigma$, containing in particular a vertical flow $\partial/\partial y$ and a horizontal flow $\partial/\partial x$.

We will be interested in the dynamics of these vector fields. By convention (and symmetry) we will generally concentrate on the vertical vector field.

1.3. Together with the complex structure on M, a translation surface structure ζ also provides an holomorphic (w.r.t that complex structure) 1-form ω, characterized by the property that it is written as dz in the charts of ζ. In particular, this holomorphic 1-form does not vanish on $M - \Sigma$. At a point $A_i \in \Sigma$, it follows from condition (ii) that ω has a zero of order $(\kappa_i - 1)$. The relation (1) between g and κ is thus a consequence of the Riemann-Roch formula.

We have just seen that a translation surface structure determine a complex structure on M and a holomorphic 1-form ω with prescribed zeros. Conversely, such data determine a translation surface structure ζ : the charts of ζ are obtained by local integration of the 1-form ω.

The last remark is also a first way to provide explicit examples of translation surfaces. Another very important way, that will be presented in Section 5, is by suspension of one-dimensional maps called interval exchange maps. A third way, which however only gives rise to a restricted family of translation surfaces, is presented in the next section.

2. The translation surface associated to a rational polygonal billiard

2.1. Let U be a bounded connected open subset in $\mathbb{R}^2 \simeq \mathbb{C}$ whose boundary is a finite union of line segments ; we say that U is a polygonal billiard table. We say that U is **rational** if the angle between any two segments in the boundary is commensurate with π.

The billiard flow associated to the billiard table U is governed by the laws of optics (or mechanics) : point particles move linearly at unit speed inside U, and reflect on the smooth parts of the boundary ; the motion is stopped if the boundary is hit at a non smooth point, but this only concerns a codimension one subset of initial conditions.

The best way to study the billiard flow on a rational polygonal billiard table is to view it as the geodesic flow on a translation surface constructed from the table; this is the construction that we now explain.

2.2. Let $\widehat{U}$ be the **prime end compactification** of U : a point of $\widehat{U}$ is determined by a point z_0 in the closure $\overline{U}$ of U in $\mathbb{C}$ and a component of $B(z_0,\varepsilon)\cap U$ with ε small enough (as U is polygonal, this does not depend on ε if ε is small enough).

EXERCISE 2.1. Define the natural topology on $\widehat{U}$; prove that $\widehat{U}$ is compact, and that the natural map from U into $\widehat{U}$ is an homeomorphism onto a dense open subset of $\widehat{U}$.

EXERCISE 2.2. Show that the natural map from $\widehat{U}$ onto $\overline{U}$ is injective (and then a homeomorphism) iff the boundary of U is the disjoint union of finitely many polygonal Jordan curves.

A point in $\widehat{U}-U$ is **regular** if the corresponding sector in $B(z_0,\varepsilon)\cap U$ is flat; the non regular points of $\widehat{U}-U$ are the vertices of $\widehat{U}$.

EXERCISE 2.3. Show that every component of $\widehat{U}-U$ is homeomorphic to a circle and contain at least two vertices. Show that there are only finitely many vertices.

A connected component of regular points in $\widehat{U}-U$ is a **side** of $\widehat{U}$. The closure in $\widehat{U}$ of a side C of $\widehat{U}$ is the union of C and two distinct vertices called the **endpoints** of C. A vertex is the endpoint of exactly two sides.

2.3. The previous considerations only depend on U being a polygonal billiard table ; we now assume that U is rational. For each side C of $\widehat{U}$, let $\sigma_C \in O(2,\mathbb{R})$ the orthogonal symmetry with respect to the direction of the image of C in $\overline{U}\subset\mathbb{R}^2$. Let G be the subgroup of $O(2,\mathbb{R})$ generated by the σ_C.

As U is rational, G is finite. More precisely, if N is the smallest integer such that the angle between any two sides of $\widehat{U}$ can be written as $\pi m/N$ for some integer m, G is a dihedral group of order $2N$, generated by the rotations of order N and a symmetry σ_C.

For any vertex $q\in\widehat{U}$, we denote by G_q the subgroup of G generated by σ_C and $\sigma_{C'}$, where C and C' are the sides of $\widehat{U}$ having q as endpoint ; if the angle of C and C' is $\pi\, m_q/N_q$ with $m_q\wedge N_q=1$, G_q is dihedral of order $2N_q$.

We now define a topological space M as the quotient of $\widehat{U}\times G$ by the following equivalence relation : two points $(z,g),(z',g')$ are equivalent iff $z=z'$ and moreover

- $g^{-1}g'=\mathbf{1}_G$ if $z\in U$;
- $g^{-1}g'\in\{\mathbf{1}_G,\sigma_C\}$ if z belongs to a side C of $\widehat{U}$;
- $g^{-1}g'\in G_z$ if z is a vertex of $\widehat{U}$.

We also define a finite subset Σ of M as the image in M of the vertices of $\widehat{U}$.

EXERCISE 2.4. Prove that M is a compact topological orientable surface.

To define a structure of translation surface on (M,Σ) (with appropriate ramification indices), we consider the following atlas on $M-\Sigma$.

- for each $g \in G$, we have a chart
$$U \times \{g\} \to \mathbb{R}^2$$
$$(z, g) \mapsto g(z) ;$$
- for each z_0 belonging to a side C of $\widehat{U}$, and each $g \in G$, let $\tilde{z}_0$ be the image of z_0 in $\overline{U}, \varepsilon$ be small enough, V be the component of $B(\tilde{z}_0, \varepsilon) \cap U$ corresponding to $z_0, \widehat{V}$ be interior of the closure of the image of V in $\widehat{U}$; we have a map
$$\widehat{V} \times \{g, g\, \sigma_c\} \to \mathbb{R}^2$$
sending (z, g) to $g(z)$ and $(z, g\sigma_c)$ to $g(\tilde{\sigma}_c(z))$, where $\tilde{\sigma}_c$ is the **affine** orthogonal symmetry with respect to the line containing the image of C in $\mathbb{R}^2$. This map is compatible with the identifications defining M and defines a chart from a neighbourhood of (z, g) in M onto an open subset of $\mathbb{R}^2$.

One checks easily that the coordinate changes between the charts considered above are translations. One then completes this atlas to a maximal one with property (i) of the definition of translation surfaces.

EXERCISE 2.5. Let q be a vertex of $\widehat{U}$, and let $\pi m_q/N_q$ be the angle between the sides at q and G_q the subgroup of G as above. Show that property (ii) in the definition of a translation surface is satisfied at any point $(q, g\, G_q) \in \Sigma$, with ramification index m_q (independent of the coset $g\, G_q$ under consideration).

We have therefore defined the ramification indices κ_i at the points of Σ and constructed a translation surface structure on (M, Σ, κ).

2.4. The relation between the trajectories of the billiard flow on U and the geodesics on $M - \Sigma$ is as follows.

Let $z(t), 0 \leqslant t \leqslant T$ be a billiard trajectory ; let $t_1 < \cdots < t_N$ be the successive times in $(0, T)$ where the trajectory bounces on the sides of $\widehat{U}$ (by hypothesis, the trajectory does not go through a vertex, except perhaps at the endpoints 0 and T). Denote by C_i the side met at time t_i and define inductively $g_0, \ldots, g_N$ by
$$g_0 = \mathbf{1}_G ,$$
$$g_{i+1} = g_i\, \sigma_{C_{i+1}} .$$
For any $g \in G$, the formulas
$$z_g(t) = \begin{cases} (z(t), gg_0), & \text{for } 0 \leqslant t \leqslant t_1, \\ (z(t), gg_i), & \text{for } t_i \leqslant t \leqslant t_{i+1}\ (1 \leqslant i < N), \\ (z(t), gg_N), & \text{for } t_N \leqslant t \leqslant T, \end{cases}$$
define a geodesic path on M. Conversely, every geodesic path on M (contained in $M - \Sigma$ except perhaps for its endpoints) defines by projection on the first coordinate a trajectory of the billiard flow on U.

2.5. The left action

$$g_0(z,g) = (z, g_0\, g)$$

of G on $\widehat{U} \times G$ is compatible with the equivalence relation defining M and therefore defines a left action of G on M. The corresponding transformations of M are isometries of the flat metric of M but not isomorphisms of the translation surface structure (except for the identity !). The existence of such a large group of isometries explain why the translation surfaces constructed from billiard tables are special amongst general translation surfaces.

2.6. On the other hand, when a billiard table admits non trivial symmetries, this gives rise to isomorphisms of the translation surface structures. More precisely, let H be the subgroup of G formed of the $h \in G$ such that $h(U)$ is a translate $U + t_h$ of U. The group H acts on the left on M through the formula

$$h(z,g) = (h(z) - t_h, g\, h^{-1}),$$

which is compatible with the equivalence relation defining M. Each $h \in H$ acts through an isomorphism of the translation surface structure (permuting the points of Σ). This allows to consider the quotient under the action of H to get a reduced translation surface $(M', \Sigma', \kappa', \zeta')$ and a ramified covering from (M, Σ) onto (M', Σ').

2.7. To illustrate all this, consider the case where U is a regular n-gon, $n \geqslant 3$. The angle at each vertex is then $\pi\frac{n-2}{n}$.

EXERCISE 2.6. Show that $G = G_q$ for every vertex q and that G has order n if n is even, $2n$ if n is odd. Show that Σ has n points, each having ramification index $n-2$ if n is odd, $\frac{n-2}{2}$ if n is even. Conclude that the genus of M is $\frac{(n-1)(n-2)}{2}$ if n is odd, $(\frac{n}{2}-1)^2$ if n is even.

EXERCISE 2.7. Show that the subgroup H of subsection 2.6. is equal to G if n is even, and is of index 2 if n is odd. Show that the reduced translation surface satisfies $\#\Sigma' = 2$ if $N-2$ is divisible by 4, $\#\Sigma' = 1$ otherwise. Show that the corresponding ramification index is $n-2$ if n odd, $\frac{(n-2)}{2}$ if n is divisible by 4, $\frac{(n-2)}{4}$ if $n-2$ is divisible by 4. Conclude that the genus g' is $\frac{(n-1)}{2}$ if n is odd, $\frac{n}{4}$ if n is divisible by 4 , $\frac{(n-2)}{4}$ if $n-2$ is divisible by 4.

3. Interval exchange maps : basic definitions

3.1. Let $(M, \Sigma, \kappa, \zeta)$ be a translation surface and let X be one of the non zero constant vector fields on $M - \Sigma$ defined by ζ.

DEFINITION 3.1. An **incoming (resp. outgoing) separatrix** for X is an orbit of X ending (resp. starting) at a marked point in Σ. A **connection** is an orbit of X which is both an incoming and outgoing separatrix.

At a point $A_i \in \Sigma$, there are κ_i incoming separatrices and κ_i outgoing separatrices.

Let S be an open bounded geodesic segment in $M - \Sigma$, parametrized by arc length, and transverse to X. Consider the first return map T_S to S of the flow generated by the vectorfield X.

As X is area-preserving, the Poincaré recurrence theorem guarantees that the map T_S is defined on a subset D_{T_S} of S of full 1-dimensional Lebesgue measure.