

陈文灯 黄先开 朱庆宇◎主编

2020 考研 数学 复习指南

(数学二) 课后习题答案详解

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课后习题答案详解

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第一篇 高等数学



第一章 函数、极限和连续

习题一

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$$(5) \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x}} = 1$$

$$(6) \lim_{x \rightarrow 0} (\sqrt{x+2\sqrt{x}} - \sqrt{x-\sqrt{x}}) = \lim_{x \rightarrow 0} \frac{\sqrt{x+2\sqrt{x}} - \sqrt{x-\sqrt{x}}}{\sqrt{x+2\sqrt{x}} + \sqrt{x-\sqrt{x}}} = \lim_{x \rightarrow 0} \frac{1-\sqrt{x}}{\sqrt{x+2\sqrt{x}} + \sqrt{x-\sqrt{x}}} = 2$$

$$(7) \lim_{x \rightarrow 0} \frac{f(x) + a \sin x}{x} = \lim_{x \rightarrow 0} \frac{f(x) + a \cos x}{1} = f'(0) + a = 1 - 1 + 1 = 1$$

$$(8) \lim_{x \rightarrow 0} \frac{x^2 - 1 + \arctan x}{1 + \arctan x} = \lim_{x \rightarrow 0} \frac{(x^2 + \arctan^2 x) - 1 - \arctan x}{1 + \arctan x} = \lim_{x \rightarrow 0} \frac{x^2 + \arctan^2 x + \arctan x - 1}{1 + \arctan x} = 0$$

$$(9) \lim_{x \rightarrow 0} \frac{x^2 + \arctan^2 x + \arctan x - 1}{1 + \arctan x} = 0$$

第一篇 高等数学

第一章 函数、极限和连续

习题一

1. 填空题

(1)【解】可得 $e^a = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{ax} = \int_{-\infty}^a te^t dt = (te^t - e^t) \Big|_{-\infty}^a = ae^a - e^a$, 所以 $a = 2$.

(2)【解】 $\sum_{i=1}^n \frac{i}{n^2 + n + n} < \sum_{i=1}^n \frac{i}{n^2 + n + i} < \sum_{i=1}^n \frac{i}{n^2 + n + 1}$,

又 $\lim_{n \rightarrow \infty} \frac{1+2+\dots+n}{n^2+n+n} = \lim_{n \rightarrow \infty} \frac{\frac{n(1+n)}{2}}{n^2+n+n} = \frac{1}{2}$,

$\lim_{n \rightarrow \infty} \frac{1+2+\dots+n}{n^2+n+1} = \lim_{n \rightarrow \infty} \frac{\frac{n(1+n)}{2}}{n^2+n+1} = \frac{1}{2}$,

所以 $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \dots + \frac{n}{n^2+n+n}\right) = \frac{1}{2}$.

(3)【解】因为 $\lim_{x \rightarrow 0} \int_b^x \frac{ax - \sin x}{\ln(1+t^3)} dt = c \neq 0$, 所以 $b = 0$.

$\lim_{x \rightarrow 0} \int_0^x \frac{ax - \sin x}{\ln(1+t^3)} dt \xrightarrow{\text{洛必达法则}} \lim_{x \rightarrow 0} \frac{a - \cos x}{\frac{3x^2}{1+x^3}} = \lim_{x \rightarrow 0} \frac{a - \cos x}{x^2} \stackrel{a=1}{=} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

$= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2} = c \neq 0$. 即 $a = 1, b = 0, c = \frac{1}{2}$.

(4)【解】 $\lim_{h \rightarrow 0} \frac{f(3-h) - f(3)}{2h} = -\frac{1}{2} \lim_{h \rightarrow 0} \frac{f(3-h) - f(3)}{-h} = -\frac{1}{2} f'(3) = -1$.

(5)【解】 $f[f(x)] = 1$.

(6)【解】 $\lim_{n \rightarrow \infty} (\sqrt{n+3\sqrt{n}} - \sqrt{n-\sqrt{n}}) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3\sqrt{n}} - \sqrt{n-\sqrt{n}})(\sqrt{n+3\sqrt{n}} + \sqrt{n-\sqrt{n}})}{\sqrt{n+3\sqrt{n}} + \sqrt{n-\sqrt{n}}}$
 $= \lim_{n \rightarrow \infty} \frac{4\sqrt{n}}{\sqrt{n+3\sqrt{n}} + \sqrt{n-\sqrt{n}}} = 2$.

(7)【解】 $\lim_{x \rightarrow 0} \frac{f(x) + a \sin x}{x} = \lim_{x \rightarrow 0} \frac{f'(x) + a \cos x}{1} = f'(0) + a = b + a = A$.

(8)【解】 $0 \neq k = \lim_{x \rightarrow 0} \frac{e^x - \frac{1+ax}{1+bx}}{x^3} = \lim_{x \rightarrow 0} \frac{(e^x + bx e^x) - 1 - ax}{x^3(1+bx)} = \lim_{x \rightarrow 0} \frac{e^x + bx e^x - 1 - ax}{x^3}$
 $\xrightarrow{\text{洛必达法则}} \lim_{x \rightarrow 0} \frac{e^x + bx e^x + be^x - a}{3x^2} \stackrel{b+1=a}{=} \lim_{x \rightarrow 0} \frac{e^x + 2be^x + bx e^x}{6x}$

所以 $2b+1=0, b=-\frac{1}{2}, a=\frac{1}{2}$.

(9)【解】 $\lim_{x \rightarrow 0} \cot x \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{\cos x}{\sin x} \cdot \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{3x^2} = \frac{1}{6}$.

(10)【解】 $\lim_{n \rightarrow \infty} \frac{n^{1990}}{n^k - (n-1)^k} = \lim_{n \rightarrow \infty} \frac{1990n^{1989}}{kn^{k-1} + \dots} = A$,

所以 $k-1=1990, k=1991; \frac{1}{k}=A, A=\frac{1}{1991}$.

2. 选择题

(1)【解】 令 $f(x)=1, \varphi(x)=\begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \in \mathbf{R}-\mathbf{Q} \end{cases}$, 则 $\varphi[f(x)]=1, f[\varphi(x)]=1$, 排除 A, C. 令 $\varphi(x)=\begin{cases} 1 & x \in \mathbf{Q} \\ -1 & x \in \mathbf{R}-\mathbf{Q} \end{cases}$, $[\varphi(x)]^2=1$, 排除 B, D. 若 $g(x)=\frac{\varphi(x)}{f(x)}$ 在 $(-\infty, +\infty)$ 内连续, 则 $\varphi(x)=g(x)f(x)$ 在 $(-\infty, +\infty)$ 内连续, 矛盾. 所以应选 D.

(2)【解】 B.

(3)【解】 $\lim_{x \rightarrow 1^+} \frac{x^2-1}{x-1} e^{\frac{1}{x-1}} = \lim_{x \rightarrow 1^+} (x+1) e^{\frac{1}{x-1}} = +\infty, \lim_{x \rightarrow 1^-} \frac{x^2-1}{x-1} e^{\frac{1}{x-1}} = 0$, 故选 D.

(4)【解】 $\lim_{x \rightarrow 0} \frac{\int_0^x (e^t - 1) dt}{x^2} = \lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{2x} = \lim_{x \rightarrow 0} \frac{x^2}{2x} = 0 = f(0) = a$, 故选 A.

(5)【解】 $\lim_{x \rightarrow \infty} \left[\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \dots + \frac{2n+1}{n^2 \times (n+1)^2} \right] =$
 $\lim_{x \rightarrow \infty} \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots + \frac{1}{n^2} - \frac{1}{(n+1)^2} \right] = \lim_{x \rightarrow \infty} \left[1 - \frac{1}{(n+1)^2} \right] = 1$, 故选 B.

(6)【解】 $8 = \lim_{x \rightarrow \infty} \frac{a^5 x^{100} + \dots}{x^{100} + \dots} = a^5, a = \sqrt[5]{8}$, 故选 C.

(7)【解】 C 为答案.

(8)【解】 $\lim_{x \rightarrow 0} \frac{2^x + 3^x - 2}{x} = \lim_{x \rightarrow 0} (2^x \ln 2 + 3^x \ln 3) = \ln 2 + \ln 3 = \ln 6 \neq 1$, 故选 B.

(9)【解】 $\lim_{x \rightarrow 0} (1+x)(1+2x)(1+3x) + a = 0, 1+a=0, a=-1$, 故选 A.

(10)【解】 $2 = \lim_{x \rightarrow 0} \frac{a \tan x + b(1 - \cos x)}{c \ln(1 - 2x) + d(1 - e^{-x^2})} \stackrel{\text{洛必达法则}}{=} \lim_{x \rightarrow 0} \frac{\frac{a}{\cos^2 x} + b \sin x}{-2c + 2x d e^{-x^2}} = \frac{a}{2c}$,

所以 $a = -4c$, 故选 D.

3. 计算题

(1)①【解】 $\lim_{x \rightarrow +\infty} (x + e^x)^{\frac{1}{x}} = \lim_{x \rightarrow +\infty} e^{\frac{\ln(x+e^x)}{x}} = e^{\lim_{x \rightarrow +\infty} \frac{\ln(x+e^x)}{x}} = e^{\lim_{x \rightarrow +\infty} \frac{1+e^x}{x+e^x}} = e$.

②【解】 $\lim_{x \rightarrow 0} (2 \sin x + \cos x)^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0} \frac{2 \sin x + \cos x - 1}{x}} = e^{\lim_{x \rightarrow 0} \frac{2 \cos x - \sin x}{1}} = e^2$.

③【解】 令 $y = \frac{1}{x}$, 则

$\lim_{x \rightarrow \infty} \left(\sin \frac{2}{x} + \cos \frac{1}{x} \right)^x = \lim_{y \rightarrow 0} (\sin 2y + \cos y)^{\frac{1}{y}} = e^{\lim_{y \rightarrow 0} \frac{\ln(\sin 2y + \cos y)}{y}} = e^{\lim_{y \rightarrow 0} \frac{2 \cos y - \sin y}{\sin 2y + \cos y}} = e^2$.

④【解】 $\lim_{x \rightarrow 0} \left(\frac{1 + \tan x}{1 + \sin x} \right)^{\frac{1}{x^3}} = \lim_{x \rightarrow 0} \left(1 + \frac{\tan x - \sin x}{1 + \sin x} \right)^{\frac{1}{x^3}} = e^{\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}}$
 $= e^{\lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{x^3}} = e^{\lim_{x \rightarrow 0} \frac{x \cdot \frac{x^2}{2}}{x^3}} = e^{\frac{1}{2}}$.

(2)①【解】 $\lim_{x \rightarrow 1} \frac{\ln(1 + \sqrt[3]{x-1})}{\arcsin 2 \sqrt[3]{x^2-1}} = \lim_{x \rightarrow 1} \frac{\sqrt[3]{x-1}}{2 \sqrt[3]{x^2-1}} = \lim_{x \rightarrow 1} \frac{1}{2 \sqrt[3]{x+1}} = \frac{1}{2 \sqrt[3]{2}}$.

$$\begin{aligned}
 \textcircled{2} \text{【解】} \lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \cot^2 x \right) &= \lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x} \right) = \lim_{x \rightarrow 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{x^2 - x^2(1 - \cos^2 x)}{x^4} = \lim_{x \rightarrow 0} \frac{1 - (x^2 + 1)\cos^2 x}{x^4} = \lim_{x \rightarrow 0} \frac{-2x\cos^2 x + 2(x^2 + 1)\cos x \sin x}{4x^3} \\
 &= \lim_{x \rightarrow 0} \frac{-2x\cos^2 x + 2\sin 2x}{4x^3} + \lim_{x \rightarrow 0} \frac{2x^2 \cos x \sin x}{4x^3} \\
 &= \lim_{x \rightarrow 0} \frac{-2\cos^2 x + 4x\cos x \sin x + 2\cos 2x}{12x^2} + \frac{1}{2} \\
 &= \lim_{x \rightarrow 0} \frac{-2\cos^2 x + 2\cos 2x}{12x^2} + \frac{1}{3} + \frac{1}{2} \\
 &= \lim_{x \rightarrow 0} \frac{4\cos x \sin x - 4\sin 2x}{24x} + \frac{5}{6} \\
 &= \lim_{x \rightarrow 0} \frac{-2\sin 2x}{24x} + \frac{5}{6} = -\frac{1}{6} + \frac{5}{6} = \frac{2}{3}.
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{3} \text{【解】} \lim_{x \rightarrow 0} \frac{\ln(\cos x \cdot \sqrt{1-x^2})}{\tan x \ln(1+x)} &= \lim_{x \rightarrow 0} \frac{\ln(\cos x \cdot \sqrt{1-x^2})}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2} \cos x} \left(-\sqrt{1-x^2} \sin x - \frac{x}{\sqrt{1-x^2}} \cos x \right)}{2x} \\
 &= -\lim_{x \rightarrow 0} \left[\frac{\sin x}{2x \cos x} + \frac{1}{2(1-x^2)} \right] = -1.
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{3} \text{①【解】} \lim_{n \rightarrow \infty} \left[e^2 - \left(1 + \frac{1}{n} \right)^{2n} \right] &= \lim_{n \rightarrow \infty} \frac{e^2 - \left(1 + \frac{1}{n} \right)^{2n}}{\frac{1}{n}} \stackrel{\text{令 } x = \frac{1}{n}}{=} \lim_{x \rightarrow 0} \frac{e^2 - (1+x)^{\frac{2}{x}}}{x} \\
 &= \lim_{x \rightarrow 0} \frac{2(1+x)^{\frac{2}{x}} \ln(1+x)}{x^2} \stackrel{\text{等价无穷小}}{=} \lim_{x \rightarrow 0} \frac{2(1+x)^{\frac{2}{x}}}{x} = \infty.
 \end{aligned}$$

$$\textcircled{2} \text{【解】} \lim_{n \rightarrow \infty} \frac{n}{\ln n} (\sqrt[n]{n} - 1) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n} - 1}{\ln \sqrt[n]{n}} \stackrel{\text{令 } \sqrt[n]{n} = x}{=} \lim_{x \rightarrow 1} \frac{x}{\ln(1+x)} = 1.$$

$$\lim_{x \rightarrow 1} \frac{x-1}{\ln x} = \lim_{x \rightarrow 1} x = 1.$$

$$\begin{aligned}
 \textcircled{3} \text{【解】} \lim_{n \rightarrow \infty} \left\{ \left(x + \frac{a}{n} \right) + \left(x + \frac{2a}{n} \right) + \cdots + \left[x + \frac{(n-1)a}{n} \right] \right\} \cdot \frac{1}{n} \\
 = \lim_{n \rightarrow \infty} \left((n-1)x + \frac{1+1+\cdots+(n-1)a}{n} \right) \frac{1}{n} \\
 = \lim_{n \rightarrow \infty} \frac{(n-1)x}{n} + \lim_{n \rightarrow \infty} \frac{n(n-1)}{2n^2} a = x + \frac{1}{2} a.
 \end{aligned}$$

$$\textcircled{4} \text{【解】} \frac{1}{n+1} + \frac{1}{(n^2+1)^{\frac{1}{2}}} + \cdots + \frac{1}{(n^n+1)^{\frac{1}{n}}} = \frac{1}{n} \left[\frac{1}{1+\frac{1}{n}} + \frac{1}{\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}}} + \cdots + \frac{1}{\left(1+\frac{1}{n^n}\right)^{\frac{1}{n}}} \right]$$

$$\forall i < j, \frac{1}{1+\frac{1}{n^i}} < \frac{1}{1+\frac{1}{n^j}}, \text{ 从而 } \frac{1}{\left(1+\frac{1}{n^i}\right)^{\frac{1}{i}}} < \frac{1}{\left(1+\frac{1}{n^j}\right)^{\frac{1}{j}}} < \frac{1}{\left(1+\frac{1}{n^j}\right)^{\frac{1}{j}}}$$

$$\text{因此 } \frac{1}{\left(1+\frac{1}{n^n}\right)^{\frac{1}{n}}} > \frac{1}{n} \left(\frac{1}{1+\frac{1}{n}} + \frac{1}{\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}}} + \cdots + \frac{1}{\left(1+\frac{1}{n^n}\right)^{\frac{1}{n}}} \right) > \frac{1}{1+\frac{1}{n}}$$

$$\text{因为 } \lim_{n \rightarrow \infty} \frac{1}{\left(1+\frac{1}{n^n}\right)^{\frac{1}{n}}} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n}} = 1, \text{ 所以 } \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{(n^2+1)^{\frac{1}{2}}} + \cdots + \frac{1}{(n^n+1)^{\frac{1}{n}}} \right) = 1.$$

$$⑤【解】 \lim_{n \rightarrow \infty} \frac{1 - e^{-nx}}{1 + e^{-nx}} = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$⑥【解】 \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \right)^n \stackrel{x=1/n, c=b/a}{=} a \lim_{x \rightarrow 0^+} \left(\frac{1+c^x}{2} \right)^{\frac{1}{x}} \\ = a e^{\lim_{x \rightarrow 0^+} \frac{(1+c^x)-1}{2} \cdot \frac{1}{x}} = a e^{\lim_{x \rightarrow 0^+} \frac{1+c^{2x}-2}{2x}} = a e^{\lim_{x \rightarrow 0^+} \frac{c^{2x} \ln c}{2}} = a e^{\frac{\ln c}{2}} = a \sqrt{\frac{b}{a}} = \sqrt{ab}.$$

$$4.【解】 f'(0^+) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x} \int_0^x \cos t^2 dt - 1}{x} = \lim_{x \rightarrow 0^+} \frac{\int_0^x \cos t^2 dt - x}{x^2} \\ = \lim_{x \rightarrow 0^+} \frac{\cos x^2 - 1}{2x} = \lim_{x \rightarrow 0^+} \frac{-\frac{1}{2}x^4}{2x} = 0.$$

$$f'(0^-) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{\frac{2}{x^2}(1 - \cos x) - 1}{x} = \lim_{x \rightarrow 0^-} \frac{2(1 - \cos x) - x^2}{x^3} \\ = \lim_{x \rightarrow 0^-} \frac{2\sin x - 2x}{3x^2} = \lim_{x \rightarrow 0^-} \frac{\cos x - 1}{3x} = \lim_{x \rightarrow 0^-} -\frac{\sin x}{3} = 0.$$

所以, $f'(0) = 0$, 所以 $f(x)$ 在 $x=0$ 处可导, 因此 $f(x)$ 在 $x=0$ 处连续.

5. 求下列函数的间断点并判别类型

$$①【解】 f(0^+) = \lim_{x \rightarrow 0^+} \frac{2^{\frac{1}{x}} - 1}{2^{\frac{1}{x}} + 1} = 1, f(0^-) = \lim_{x \rightarrow 0^-} \frac{2^{\frac{1}{x}} - 1}{2^{\frac{1}{x}} + 1} = -1,$$

所以 $x=0$ 为第一类间断点.

$$②【解】 f(x) = \lim_{n \rightarrow \infty} \frac{1 - x^{2n}}{1 + x^{2n}} \cdot x = \begin{cases} -x, & |x| \geq 1 \\ x, & |x| < 1 \end{cases}$$

显然, $\lim_{x \rightarrow 1^+} f(x) = -1, \lim_{x \rightarrow 1^-} f(x) = 1, \lim_{x \rightarrow -1^+} f(x) = -1, \lim_{x \rightarrow -1^-} f(x) = 1$, 所以, $x=1, x=-1$ 为第一类间断点.

$$③【解】 \lim_{x \rightarrow 0^+} f(x) = -\sin 1, \lim_{x \rightarrow 0^-} f(x) = 0. \text{ 所以 } x=0 \text{ 为第一类跳跃间断点;}$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \sin \frac{1}{x^2 - 1} \text{ 不存在, 所以 } x=1 \text{ 为第二类间断点;}$$

$$f\left(-\frac{\pi}{2}\right) \text{ 不存在, 而 } \lim_{x \rightarrow -\frac{\pi}{2}} \frac{x(2x + \pi)}{2\cos x} = \lim_{x \rightarrow -\frac{\pi}{2}} \frac{4x + \pi}{-2\sin x} = -\frac{\pi}{2}, \text{ 所以 } x = -\frac{\pi}{2} \text{ 为第一类可去间断点;}$$

$$\lim_{x \rightarrow -k\pi - \frac{\pi}{2}} \frac{x(2x + \pi)}{2\cos x} = \infty, (k=1, 2, \dots) \text{ 所以 } x = -k\pi - \frac{\pi}{2} \text{ 为第二类无穷间断点.}$$

$$6.【解】 \lim_{x \rightarrow 0} \left(\frac{a}{x^2} + \frac{1}{x^4} + \frac{b}{x^5} \int_0^x e^{-t^2} dt \right) = \lim_{x \rightarrow 0} \frac{ax^3 + x + b \int_0^x e^{-t^2} dt}{x^5} \\ = \lim_{x \rightarrow 0} \frac{3ax^2 + 1 + be^{-x^2}}{5x^4} \quad \text{分子极限为 } 0, \text{ 所以 } b = -1 \quad \lim_{x \rightarrow 0} \frac{3ax^2 + 1 - e^{-x^2}}{5x^4}$$

$$= \lim_{x \rightarrow 0} \frac{6ax + 2xe^{-x^2}}{20x^3} = \lim_{x \rightarrow 0} \frac{3a + xe^{-x^2}}{10x^2} \quad \text{分子极限为 } 0, \text{ 所以 } a = -\frac{1}{3} \\ \lim_{x \rightarrow 0} \frac{-\frac{1}{3} + xe^{-x^2}}{10x^2} = -\frac{1}{10}.$$

7.【解】 由于 $x=0$ 是 $f(x)$ 的可去间断点,

因此 $\lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x + \sin^2 x} - (\alpha + \beta \sin x)}{\sin^2 x}$ 存在. 所以

$$\lim_{x \rightarrow 0} [\sqrt{1 + \sin x + \sin^2 x} - (\alpha + \beta \sin x)] = 0. \text{ 即}$$

$$0 = \lim_{x \rightarrow 0} \frac{1 + \sin x + \sin^2 x - (\alpha + \beta \sin x)^2}{\sqrt{1 + \sin x + \sin^2 x} + (\alpha + \beta \sin x)}$$

$$= \lim_{x \rightarrow 0} \frac{(1 - \alpha^2) + (1 - 2\alpha\beta)\sin x + (1 - \beta^2)\sin^2 x}{\sqrt{1 + \sin x + \sin^2 x} + (\alpha + \beta \sin x)} = \frac{1 - \alpha^2}{1 + \alpha} = 1 - \alpha.$$

所以 $\alpha = 1$.

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x + \sin^2 x} - (\alpha + \beta \sin x)}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{(1 - 2\beta) + (1 - \beta^2)\sin x}{\sin x \cdot [\sqrt{1 + \sin x + \sin^2 x} + (1 + \beta \sin x)]}.$$

若上式极限存在, 必须分子为 0, 即 $1 - 2\beta = 0, \beta = \frac{1}{2}$.

8.【解】因为极限存在, 从而 $a = \frac{1}{5}$.

$$\text{所以 } \lim_{x \rightarrow \infty} [(x^5 + 7x^4 + 2)^a - x] = \lim_{x \rightarrow \infty} \frac{\left(1 + \frac{7}{x} + \frac{2}{x^5}\right)^{\frac{1}{5}} - 1}{\frac{1}{x}} \stackrel{\text{令 } y = \frac{1}{x}}{=} \lim_{y \rightarrow 0} \frac{(1 + 7y + 2y^5)^{\frac{1}{5}} - 1}{y}$$

$$= \lim_{y \rightarrow 0} \frac{1}{5} (1 + 7y + 2y^5)^{-\frac{4}{5}} (7 + 10y^4) = \frac{7}{5}. \text{ 所以 } b = \frac{7}{5}.$$

9.【解】当 $\alpha \leq 0$ 时, $\lim_{x \rightarrow 0^+} (x^\alpha \sin \frac{1}{x})$ 不存在, 所以 $x = 0$ 为第二类间断点;

当 $\alpha > 0$ 时, $\lim_{x \rightarrow 0^+} (x^\alpha \sin \frac{1}{x}) = 0$, 所以 $\beta = -1$ 时, $f(x)$ 在 $x = 0$ 连续; $\beta \neq -1$ 时, $x = 0$ 为第一类跳跃间断点.

10.【解】 $\lim_{x \rightarrow 0} \left(\frac{\sin 3x}{x^3} + \frac{f(x)}{x^2} \right) = \lim_{x \rightarrow 0} \frac{\sin 3x + f(x)}{x^2} = 0.$

所以, $\lim_{x \rightarrow 0} \left(\frac{\sin 3x}{x} + f(x) \right) = 0$. $f(x)$ 在 $x = 0$ 的某邻域内二阶可导, 所以 $f(x), f'(x)$ 在 $x = 0$ 处连续. 因此

$$f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{-\sin 3x}{x} = -3.$$

因为 $\lim_{x \rightarrow 0} \frac{\sin 3x}{x} + f(x) = 0$, 所以 $\lim_{x \rightarrow 0} \frac{\sin 3x - 3 + f(x) + 3}{x^2} = 0$, 即

$$\lim_{x \rightarrow 0} \frac{f(x) + 3}{x^2} = \lim_{x \rightarrow 0} \frac{3 - \frac{\sin 3x}{x}}{x^2} = \lim_{x \rightarrow 0} \frac{3x - \sin 3x}{x^3} = \lim_{x \rightarrow 0} \frac{3 - 3\cos 3x}{3x^2} = \lim_{x \rightarrow 0} \frac{\frac{9}{2}x^2}{x^2} = \frac{9}{2}.$$

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x) + 3}{x} = \lim_{x \rightarrow 0} x \cdot \frac{f(x) + 3}{x^2} = 0 \times \frac{9}{2} = 0.$$

$$f''(0) = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f'(x)}{x}.$$

$$\text{因为 } \frac{9}{2} = \lim_{x \rightarrow 0} \frac{f(x) + 3}{x^2} = \lim_{x \rightarrow 0} \frac{f'(x)}{2x},$$

$$\text{从而 } \lim_{x \rightarrow 0} \frac{f'(x)}{x} = 9, \text{ 即 } f''(0) = 9.$$

第二章 导数与微分

习题二

1. 填空题

(1)【解】 $\frac{d}{dx} \int_x^0 x \cos t^2 dt = \int_x^0 \cos t^2 dt - 2x^2 \cos x^4.$

(2)【解】 $f'(x) = \frac{-1 - x - 1 + x}{(1+x)^2} = \frac{(-1)^1 2 \cdot 1!}{(1+x)^{1+1}}$, 假设 $f^{(k)}(x) = \frac{(-1)^k 2 \cdot k!}{(1+x)^{k+1}}$, 则

$$f^{(k+1)}(x) = \frac{(-1)^{k+1} \cdot 2 \cdot (k+1)!}{(1+x)^{k+1+1}}, \text{ 所以 } f^{(n)}(x) = \frac{(-1)^n \cdot 2 \cdot n!}{(1+x)^{n+1}}.$$

(3)【解】 $\frac{dy}{dx} = \frac{-\sin t}{2t}, \frac{dt}{dx} = \frac{1}{2t}, \frac{d^2y}{dx^2} = \left(\frac{-\sin t}{2t}\right)' \frac{dt}{dx} = \frac{2t\cos t - 2\sin t}{4t^2} \cdot \frac{1}{2t} = \frac{\sin t - t\cos t}{4t^3}.$

(4)【解】 隐函数求导得: $e^{xy}(1+y') - (y+xy')\sin xy = 0,$

即得: $\frac{dy}{dx} = y' = \frac{y\sin xy - e^{xy}}{e^{xy} - x\sin xy}.$

(5)【解】 由 $f(-x) = -f(x),$ 得 $-f'(-x) = -f'(x),$ 所以 $f'(-x) = f'(x),$

所以 $f'(x_0) = f'(-x_0) = k.$

(6)【解】 $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + m\Delta x) - f(x_0 - n\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + m\Delta x) - f(x_0) + f(x_0) - f(x_0 - n\Delta x)}{\Delta x}$

$$= m \lim_{m\Delta x \rightarrow 0} \frac{f(x_0 + m\Delta x) - f(x_0)}{m\Delta x} + n \lim_{n\Delta x \rightarrow 0} \frac{f(x_0 - n\Delta x) - f(x_0)}{-n\Delta x} = (m+n)f'(x_0).$$

(7)【解】 $k \lim_{k\Delta x \rightarrow 0} \frac{f(x_0 + k\Delta x) - f(x_0)}{k\Delta x} = \frac{1}{3}f'(x_0),$ 所以 $kf'(x_0) = \frac{1}{3}f'(x_0),$ 所以 $k = \frac{1}{3}.$

(8)【解】 $\frac{d}{dx} \left[f\left(\frac{1}{x^2}\right) \right] = -f'\left(\frac{1}{x^2}\right) \cdot \frac{2}{x^3} = \frac{1}{x},$ 所以 $f'\left(\frac{1}{x^2}\right) = -\frac{x^2}{2}.$

令 $x^2 = 2,$ 所以 $f'\left(\frac{1}{2}\right) = -1.$

(9)【解】 $\frac{dy}{dx} = f'(x)\cos f(x)f'[\sin f(x)]\cos f[\sin f(x)].$

(10)【解】 对隐函数求导得: $e^{2+xy}(2+y') - (y+xy')\sin(xy) = 0.$

所以切线斜率 $k = y'(0) = -2.$ 法线斜率为 $\frac{1}{2},$ 法线方程为

$$y - 1 = \frac{1}{2}x, \text{ 即 } x - 2y + 2 = 0.$$

2. 选择题

(1)【解】“ \leftarrow ”: 因为 $F'(0)$ 存在, 所以 $F'(0^+) = F'(0^-),$ 于是

$$F'_+(0) = \lim_{x \rightarrow 0^+} \frac{F(x) - F(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{f(x)(1 + \sin x) - f(0)}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{[f(x) - f(0)] + f(x)\sin x}{x} = f'(0) + f(0).$$

$$F'_-(0) = \lim_{x \rightarrow 0^-} \frac{F(x) - F(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{f(x)(1 - \sin x) - f(0)}{x}$$

$$= \lim_{x \rightarrow 0^-} \frac{[f(x) - f(0)] - f(x)\sin x}{x} = f'(0) - f(0).$$

所以 $f'(0) + f(0) = f'(0) - f(0), 2f(0) = 0, f(0) = 0.$

“ \Rightarrow ”: 已知 $f(0) = 0,$ 所以

$$F'_+(0) = \lim_{x \rightarrow 0^+} \frac{F(x) - F(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{f(x)(1 + \sin x) - f(0)}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{[f(x) - f(0)] + f(x)\sin x}{x} = f'(0) + f(0) = f'(0).$$

$$F'_-(0) = \lim_{x \rightarrow 0^-} \frac{F(x) - F(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{f(x)(1 - \sin x) - f(0)}{x}$$

$$= \lim_{x \rightarrow 0^-} \frac{[f(x) - f(0)] - f(x)\sin x}{x} = f'(0) - f(0) = f'(0).$$

所以 $F'(0) = f'(0)$ 存在, 即答案为 A.

(2)【解】 因为 $f(x)$ 是连续函数, $F'(x) = f(e^{-x})(e^{-x})' - f(x) = -e^{-x}f(e^{-x}) - f(x),$ 所以答案是 A.

(3)【解】 因为 $f(x) = [f(x)]^2,$ 且 $f(x)$ 具有任意阶导数,

所以 $f''(x) = 2f(x)f'(x) = 2![f(x)]^3.$ 假设 $f^{(k)}(x) = k![f(x)]^{k+1},$

所以 $f^{(k+1)}(x) = (k+1)k![f(x)]^k f'(x) = (k+1)![f(x)]^{k+2},$ 由数学归纳法知:

$f^{(n)}(x) = n![f(x)]^{n+1}$ 对一切正整数成立, 即答案为 A.

(4)【解】因为 $f(1+x) = af(x)$, 且 $f'(0) = b$,

$$\text{所以, } b = f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\frac{1}{a}f(1+x) - \frac{1}{a}f(1)}{x} = \frac{1}{a}f'(1),$$

所以 $f'(1) = ab$. 即答案为 D.

(5)【解】依题意知: $f(x) = \begin{cases} 4x^3, & x \geq 0 \\ 2x^3, & x < 0 \end{cases}$, $f'(x) = \begin{cases} 24x, & x \geq 0 \\ 12x, & x < 0 \end{cases}$.

$$\text{所以, } f''(0^+) = \lim_{x \rightarrow 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{24x - 0}{x} = 24,$$

$$f''(0^-) = \lim_{x \rightarrow 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{12x - 0}{x} = 12,$$

所以 $n = 2$, 应选 C.

(6)【解】因为 $f(x)$ 可导, 所以由微分定义 $\Delta y = dy + o(\Delta x)$. 即

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y - dy}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{o(\Delta x)}{\Delta x} = 0.$$

即答案为 B.

(7)【解】在 $x = 0$ 处可导一定在 $x = 0$ 处连续, 所以 $\lim_{x \rightarrow 0^+} x^2 \sin \frac{1}{x} = \lim_{x \rightarrow 0^+} (ax + b)$,

$$\text{所以 } b = 0, f'(0^+) = f'(0^-), \lim_{x \rightarrow 0^+} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \rightarrow 0^+} \frac{ax}{x}, \text{ 所以 } a = 0, \text{ 即答案为 C.}$$

(8)【解】(A), (C) 项极限式的分母均为 h^2 , 而 $\lim_{h \rightarrow 0} h^2 = 0^+$, 所以可排除 (A)(C).

对于 (D) 项, 令 $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$, 则 $f(x)$ 在点 $x = 0$ 不可导, 但

$$\lim_{h \rightarrow 0} \frac{1}{h} [f(2h) - f(h)] = \lim_{h \rightarrow 0} \frac{1 - 1}{h} = 0 \text{ 存在.}$$

故选 (B).

(9)【解】若取 $y = x$, 则 A 不正确; 若取 $y = x^2$, 则 B 不正确; 若取 $y = x$, 则 C 不正确. 故应选 D.

(10)【解】 $f(x) = 0$, 取 $a = 0$. 排除 A; $f(x) = x^2 + x + 1$, 取 $a = 0$. $f(0) = 1 > 0$, $f'(0) = 1 > 0$, $|f(x)| = f(x)$, 在 $x = 0$ 处可导. 排除 C; $f(x) = -x^2 - x - 1$, 取 $a = 0$. 排除 D; 所以应选 B.

3. 计算题.

$$(1)【解】y' = \frac{-\sin(10 + 3x^2) \cdot 6x}{\cos(10 + 3x^2)} = -6x \tan(10 + 3x^2).$$

$$(2)【解】y' = f'[\ln(x + \sqrt{a+x^2})] \cdot \frac{1}{x + \sqrt{a+x^2}} \left(1 + \frac{2x}{2\sqrt{a+x^2}}\right) \\ = \frac{f'[\ln(x + \sqrt{a+x^2})]}{\sqrt{a+x^2}}.$$

$$(3)【解】e^{y^2} y' = 2x \cos x^2 + 2yy' \cos y^2 \Rightarrow y' = \frac{2x \cos x^2}{e^{y^2} - 2y \cos y^2}.$$

$$(4)【解】\frac{2x + 2yy'}{\sqrt{x^2 + y^2} \cdot 2\sqrt{x^2 + y^2}} = \frac{y'x - y}{x^2} = \frac{y'}{1 + \frac{y^2}{x^2}}.$$

$$x + yy' = y'x - y, \text{ 所以 } y' = \frac{x + y}{x - y}.$$

$$(5)【解】\frac{dy}{dx} = \frac{e^{\cos t} - e^{\sin t}}{e^{\cos t} + e^{\sin t}} = \frac{\cos t - \sin t}{\cos t + \sin t}, \frac{dx}{dt} = e^{\sin t} + e^{\cos t},$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dt} \left(\frac{\cos t - \sin t}{\cos t + \sin t} \right) \cdot \frac{dt}{dx} = \frac{-(\cos t + \sin t)^2 - (\cos t - \sin t)^2}{(\cos t + \sin t)^2} \cdot \frac{1}{\frac{dx}{dt}} = -\frac{2}{e^{\sin t} (\cos t + \sin t)^3}.$$

(6)【解】 $dx = (2y+1)dy, du = \frac{3}{2}(x^2+x)^{\frac{1}{2}}(2x+1)dx$

$$\frac{(2y+1)dy}{du} = \frac{dx}{\frac{3}{2}\sqrt{x^2+x}(2x+1)dx}$$

$$\frac{dy}{du} = \frac{2}{3(2y+1)\sqrt{x^2+x}(2x+1)}$$

(7)【解】由题意知： $\frac{dy}{dx} = \frac{f'(e^{3t}-1)3e^{3t}}{f'(t)}$ ，所以 $\frac{dy}{dx}\Big|_{t=0} = 3$ 。

$$\frac{d^2y}{dx^2} = 3 \frac{[f''(e^{3t}-1)3(e^{3t})^2 + 3e^{3t}f'(e^{3t}-1)]f'(t) - e^{3t}f'(e^{3t}-1)f''(t)}{[f'(t)]^3}$$

所以 $\frac{d^2y}{dx^2}\Big|_{t=0} = 3 \frac{[3f''(0) + 3f'(0)]f'(0) - f'(0)f''(0)}{[f'(0)]^3} = \frac{9f'(0) + 6f''(0)}{[f'(0)]^2}$ 。

(8)【解】由 $\begin{cases} x = te^t \\ e^t + e^y = 2e \end{cases}$ 得 $y' = -\frac{e^t}{e^t - 2e}$

所以 $\frac{dy}{dx} = \frac{y'}{x'} = \frac{e^t - 2e}{e^t + te^t} = \frac{1}{(1+t)(e^t - 2e)}$ ，

所以 $\frac{dy}{dx}\Big|_{t=1} = -\frac{1}{2e}$ 。

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{1}{(1+t)(e^t - 2e)} \right) \frac{dt}{dx} = -\frac{2e^t - 2e + te^t}{(1+t)^2(e^t - 2e)^2 e^t}$$

所以 $\frac{d^2y}{dx^2}\Big|_{t=1} = -\frac{1}{8e^2}$ 在 $t=1$ 处的曲率为 $k = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}\Big|_{t=1} = \frac{8e^2}{\left(1 + \frac{1}{4e^2}\right)^{\frac{3}{2}}} = e(1+4e^2)^{-\frac{3}{2}}$ 。

4.【解】(1) $f(x)$ 在 $x=0$ 点连接，所以 $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{g(x) - \cos x}{x} = a, \frac{1}{A}$

所以 $\lim_{x \rightarrow 0} (g(x) - \cos x) = 0, g(0) = \cos 0 = 1$ ，

$$\begin{aligned} \text{所以 } a &= \lim_{x \rightarrow 0} \frac{g(x) - \cos x}{x} = \lim_{x \rightarrow 0} \frac{g(x) - g(0) + 1 - \cos x}{x} \\ &= \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x} + \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = g'(0) + 0 = g'(0). \end{aligned}$$

$$(2) f'(x) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\frac{g(x) - \cos x - a}{x} + \frac{g(x) - \cos x - ax}{x}}{x} = \lim_{x \rightarrow 0} \frac{g(x) - \cos x - ax}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{g(x) - \cos x - ax}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{g(0) + g'(0)x + \frac{1}{2}g''(\xi)x^2 - \cos x - ax}{x^2} \quad (0 < \xi < x)$$

$$= \lim_{x \rightarrow 0} \frac{1 + \frac{1}{2}g''(\xi)x^2 - \cos x}{x^2} = \frac{1}{2}(g''(0) + 1),$$

$$\text{所以 } f'(x) = \begin{cases} \frac{x[g'(x) + \sin x] - [g(x) - \cos x]}{x^2}, & x \neq 0 \\ \frac{1}{2}(g''(0) + 1), & x = 0 \end{cases}$$

5.【解】 $F(x)$ 连续，所以 $\lim_{x \rightarrow 0^-} F(x) = \lim_{x \rightarrow 0^+} F(x)$ ，所以 $c = f(-0) = f(0)$ ；

因为 $F(x)$ 二阶可导，所以 $F'(x)$ 连续，所以 $b = f'_-(0) = f'(0)$ ，且

$$F'(x) = \begin{cases} f'(x), & x \leq 0 \\ 2ax + f'_-(0), & x > 0 \end{cases}, F''(0) \text{ 存在, 所以 } F''_-(0) = F''_+(0),$$

所以 $\lim_{x \rightarrow 0^-} \frac{f'(x) - f'(0)}{x} = \lim_{x \rightarrow 0^+} \frac{2ax + f'(0) - f'(0)}{x} = 2a$, 所以 $a = \frac{1}{2} f''(0)$.

6. 【解】 $f(x) = -1 + \frac{1}{2} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{1+x}$,

$f^{(n)}(x) = \frac{1}{2} \cdot \frac{n!}{(1-x)^{n+1}} + \frac{1}{2} \cdot \frac{(-1)^n n!}{(1+x)^{n+1}}$,

$f^{(2k+1)}(0) = 0, k = 0, 1, 2, \dots, f^{(2k)}(0) = n!, k = 0, 1, 2, \dots$

7. 【解】 使用莱布尼茨高阶导数公式

$f^{(n)}(x) = x \cdot (\ln x)^{(n)} + n(\ln x)^{(n-1)} = x(-1)^{n-1} \frac{(n-1)!}{x^n} + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$
 $= (-1)^{n-2} (n-2)! \left[\frac{-(n-1)}{x^{n-1}} + \frac{n}{x^{n-1}} \right] = (-1)^{n-2} (n-2)! \frac{1}{x^{n-1}}$,

所以 $f^{(n)}(1) = (-1)^{n-2} (n-2)!$.

8. 【解】 因为 $y = (\arcsin x)^2$, 所以 $y' = 2\arcsin x \frac{1}{\sqrt{1-x^2}}$,

$y'' = 2 \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2}} + 2\arcsin x \left[-\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \right] (-2x) = \frac{2}{1-x^2} + \frac{2x\arcsin x}{(1-x^2)\sqrt{1-x^2}}$

所以 $(1-x^2)y'' = 2 + xy'$.

对上式二边求 $n-1$ 阶导数, 按莱布尼茨公式有

$(1-x^2)(y'')^{(n-1)} + (1-x^2)'C_{n-1}^1(y'')^{(n-2)} + (1-x^2)''C_{n-1}^2(y'')^{(n-3)} = x(y')^{(n-1)} + x'C_{n-1}^1(y')^{(n-2)}$,

$(1-x^2)y^{(n+1)} - 2x(n-1)y^{(n)} - 2 \frac{(n-1)(n-2)}{2!} y^{(n-1)} = xy^{(n)} + (n-1)y^{(n-1)}$,

所以 $(1-x^2)y^{(n+1)} - (2n-1)xy^{(n)} - (n-1)^2 y^{(n-1)} = 0$.

第三章 不定积分

习题三

1. 求下列不定积分:

(1) 【解】 因为 $d\left(\ln \frac{1+x}{1-x}\right) = \frac{2}{1-x^2} dx$.

所以 $\int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx = \frac{1}{2} \int \ln \frac{1+x}{1-x} d \ln \frac{1+x}{1-x} = \frac{1}{4} \left(\ln \frac{1+x}{1-x}\right)^2 + c$.

(2) 【解】 因为 $d\left(\arctan \frac{1+x}{1-x}\right) dx = \frac{(1-x)^2}{1 + \left(\frac{1+x}{1-x}\right)^2} dx = \frac{2}{(1-x)^2 + (1+x)^2} dx = \frac{dx}{1+x^2}$,

所以 $\int \frac{1}{1+x^2} \arctan \frac{1+x}{1-x} dx = \int \arctan \frac{1+x}{1-x} d\arctan \frac{1+x}{1-x} = \frac{1}{2} \left(\arctan \frac{1+x}{1-x}\right)^2 + c$.

(3) 【解】 因为 $d \frac{1+\sin x}{1+\cos x} = \frac{1+\cos x + \sin x}{(1+\cos x)^2} dx$.

所以 $\int \frac{\cos x + \sin x + 1}{(1+\cos x)^2} \cdot \frac{1+\sin x}{1+\cos x} dx = \int \frac{1+\sin x}{1+\cos x} d \frac{1+\sin x}{1+\cos x} = \frac{1}{2} \left(\frac{1+\sin x}{1+\cos x}\right)^2 + c$.

(4) 【解】 令 $x = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$,

$\int \frac{dx}{x(x^8+1)} = -\int \frac{t^7 dt}{t^8+1} = -\frac{1}{8} \int \frac{dt^8}{t^8+1} = -\frac{1}{8} \ln(1+t^8) + c = -\frac{1}{8} \ln\left(1+\frac{1}{x^8}\right) + c$.

(5) 【解】 $\int \frac{1+\sin x}{1+\sin x + \cos x} dx = \int \frac{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2}{2\sin \frac{x}{2} \cos \frac{x}{2} + 2\cos^2 \frac{x}{2}} dx = \int \frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{2\cos \frac{x}{2}} dx$

$$\text{【解】 } \int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

2. 求下列不定积分:

$$(1) \text{【解】 } \int \frac{dx}{(x+1)^2 \sqrt{x^2+2x+2}} = \int \frac{d(x+1)}{(x+1)^2 \sqrt{(x+1)^2+1}} \quad x+1 = \tan t$$

$$= \int \frac{\cos t dt}{\sin^2 t} = -\frac{1}{\sin t} + c = -\frac{\sqrt{x^2+2x+2}}{x+1} + c$$

$$(2) \text{【解】 令 } x = \tan t, \text{ 则 } dx = \frac{1}{\cos^2 t} dt.$$

$$\begin{aligned} \int \frac{dx}{x^4 \sqrt{1+x^2}} &= \int \frac{\cos^3 t dt}{\sin^4 t} = \int \frac{(1-\sin^2 t) d\sin t}{\sin^4 t} = \int \frac{d\sin t}{\sin^4 t} - \int \frac{d\sin t}{\sin^2 t} = -\frac{1}{3\sin^3 t} + \frac{1}{\sin t} + c \\ &= -\frac{1}{3} \left(\frac{\sqrt{1+x^2}}{x} \right)^3 + \frac{\sqrt{1+x^2}}{x} + c. \end{aligned}$$

$$(3) \text{【解】 令 } x = \tan t, \text{ 则 } dx = \frac{1}{\cos^2 t} dt.$$

$$\begin{aligned} \int \frac{dx}{(2x^2+1)\sqrt{1+x^2}} &= \int \frac{dt}{(2\tan^2 t+1)\cos t} = \int \frac{\cos t}{2\sin^2 t+\cos^2 t} dt = \int \frac{d\sin t}{1+\sin^2 t} \\ &= \arctan \sin t + c = \arctan \frac{x}{\sqrt{1+x^2}} + c. \end{aligned}$$

$$(4) \text{【解】 令 } x = a \sin t, dx = a \cos t dt, \sqrt{a^2-x^2} = a \cos t.$$

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{a^2-x^2}} &= \int \frac{a^2 \sin^2 t \cdot a \cos t dt}{a \cos t} = a^2 \int \frac{1-\cos 2t}{2} dt = \frac{1}{2} a^2 t - \frac{1}{4} a^2 \sin 2t + c \\ &= \frac{a^2}{2} \left(\arcsin \frac{x}{a} - \frac{x}{a^2} \sqrt{a^2-x^2} \right) + c. \end{aligned}$$

$$(5) \text{【解】 令 } x = \sin t,$$

$$\begin{aligned} \int \sqrt{1-x^2}^3 dx &= \int \cos^4 t dt = \int \left(\frac{1+\cos 2t}{2} \right)^2 dt = \int \frac{1+2\cos 2t+\cos^2 2t}{4} dt \\ &= \frac{1}{4} t + \frac{1}{4} \sin 2t + \frac{1}{8} \int (1+\cos 4t) dt = \frac{3}{8} t + \frac{1}{4} \sin 2t + \frac{1}{32} \sin 4t + c \\ &= \frac{3}{8} \arcsin x + \frac{1}{4} \sin 2t \left(1 + \frac{1}{4} \cos 2t \right) + c \\ &= \frac{3}{8} \arcsin x + \frac{1}{2} \sin t \cos t \left(1 + \frac{1}{4} - \frac{\sin^2 t}{2} \right) + c \\ &= \frac{3}{8} \arcsin x + \frac{1}{8} x \sqrt{1-x^2} (5-2x^2) + c. \end{aligned}$$

$$(6) \text{【解】 令 } x = \frac{1}{t}, \text{ 则 } dx = -\frac{1}{t^2} dt.$$

$$\begin{aligned} \int \frac{\sqrt{x^2-1}}{x^4} dx &= \int \frac{\sqrt{\frac{1-t^2}{t^2}}}{\frac{1}{t^4}} \left(-\frac{1}{t^2} \right) dt = -\int t \sqrt{1-t^2} dt \stackrel{\text{令 } t = \sin u}{=} -\int \sin u \cos^2 u du \\ &= \int \cos^2 u d\cos u = \frac{1}{3} \cos^3 u + c = \frac{\sqrt{(x^2-1)^3}}{3x^3} + c. \end{aligned}$$

$$(7) \text{【解】 令 } x = \frac{1}{t}, \text{ 则 } dx = -\frac{dt}{t^2}.$$

$$\begin{aligned} \int \frac{x+1}{x^2 \sqrt{x^2-1}} dx &= \int \frac{\frac{1}{t}+1}{\frac{1}{t^2} \sqrt{\frac{1}{t^2}-1}} \left(-\frac{1}{t^2} \right) dt = -\int \frac{t+1}{\sqrt{1-t^2}} dt \\ &\stackrel{\text{令 } t = \sin u}{=} -\int (\sin u + 1) du = \cos u - u + c = \frac{\sqrt{x^2-1}}{x} - \arcsin \frac{1}{x} + c. \end{aligned}$$

3. 求下列不定积分:

$$(1) \text{【解】} \int \frac{e^{3x} + e^x}{e^{4x} - e^{2x} + 1} dx = \int \frac{e^x + e^{-x}}{e^{2x} - 1 + e^{-2x}} dx = \int \frac{d(e^x - e^{-x})}{(e^x - e^{-x})^2 + 1} = \arctan(e^x - e^{-x}) + c.$$

$$(2) \text{【解】} \text{令 } t = 2^x, dx = \frac{dt}{t \ln 2}.$$

$$\int \frac{dx}{2^x(1+4^x)} = \int \frac{dt}{t^2(1+t^2)\ln 2} = \frac{1}{\ln 2} \int \left(\frac{1}{t^2} - \frac{1}{1+t^2} \right) dt = -\frac{1}{\ln 2} \left(\frac{1}{t} + \arctan t \right) + c \\ = -\frac{1}{\ln 2} \left(\frac{1}{2^x} + \arctan 2^x \right) + c.$$

4. 求下列不定积分:

$$(1) \text{【解】} \int \frac{x^5}{(x-2)^{100}} dx = -\frac{1}{99} \int x^5 d(x-2)^{-99} = -\frac{x^5}{99(x-2)^{99}} + \frac{5}{99} \int x^4 (x-2)^{-99} dx \\ = -\frac{x^5}{99(x-2)^{99}} - \frac{5x^4}{9702(x-2)^{98}} + \frac{10}{4851} \int x^3 (x-2)^{-98} dx \\ = -\frac{x^5}{99(x-2)^{99}} - \frac{5x^4}{9702(x-2)^{98}} - \frac{10x^3}{470547(x-2)^{97}} - \frac{5x^2}{7528752(x-2)^{96}} \\ - \frac{x}{71523144(x-2)^{95}} - \frac{1}{71523144(x-2)^{94}} + c.$$

$$(2) \text{【解】} \text{令 } x = 1/t,$$

$$\int \frac{dx}{x \sqrt{1+x^4}} = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t} \sqrt{t^4+1}} = -\int \frac{tdt}{\sqrt{1+t^4}} = -\frac{1}{2} \int \frac{dt^2}{\sqrt{1+(t^2)^2}} \\ \text{令 } t^2 = \tan u, \frac{1}{2} \int \frac{\sec^2 u du}{\sec u} du = \frac{1}{2} \ln |\tan u + \sec u| + c = -\frac{1}{2} \ln \frac{1 + \sqrt{1+x^4}}{x^2} + c.$$

5. 求下列不定积分:

$$(1) \text{【解】} \int x \cos^2 x dx = \frac{1}{2} \int x(1 + \cos 2x) dx = \frac{1}{4} x^2 + \frac{1}{4} \int x \sin 2x dx \\ = \frac{1}{4} x^2 + \frac{1}{4} x \sin 2x - \frac{1}{4} \int \sin 2x dx = \frac{1}{4} x^2 + \frac{1}{4} x \sin 2x + \frac{1}{8} \cos 2x + c.$$

$$(2) \text{【解】} \int \sec^3 x dx = \int \frac{1}{\cos x} d \tan x = \frac{\sin x}{\cos^2 x} - \int \sec x \tan^2 x dx \\ = \sec x \tan x - \int (\sec^2 x - 1) \sec x dx = \sec x \tan x + \ln |\sec x + \tan x| - \int \sec^3 x dx \\ = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + c.$$

$$(3) \text{【解】} \int \frac{(\ln x)^3}{x^2} dx = -\int (\ln x)^3 d \frac{1}{x} = -\frac{1}{x} (\ln x)^3 + \int \frac{3(\ln x)^2}{x^2} dx \\ = -\frac{(\ln x)^3}{x} - \frac{3(\ln x)^2}{x} + \int \frac{6 \ln x}{x^2} dx = -\frac{(\ln x)^3}{x} - \frac{3(\ln x)^2}{x} - \frac{6 \ln x}{x} + \int \frac{6}{x^2} dx \\ = -\frac{1}{x} [(\ln x)^3 + 3(\ln x)^2 + 6 \ln x + 6] + c.$$

$$(4) \text{【解】} \int \cos(\ln x) dx = x \cos(\ln x) - \int x d \cos(\ln x) = x \cos(\ln x) - \int \sin(\ln x) dx \\ = x[\cos(\ln x) + \sin(\ln x)] - \int \cos(\ln x) dx \\ = \frac{x}{2} [\cos(\ln x) + \sin(\ln x)] + c.$$

$$(5) \text{【解】} \int \frac{x \cos^4 \frac{x}{2}}{\sin^3 x} dx = \int \frac{x \cos^4 \frac{x}{2}}{8 \sin^3 \frac{x}{2} \cos^3 \frac{x}{2}} dx = \frac{1}{4} \int \frac{x}{\sin \frac{x}{2}} \cdot \frac{\cos \frac{x}{2}}{\sin^2 \frac{x}{2}} d \frac{x}{2} \\ = -\frac{1}{4} \int x \csc \frac{x}{2} d \csc \frac{x}{2} = -\frac{1}{8} \int x d \csc^2 \frac{x}{2} = -\frac{1}{8} x \csc^2 \frac{x}{2} + \frac{1}{4} \int \csc^2 \frac{x}{2} d \frac{x}{2}$$

$$= -\frac{1}{8}x\csc^2 \frac{x}{2} - \frac{1}{4}\cot \frac{x}{2} + c.$$

6. 求下列不定积分:

$$\begin{aligned} (1) \text{【解】} \int \frac{x \ln(x + \sqrt{1+x^2})}{(1-x^2)^2} dx &= \frac{1}{2} \int \ln(x + \sqrt{1+x^2}) d \frac{1}{1-x^2} \\ &= \frac{1}{2} \ln(x + \sqrt{1+x^2}) \cdot \frac{1}{1-x^2} - \frac{1}{2} \int \frac{1}{1-x^2} \cdot \frac{1}{\sqrt{1+x^2}} dx. \end{aligned}$$

令 $x = \tan t$, 则 $dx = \frac{dt}{\cos^2 t}$.

$$\begin{aligned} \int \frac{1}{1-x^2} \cdot \frac{1}{\sqrt{1+x^2}} dx &= \int \frac{1}{1-\tan^2 t} \cdot \frac{1}{\cos t} dt \\ &= \int \frac{\cos t}{1-2\sin^2 t} dt = \frac{1}{\sqrt{2}} \int \frac{d\sqrt{2}\sin t}{1-2\sin^2 t} \\ &= \frac{1}{\sqrt{2}} \ln \frac{1+\sqrt{2}\sin t}{1-\sqrt{2}\sin t} + c = \frac{1}{\sqrt{2}} \ln \frac{\sqrt{1+x^2}+\sqrt{2}x}{\sqrt{1+x^2}-\sqrt{2}x} + c. \end{aligned}$$

$$\text{所以} \int \frac{x \ln(x + \sqrt{1+x^2})}{(1-x^2)^2} dx = \frac{\ln(x + \sqrt{1+x^2})}{2(1-x^2)} - \frac{1}{4\sqrt{2}} \ln \frac{\sqrt{1+x^2}+\sqrt{2}x}{\sqrt{1+x^2}-\sqrt{2}x} + c.$$

$$\begin{aligned} (2) \text{【解】} \int \frac{x \arctan x}{\sqrt{1+x^2}} dx &= \int \arctan x d \sqrt{1+x^2} \\ &= \sqrt{1+x^2} \arctan x - \int \frac{1}{\sqrt{1+x^2}} dx \\ &= \sqrt{1+x^2} \arctan x - \ln(x + \sqrt{1+x^2}) + c. \end{aligned}$$

$$\begin{aligned} (3) \text{【解】} \int \frac{\arctan e^x}{e^{2x}} dx &= -\frac{1}{2} \int \arctan e^x de^{-2x} \\ &= -\frac{1}{2} e^{-2x} \arctan e^x + \frac{1}{2} \int \frac{e^{-x}}{1+e^{2x}} dx \\ &= -\frac{1}{2} e^{-2x} \arctan e^x + \frac{1}{2} \int \frac{1}{e^x(1+e^{2x})} dx \\ &= -\frac{1}{2} e^{-2x} \arctan e^x + \frac{1}{2} \int \left(\frac{1}{e^x} - \frac{e^x}{1+e^{2x}} \right) dx \\ &= -\frac{1}{2} (e^{-2x} \arctan e^x + e^{-x} + \arctan e^x) + c. \end{aligned}$$

$$\begin{aligned} 7. \text{【解】} \int f(x) dx &= \begin{cases} (x \ln(1+x^2) - 3) dx, & x \geq 0 \\ (x^2 + 2x - 3)e^{-x} dx, & x < 0 \end{cases} \\ &= \begin{cases} \frac{1}{2} x^2 \ln(1+x^2) - \frac{1}{2} [x^2 - \ln(1+x^2)] - 3x + c, & x \geq 0 \\ -(x^2 + 4x + 1)e^{-x} + c_1, & x < 0 \end{cases} \end{aligned}$$

由于 $\int f(x) dx$ 连续, 所以 $c = -1 + c_1, c_1 = 1 + c$.

$$\text{即} \int f(x) dx = \begin{cases} \frac{1}{2} x^2 \ln(1+x^2) - \frac{1}{2} [x^2 - \ln(1+x^2)] - 3x + c, & x \geq 0 \\ -(x^2 + 4x + 1)e^{-x} + 1 + c, & x < 0 \end{cases}$$

8. 【解】令 $t = e^x, x = \ln t, f'(t) = a \sin(\ln t) + b \cos(\ln t)$,

$$f(x) = \int [a \sin(\ln x) + b \cos(\ln x)] dx = \frac{x}{2} [(a+b) \sin(\ln x) + (b-a) \cos(\ln x)] + c.$$

9. 求下列不定积分:

$$(1) \text{【解】} \int 3^{2+3x} (2x+3) dx = \int 3^{2+3x} d(x^2+3x) = \frac{3^{2+3x}}{\ln 3} + c.$$

$$(2) \text{【解】} \int (3x^2 - 2x + 5)^{\frac{3}{2}} (3x - 1) dx = \frac{1}{2} \int (3x^2 - 2x + 5)^{\frac{3}{2}} d(3x^2 - 2x + 5) \\ = \frac{1}{5} (3x^2 - 2x + 5)^{\frac{5}{2}} + c.$$

$$(3) \text{【解】} \text{因为 } \ln(x + \sqrt{1+x^2}) = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}},$$

$$\text{所以 } \int \frac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx = \int \ln(x + \sqrt{1+x^2}) d\ln(x + \sqrt{1+x^2}) = \frac{1}{2} [\ln(x + \sqrt{1+x^2})]^2 + c.$$

$$(4) \text{【解】} \text{因为 } \ln(1 + \sqrt{1+x^2}) = \frac{x/\sqrt{1+x^2}}{1 + \sqrt{1+x^2}} = \frac{x}{1+x^2 + \sqrt{1+x^2}},$$

$$\text{所以 } \int \frac{xdx}{(1+x^2 + \sqrt{1+x^2}) \ln(1 + \sqrt{1+x^2})} = \int \frac{d\ln(1 + \sqrt{1+x^2})}{\ln(1 + \sqrt{1+x^2})} \\ = \ln |\ln(1 + \sqrt{1+x^2})| + c.$$

$$10. \text{【解】} \int \frac{xf'(x) - (1+x)f(x)}{x^2 e^x} dx = \int \frac{xf'(x) - f(x)}{x^2 e^x} dx - \int \frac{f(x)}{x e^x} dx \\ = \int \frac{1}{e^x} d\frac{f(x)}{x} - \int \frac{f(x)}{x e^x} dx \\ = \frac{1}{e^x} \frac{f(x)}{x} + \int \frac{f(x)}{x e^x} dx - \int \frac{f(x)}{x e^x} dx = \frac{f(x)}{x e^x} + c.$$

11. 【解】 令 $t = \cos x + 2$, 则 $\cos x = t - 2$, 从而

$$f'(t) = 1 - \cos^2 x + \frac{1}{\cos^2 x} - 1 = -(t-2)^2 + \frac{1}{(t-2)^2}.$$

$$\text{所以 } f(x) = - \int \left[(x-2)^2 - \frac{1}{(x-2)^2} \right] dx = -\frac{1}{3}(x-2)^3 - \frac{1}{x-2} + c.$$

12. 求下列不定积分:

$$(1) \text{【解】} \int \frac{x \arctan x}{(1+x^2)^2} dx = -\frac{1}{2} \int \arctan x d\frac{1}{1+x^2} \\ = -\frac{\arctan x}{2(1+x^2)} + \frac{1}{2} \int \frac{dx}{(1+x^2)^2} \\ \stackrel{\text{令 } x = \tan t}{=} -\frac{\arctan x}{2(1+x^2)} + \frac{1}{2} \int \frac{\sec^2 t}{\sec^4 t} dt \\ = -\frac{\arctan x}{2(1+x^2)} + \frac{1}{2} \int \cos^2 t dt \\ = -\frac{\arctan x}{2(1+x^2)} + \frac{1}{4} \int (\cos(2t) + 1) dt \\ = -\frac{\arctan x}{2(1+x^2)} + \frac{1}{4} t + \frac{1}{8} \sin 2t + c \\ = -\frac{\arctan x}{2(1+x^2)} + \frac{1}{4} \arctan x + \frac{x}{4(1+x^2)} + c.$$

$$(2) \text{【解】} \text{令 } u = \sqrt{\frac{x}{1+x}}, x = \frac{u^2}{1-u^2}, dx = \frac{2u}{(1-u^2)^2} du$$

$$\int \arcsin \sqrt{\frac{x}{1+x}} dx = \int \arcsin u \frac{2u}{(1-u^2)^2} du \\ \stackrel{\text{令 } u = \sin t}{=} \int t \frac{2 \sin t}{\cos^4 t} \cos t dt = 2 \int t \frac{\sin t}{\cos^3 t} dt = \int t \tan^2 t dt \\ = t \tan^2 t - \int \tan^2 t dt \\ = t \tan^2 t - \int \left(\frac{1}{\cos^2 t} - 1 \right) dt$$

$$= t \tan^2 t - \tan t + t + c$$

$$= x \arcsin \sqrt{\frac{x}{1+x}} - \sqrt{x} + \arcsin \sqrt{\frac{x}{1+x}} + c.$$

$$\begin{aligned} (3) \text{【解】} \int \frac{\arcsin x}{x^2} \cdot \frac{1+x^2}{\sqrt{1-x^2}} dx &= \int \frac{\arcsin x}{x^2 \sqrt{1-x^2}} dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx \\ &\stackrel{\text{令 } x = \sin t}{=} \int \frac{t}{\sin^2 t \cos t} \cos t dt + \frac{1}{2} (\arcsin x)^2 \\ &= - \int t \cot t + \frac{1}{2} (\arcsin x)^2 \\ &= -t \cot t + \int \cot t dt + \frac{1}{2} (\arcsin x)^2 \\ &= -t \cot t + \ln |\sin t| + \frac{1}{2} (\arcsin x)^2 + c \\ &= -\frac{\sqrt{1-x^2}}{x} \arcsin x + \ln |x| + \frac{1}{2} (\arcsin x)^2 + c. \end{aligned}$$

(4) 令 $t = \arctan x, x = \tan t, dx = \sec^2 t dt,$

$$\begin{aligned} \int \frac{\arctan x}{x^2(1+x^2)} dx &= \int \frac{t \sec^2 t}{\tan^2 t \sec^2 t} dt = \int \frac{t \cos^2 t}{\sin^2 t} dt = \int \frac{t(1-\sin^2 t)}{\sin^2 t} dt \\ &= - \int t \cot t - \int t dt = -t \cot t + \int \cot t dt - \frac{1}{2} t^2 = -t \cot t + \ln |\sin t| - \frac{1}{2} t^2 + c \\ &= -\frac{\arctan x}{x} + \frac{1}{2} \ln \frac{x^2}{1+x^2} - \frac{1}{2} (\arctan x)^2 + c. \end{aligned}$$

13. 求下列不定积分:

(1)【解】令 $x = 2 \sin t, dx = 2 \cos t dt,$

$$\begin{aligned} \int x^3 \sqrt{4-x^2} dx &= 32 \int \sin^3 t \cos^2 t dt = -32 \int (1-\cos^2 t) \cos^2 t \cos t dt \\ &= -\frac{32}{3} \cos^3 t + \frac{32}{5} \cos^5 t + c = \frac{1}{5} (4-x^2)^{\frac{5}{2}} - \frac{4}{3} (4-x^2)^{\frac{3}{2}} + c. \end{aligned}$$

(2)【解】令 $x = a \sec t, dx = a \sec t \tan t dt,$

$$\begin{aligned} \int \frac{\sqrt{x^2-a^2}}{x} dx &= \int \frac{a \tan t}{a \sec t} a \sec t \tan t dt = a \int \tan^2 t dt \\ &= a \int \frac{1-\cos^2 t}{\cos^2 t} dt = a \int \frac{dt}{\cos^2 t} - at = a \tan t - at + c \\ &= \sqrt{x^2-a^2} - a \arccos \frac{a}{x} + c. \end{aligned}$$

$$\begin{aligned} (3) \text{【解】} \int \frac{e^x(1+e^x)}{\sqrt{1-e^{2x}}} dx &= \int \frac{e^x}{\sqrt{1-e^{2x}}} dx + \int \frac{e^{2x}}{\sqrt{1-e^{2x}}} dx \\ &= \int \frac{de^x}{\sqrt{1-e^{2x}}} - \frac{1}{2} \int \frac{d(1-e^{2x})}{\sqrt{1-e^{2x}}} dx \\ &= \arcsin e^x - \sqrt{1-e^{2x}} + c. \end{aligned}$$

$$\begin{aligned} (4) \text{【解】} \int x \sqrt{\frac{x}{2a-x}} dx &\stackrel{\text{令 } u = \sqrt{x}}{=} 2 \int \frac{u^4}{\sqrt{2a-u^2}} du \stackrel{\text{令 } u = \sqrt{2a} \sin t}{=} 8a^2 \int \sin^4 t dt \\ &= 8a^2 \int \left(\frac{1-\cos 2t}{2} \right)^2 dt = 2a^2 \int (1-2\cos 2t + \cos^2 2t) dt \\ &= 2a^2 t - 2a^2 \sin 2t + 2a^2 \int \frac{1+\cos 4t}{2} dt = 3a^2 t - 2a^2 \sin 2t + \frac{a^2}{4} \sin 4t + c \\ &= 3a^2 t - 4a^2 \sin t \cos t + a^2 \sin t \cos t (1-2\sin^2 t) + c \\ &= 3a^2 t - 3a^2 \sin t \cos t - 2a^2 \sin^3 t \cos t + c \\ &= 3a^2 \arcsin \sqrt{\frac{x}{2a}} - 3a^2 \sqrt{\frac{x}{2a}} \sqrt{\frac{2a-x}{2a}} - 2a^2 \frac{x}{2a} \sqrt{\frac{x}{2a}} \sqrt{\frac{2a-x}{2a}} + c \end{aligned}$$