



The Complete Works of Wu Wen-Tsun  
Mathematics Mechanization V

# 吴文俊全集

◆ 数学机械化卷V

吴文俊/著 高小山/编订



科学出版社



龙门书局



国家出版基金项目  
NATIONAL PUBLICATION FOUNDATION

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北 京

## 内 容 简 介

本卷收录了吴文俊在数学机械化领域发表的46篇论文,内容包括:几何定理机器证明的吴方法、多项式系统符号求解的Ritt-吴特征列方法、构造性微分代数几何理论、不等式机器证明与优化问题的有限核定理等数学机械化领域的奠基性成果,还包括数学机械化方法在数学定理的自动发现、天体中心构型问题求解、平面机构定理的机器证明、机器人的运动学问题的自动求解、几何设计中的曲面拼接等问题中的应用。

本书可作为数学与相关领域研究人员、教师、研究生的参考文献,也可以作为计算机数学专业的参考书。

### 图书在版编目(CIP)数据

吴文俊全集. 数学机械化卷. V / 吴文俊著; 高小山编订. —北京: 龙门书局, 2019.5

国家出版基金项目

ISBN 978-7-5088-5554-7

I. ①吴… II. ①吴… ②高… III. ①数学理论—文集 IV. ①O1-53

中国版本图书馆CIP数据核字(2019)第074646号

责任编辑: 李兴 / 封面设计: 彭珍珍

责任印制: 肖兴 / 封面设计: 王超书装



科学出版社 出版

北京东黄城根北街16号

邮政编码: 100717

<http://www.sciencep.com>

北京通州皇家印刷厂印刷

科学出版社发行 各地新华书店经销

\*

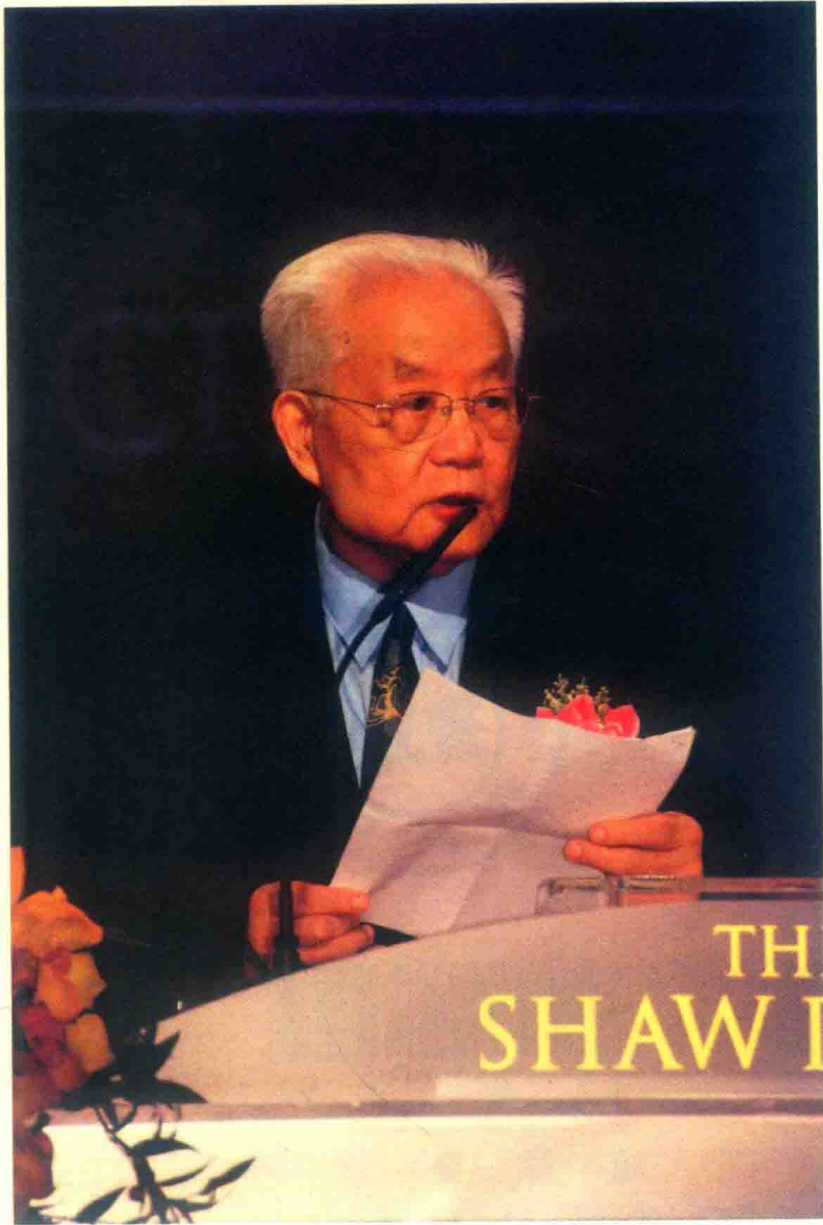
2019年5月第一版 开本: 720×1000 1/16

2019年5月第一次印刷 印张: 43 3/4

字数: 880 000

定价: 298.00元

(如有印装质量问题, 我社负责调换)



纪念吴文俊先生诞辰 100 周年

## 编者序

中国现代数学的崛起, 开始于 20 世纪初, 经历了几代人坚苦卓绝的努力. 在这百年奋战中涌现出来的数学家中, 吴文俊是最杰出的代表之一. 他早年留学法国, 留学期间就已在拓扑学方面做出了杰出贡献, 提出了后来以他的名字命名的“吴公式”和“吴示性类”. 回国后提出了“吴示嵌类”等拓扑不变量, 发展了统一的嵌入理论. 他关于示性类与示嵌类的研究, 已成为 20 世纪拓扑学的经典, 至今还在前沿研究中使用. 20 世纪 70 年代以来, 吴文俊院士在汲取中国古代数学精髓的基础上, 开创了崭新的现代数学领域——数学机械化. 他发明的被国际上誉为“吴方法”的数学机械化方法, 改变了国际自动推理的面貌, 形成了自动推理的中国学派, 已使中国在数学机械化领域处于国际领先地位. 上述工作无疑属于 20 世纪中国数学赶超国际先进水平的标志性成果, 而吴文俊院士博大精深的科学研究, 除了拓扑学与数学机械化以外, 还跨越了代数几何、博弈论、中国数学史、计算图论、人工智能等众多领域, 并在每个领域都留下了这位多能数学家的重要贡献.

吴文俊先生是一位具有强烈爱国精神的数学家. 自 1950 年谢绝法国师友的挽留回到祖国后, 半个世纪如一日, 为在他深爱的中华故土发展数学事业而鞠躬尽瘁. 除了第一流的科研成果, 吴文俊先生长期身处中国数学界领导地位, 在团结带领整个中国数学界赶超世界先进水平方面, 也做出了不可磨灭的贡献. 特别是, 吴文俊先生在担任中国数学会理事长期间, 领导中国数学最终成功地加入了国际数学联盟, 此举大大提高了我国数学界的国际地位, 同时也为我国成功举办 2002 年国际数学家大会铺平了道路.

吴文俊治学严谨, 学术思想活跃, 无论获得多么高的声誉, 他总是勤奋地在科研第一线工作, 一生积极进取, 锲而不舍, 不断取得新的成就. 在开始从事机器证明时, 他已近花甲之年, 从零开始学习编写计算机程序, 每天十多个小时在机房连续工作, 终于在几何定理机器证明这一难题上取得成功.

吴文俊先生为中国现代数学的发展建立了丰功伟绩, 而他本人却始终淡泊、谦逊. 他处事公正豁达, 待人充满善意, 受过他帮助的人可以说不计其数. 正因如此, 这位有着崇高国际声望而平易近人的学者, 受到了每一个认识他的人格外的爱戴与尊敬.

2019 年 5 月 12 日是吴文俊先生百年诞辰. 为了纪念这个特殊的日子, 我们编辑出版了《吴文俊全集》, 通过系统地收录、整理吴文俊先生的学术著作和论文, 纪

念吴先生的学术思想及学术成就. 全集共计 13 卷, 包括拓扑学 4 卷、数学机械化 5 卷以及数学史、博弈论与代数几何、数学思想各 1 卷; 同时, 全集还设有附卷, 收录吴文俊先生的同事、学生和其他社会各界人士发表过的与吴先生有关的各类文献资料.

最后, 我们对在全集编辑中给予帮助的各位同事表示衷心感谢; 感谢国家出版基金对于全集出版的资助; 感谢科学出版社编辑人员在出版全集时认真细致的专业精神; 感谢相关出版与新闻机构在版权方面提供的帮助.

李邦河 高小山 李文林

2019 年 3 月

## 前 言

本卷收录了吴文俊在数学机械化领域发表的 46 篇论文. 我们将根据吴的主要研究成果, 分类介绍这些论文.

发表于 1978 年的论文 [1] 是吴文俊开创数学机械化研究的开山之作, 提出了基于代数计算的几何定理机器证明的吴方法. 本文发表后的前几年并未引起学术界的重视. 在 1983 年召开的美国数学会年会上, 当时在美国得克萨斯大学攻读博士的周咸青报告了他用吴方法进行几何定理机器证明的结果, 引起了国外学术界的广泛关注. 本次会议出版的论文集 *Automated Theorem Proving: After 25 Years* 中收录了文 [1], 吴文俊关于机器证明的工作开始在国外广泛传播.

文 [1] 只包括了几何定理机器证明吴方法的核心步骤, 数学机械化方法的系统论述在文 [7] 中给出, 其核心是多项式系统符号求解的 Ritt-吴特征列方法. 吴文俊自 1952 年回国后, 除了一些特殊情况, 如被邀请撰写论文外, 所有论文都发表在国内的期刊, 论文 [7] 发表在《系统科学与数学》英文版. 1983 年美国数学会年会后, 欧美多个研究组相继开始从事相关研究, 而当时发表在国内期刊的论文在国外很难看到. 国际《自动推理杂志》因此全文转载了文 [7]. J. Moore 还专门写了“编辑说明”, 讲到: “该文不仅奠定了几何定理高效证明的基础, 而且为未来几何推理机器证明树立了一个标杆”“在将来很长一段时间, 几何定理证明研究都将基于该文, 该文将是几何定理证明不可或缺文献.”

数学机械化的一个主要分支是微分多项式系统的符号求解, 或构造性微分代数几何. 实际上, 吴文俊在发表第一篇机器证明论文的次年即发表了微分几何定理机器证明的工作 [2,3,8]. 这里主要关注的是常微分多项式系统, 相关的机器证明方法与代数情形区别不大. 在文 [16] 中, 作为其方法的应用, 吴文俊证明 Newton 的万有引力定理可以从 Kepler 行星运动定律自动推导出来. 该问题是吴访问美国阿贡国家实验室时 L. Wos 所提, 被认为是机器证明方法用于物理规律自动发现的最好示例之一. 到 1989 年, 吴文俊在文 [20] 中给出了偏微分多项式系统机器证明的系统方法. 在文 [27, 42] 中, 吴文俊指出: 多项式系统的 Gröbner 基算法可以视为偏微分多项式系统特征列方法的特例.

Ritt-吴特征列方法处理的是等式型问题, 即考虑多项式方程的复零点或微分代数方程在微分代数闭域中的零点. 在文 [17, 29, 33, 37] 中, 吴文俊进一步提出了自

动证明不等式与求解优化问题的机械化方法. 主要结果是有限核定理, 即多项式问题能取到的极值是有限的且是一个可以有效构造的单变元多项式方程的根.

在文 [35] 中, 针对符号计算容易产生计算空间膨胀的困难, 吴文俊提出了发展数值与符号混合算法的设想, 其目的是既可以像数值计算那样速度快, 又可以在一定意义下保持计算结果的准确性.

吴文俊在文 [9, 21, 24, 31, 34] 中从不同侧面改进与完善了其数学机械化方法. 文 [9] 改进了特征列方法, 使其可以更加有效地用于方程求解. 文 [21] 指出, 可以借助 Chow 形式从特征列求相应多项式理想的生成元. 文 [24] 给出了如何借助特征列方法进行复数域上的量词消去. 文 [31, 34] 提出如何借助结式理论将非线性代数方程求解转变为线性方程求解.

吴文俊特别关注数学机械化方法的应用, 在其系列论文 [11, 14, 17, 19, 22, 23] 中将自己的数学机械化方法应用于多个领域. 文 [11] 是关于数学定理的自动发现. 吴文俊特别提出, 是否可以自动推导非欧几何的体积公式. 文 [14] 给出了一类微分几何定理机器证明与发现的结果. 文 [17] 给出了利用特征列方法证明不等式的算法. 文 [19] 将特征列方法用于平面机构定理的机器证明. 文 [22] 将特征列方法用于超越方程的符号求解. 文 [23] 求解了一类串联机器人的运动学问题. 吴文俊的其他应用方面的论文还包括: 化学反映平衡点的计算 [25]、几何设计中的曲面拼接 [36,38]、天体中心构型的 Wintner 猜想的一些特殊情形的证明 [40,43]. 这些应用方面的论文反映了吴文俊数学机械化研究的一个显著特点: 不仅要 (部分) 实现数学的机械化, 而且要将机械化的数学方法用于解决高科技问题, 实现各个领域问题求解的机械化. 吴在应用方面的工作也具有很大影响, 他研究过的多个方向都有不少的追随者.

文 [4] 虽然不是一篇研究论文, 但值得特别介绍. 这是在实现了几何定理机器证明的突破后, 吴文俊思考其工作的历史定位时的一点儿发现. 本文指出, Hilbert 的名著《几何基础》中一个晦涩难懂的定理, 其实质是给出一类几何定理的机器证明算法. 实际上, 定理的自动证明是一个“古老的梦想”, 而 Hilbert 首次给出了其明确的数学描述. Hilbert 在 1928 年提出了“Hilbert 计划”, 其中的判定性问题是: 是否存在一个算法, 可以“机械化”地判定每个数学分支中所有命题的正确性. Gödel 在 1931 年证明, 即使是 Peano 算术这样简单的数学系统也是不可判定的, 从而否定地回答了 Hilbert 的判定性问题. 吴文俊观察到, 关于机器证明的著名结果, 大多是否定性的, 或者是关于单个困难问题, 如四色定理的证明. 在证明这些定理过程中发展的方法难以应用到其他问题. 吴文俊根据自己工作的特点, 提出一个更加实用的中间路线, 并给出了后来称之为“数学机械化纲领”的研究思路: “在数学的各个

学科选择适当的范围,既不至于太小以致失去意义,又不至太大以致于不可机械化,提出切实可行的方法,实现机械化,推动数学发展,并以此为基础解决高科技问题。”

高小山

2019年4月8日

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# 1. On the Decision Problem and the Mechanization of Theorem-Proving in Elementary Geometry\*

**Abstract** The idea of proving theorems mechanically may be dated back to Leibniz in the 17th century and has been formulated in precise mathematical forms in this century through the school of Hilbert as well as his followers on mathematical logic. The problem consists in essence in replacing qualitative difficulties inherited in usual mathematical proofs by quantitative complexities of calculations on standardizing the proof procedures in an algorithmic manner. Such quantitative complexities of calculations, formerly far beyond the reach of human abilities, have become more and more trivial owing to the occurrence and rapid development of computers. In spite of vigorous efforts, however, researches in this direction give rise quite often to negative results in the form of undecidable mathematical theories. To cite a notable positive result, we may mention Tarski's method of proving theorems mechanically in elementary geometry and elementary algebra. The methods of Tarski as well as later ones are largely based on a generalization of Sturm theorem and are still too complicated to be feasible, even with the use of computers. The present paper, restricted to theorems with betweenness out of consideration and based on an entirely different principle, aims at giving a mechanical procedure which permits to prove quite non-trivial theorems in elementary geometry even by hands.

## I. Formulation of the problem

A. Tarski in a classic paper<sup>[14]</sup> of 1948 has settled the decision problem of real closed field with one of its main aims to give mechanical proofs of theorems in elementary geometry. Alternative proofs of Tarski's result have later been given by

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\* *Scientia Sinica*, 1978, 21: 159-172. Re-published in *Automated Theorem Proving: After 25 Years*, 213-234. AMS Press.

Seidenberg, A. Robinson and P. J. Cohen, ef [12, 9, 2]. These authors have even suggested construction of certain decision machine to carry out such mechanical proofs. However, such a procedure seems to be far from being realized. In fact, only proofs of very trivial theorems in elementary geometry have actually been carried out on computers, ef. e.g.[6, 7]. The purpose of the present paper is, leaving aside questions involving betweenness of points, to give an alternative solution of the decision problem of elementary geometry based on a principle entirely different from those employed by the authors above-mentioned. Our method permits to furnish mechanical proofs of quite difficult geometrical theorems which can be practised even by hands, i.e., by means of papers and pencils only. The programming on a computer, based on such a method, though has not yet been done, will present no actual difficulties at all.

We shall restrict our considerations wholly to plane elementary geometry, though our method may be applied to the consideration of various other kinds of geometry. The first step of our method consists in the algebraic formalization of the geometrical problems involved. Points in the plane are to be defined as ordered pairs of numbers in a fixed field, say the field of rational numbers  $R$ . A dictionary is then set up turning geometrical relations into algebraic expressions which may be considered as either definitions or axioms. For example, for points  $A_i = (x_i, y_i)$ , distinct or not, we shall say:

$A_1A_2$  is parallel to  $A_3A_4$  if  $(x_1 - x_2)(y_3 - y_4) - (x_3 - x_4)(y_1 - y_2) = 0$ ,

$A_1A_2$  is orthogonal to  $A_3A_4$  if  $(x_1 - x_2)(x_3 - x_4) + (y_1 - y_2)(y_3 - y_4) = 0$ ,

the length - square of  $A_1A_2$  is  $r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ , etc.

We may replace the basic field  $R$  by other fields, make correspondence the points to other kinds of number-sets, or modify the algebraic expressions in the axioms, e.g., instead of the length-square function  $r^2$  given above, we consider the function  $r^4 = (x_1 - x_2)^4 + (y_1 - y_2)^4$ . We then go to other realms of geometry, non-euclidean geometry, real or complex projective geometry, finite geometry, etc. We shall however stick ourselves in what follows to plane elementary geometry only which has some representative character.

To illustrate our method of treatment, let us cite first a simple example. Consider the following statement:

( $S_n$ ) Let  $A_0A_1A_2$  be a right-angled triangle with right angle at  $A_0$ . If  $x_1, x_2$  denote the lengths of sides  $A_0A_1, A_0A_2$  and  $x_3$  is the length of the hypotenuse, then

$$x_1^n + x_2^n = x_3^n.$$

The problem is to decide whether the statement  $(S_n)$  is true or not and to give an algorithmic procedure of proving or disproving  $(S_n)$  which holds good for all statements alike in elementary geometry (with betweenness out of consideration) .

To solve this problem, let us remark first that the points, etc., occurring in the statement have a generic character subjected to the conditions implied in the hypothesis of the statement. Thus, if we represent the points in question in coordinates with  $A_0 = (x_0, v_0), A_1 = (u_1, v_1), A_2 = (u_2, v_2)$ , the coordinates  $v_0, u_1, v_1, u_2, v_2$  can be considered as indeterminates. On the other hand, the other coordinates and geometric entities  $x_0, x_1, x_2, x_3$  are then algebraically dependent on these indeterminates, being restricted by following algebraic equations according to hypothesis of the statement  $(S_n)$ :

$$\begin{aligned} f_0 &\equiv (u_1 - x_0)(u_2 - x_0) + (v_1 - v_0)(v_2 - v_0) = 0, \\ f_1 &\equiv x_1^2 - (u_1 - x_0)^2 - (v_1 - v_0)^2 = 0, \\ f_2 &\equiv x_2^2 - (u_2 - x_0)^2 - (v_2 - v_0)^2 = 0, \\ f_3 &\equiv x_3^2 - (u_1 - u_2)^2 - (v_1 - v_2)^2 = 0. \end{aligned}$$

The conclusion in the statement  $(S_n)$  is equivalent to

$$g_n \equiv x_3^n - x_1^n - x_2^n = 0.$$

Let us now take once for all the rational number field  $R$  as the base field. Let  $A$  be the affine space on  $R$  with coordinates  $(v_0, u_1, v_1, u_2, v_2, x_0, x_1, x_2, x_3)$  arranged in that definite order. Then the above equations  $f_i = 0$  define an algebraic variety  $V$  of dimension 5, in the present case irreducible over  $R$ , with some generic point  $(v_0, u_1, v_1, u_2, v_2, \bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$  of which  $v_0, u_1, v_1, u_2, v_2$  are indeterminates. The truth or untruth of the statement  $(S_n)$  amounts to  $g_n \equiv 0$  or  $\neq 0$  on  $V$  respectively.

It turns out that the general decision problem can be formulated in the following manner.

*Problem.* In a certain affine space  $A^n$  of dimension  $n = r + d$  over  $R$  with coordinates  $(u_1, \dots, u_d, x_1, \dots, x_r)$ , consider an algebraic variety  $V$  with defining equations ( $u_i$  being independent indeterminates)

$$(I) \quad \begin{cases} f_1(u_1, \dots, u_d, x_1) = 0, \\ f_2(u_1, \dots, u_d, x_1, x_2) = 0, \\ \dots\dots\dots \\ f_r(u_1, \dots, u_d, x_1, x_2, \dots, x_r) = 0. \end{cases}$$

The variety  $V$  may eventually split into irreducible components, all of (real) dimension  $\leq d$ . Those of dimension  $= d$ , with generic points of the form  $(u_1, \dots, u_d, \bar{x}_1, \dots, \bar{x}_r)$  for which  $\bar{x}_i$  are algebraic over the field  $K = R(u_1, \dots, u_d)$  will have a union  $V^*$  usually coincident with  $V$ . Let a polynomial  $g(u_1, \dots, u_d, x_1, \dots, x_r)$  (or a set of such polynomials  $g_k$ ) in  $R[u_1, \dots, u_d, x_1, \dots, x_r]$  be given. It is to decide in an algorithmic manner whether

$$g \equiv 0$$

(or all  $g_k \equiv 0$ ) on  $V^*$  or not.

In the above formulation the algebraic variety  $V$ , or preferably  $V^*$  reflects the hypothesis of the geometric statement considered. Either  $V$  or  $V^*$  will be called the *associated variety* of the statement in question. The variety  $V^*$  considered as one defined on the field  $K = R(u_1, \dots, u_d)$  is of dimension 0. The form of equations (I) shows that the algebraically dependent variables  $x_1, \dots, x_r$  are to be adjoined to  $K$  *successively* which reflects the geometrical fact that, starting from some generic points on certain generic lines, circles, etc., new points are to be *successively* adjoined in an algebraic manner by various geometric operations of joining points, drawing parallels, perpendiculars or circles, forming intersections of lines and circles, etc. These geometrical constructions give rise to algebraic equations involving  $x$  and  $u$  which can easily be turned into the form (I) by simple elimination procedure. In fact, the starting equations in  $x$  and  $u$  are rarely higher than 2. We shall call the variables  $u$  as the *parameters* and  $x$  as the *dependents*. It is also to be remarked that the condition for all components of  $V$  to be of dimension  $\leq d$  over  $R$  reflects just the *determinate* character of the geometric statements to be considered, and the restriction to  $V^*$  reflects the depriving of degeneracies in our consideration. Both of these are, in reality, implicitly implied in the hypothesis of ordinary geometrical theorems. On the other hand, the equation  $g \equiv 0$  on  $V^*$  (or set of  $g_k \equiv 0$  on  $V^*$ ) is the algebraic equivalent of the conclusion of the statement to be proved or disproved. We shall call  $g$  or set of  $g_k$  in what follows the *deciding polynomial (s)* of the geometrical statement in question.

Theoretically, the methods given by Hermann in [4] permit already to solve the above decision problem in an algorithmic manner. However, his methods are so complicated to give rise to astronomical expansions that even the simplest geometrical theorems can hardly be proved. On the contrary the decision procedure given below takes advantages of the particular character of the equations (I) and permits to prove mechanically quite non-trivial theorems even by hands, i.e. by means of pencils and

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