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David A. Ellwood
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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于1888年，是国际上极具影响力的专业学术组织，目前拥有近30000会员和580余个机构成员，出版图书3500多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

高 等 教 育 出 版 社

2016年12月

Introduction

In 2006, Peter E. Newstead paid his first academic visit to North America after the 1960s, and the occasion originated a number of workshops and conferences in his honor. The editors of this volume, together with Montserrat Teixidor i Bigas, organized a Clay Mathematics Institute workshop, "Moduli spaces of vector bundles, with a view toward coherent sheaves" (October 6-10, 2006). The experts convened produced a vigorous confluence of so many different techniques and discussed such deep connections that we felt a proceedings volume would be a valuable asset to the community of mathematicians and physicists; when a participant was not available to write for this volume, state-of-the-art coverage of the topic was provided through the generosity of an alternative expert.

Peter E. Newstead earned his Ph.D. from the University of Cambridge in 1966; both John A. Todd and Michael F. Atiyah supervised his doctoral work. From the beginning of his career, he was interested in topological properties of classification spaces of vector bundles. Geometric Invariant Theory was re-invigorated around that time by the refinement of concepts of (semi-)stability and projective models, and Newstead contributed some of the more original and deeper constructions, for example the projective models for rank-2 bundles of fixed determinant of odd degree over a curve of genus two, a quadratic complex (obtained also, using a different method, by M.S. Narasimhan with S. Ramanan), and a topological proof for non-existence of universal bundles in certain cases. He played a prominent role in the development of a major area by focusing on moduli of vector bundles over an algebraic curve, was a main contributor to a Brill-Noether geography for these spaces, and his topological results led him to make major contributions (cf. [N1-2] and [KN]) to the description, by generators and relations, of the rational cohomology algebra $H^*(\mathcal{SU}_X(2,L))$ for the moduli space $\mathcal{SU}_X(2,L)$ of stable rank-2 bundles over X with fixed determinant L, where X is a compact Riemann surface of genus $g \geq 2$, and L a line bundle of odd degree over X (the higher-rank case was then settled in [EK]). The main concerns of his current work are coherent systems on algebraic curves and Picard bundles. This bird's-eye view of Newstead's work omits several topics, ranging from invariants of group action to algebraic geometry over the reals, conic bundles and other special projective varieties defined by quadrics, and compactifications of moduli spaces, and is merely intended to serve as orientation for the readers of the present volume.

This volume of cutting-edge contributions provides a collection of problems and methods that are greatly enriching our understanding of moduli spaces and their applications. It should be accessible to non-experts, as well as further the interaction among researchers specializing in various aspects of these spaces. Indeed, we hope this volume will impress the reader with the diversity of ideas and techniques that are brought together by the nature of these varieties.

In brief and non-technical terms, the volume covers the following areas. An aspect of moduli spaces that recently emerged is the disparate set of dualities that parallel the classical Hecke correspondence of number theory. In modern terms, such a pairing of two variables (or two categories) is a Fourier-Mukai-Laumon transform; implications go under the heading of geometric Langlands program, the area of Kamnitzer's article. Pareschi and Popa offer original techniques in the derived-theoretic study of regularity and generic vanishing for coherent sheaves on abelian varieties, with applications to the study of vector bundles, as well as that of linear series on irregular varieties. Also on the theme of moduli spaces, Aprodu and Farkas on the one hand, Jeffrey on the other, provide techniques to analyze different properties: respectively, applications of Koszul cohomology to the study of various moduli spaces, and symplectic-geometric methods for intersection cohomology over singular moduli spaces. Teixidor treats moduli spaces of vector bundles over reducible curves, a delicate issue with promising applications to integrable systems. Arcara and Bertram work torwards a concept of stability for bundles over surfaces and conjecturally over threefolds. Using the Brauer group, Lieblich relates the geometry of moduli spaces to the properties of certain non-commutative algebras and to arithmetic local-to-global principles. Andersen and Gammelgaard's paper addresses a quantization of the moduli space: the fibration of the moduli space of curves given by the moduli of bundles admits a projective connection whose associated operator generalizes the heat equation – it was defined independently by N. Hitchin, and by S. Axelrod with S. Della Pietra and E. Witten. Finally, the workshop's guest of honor, Peter Newstead himself, offers a cutting-edge overview of his current area of work, coherent systems over algebraic curves; may we salute it as the Brill-Noether theory of the XXI century?

We hope you enjoy the book and find it as inspiring as we do.

David A. Ellwood, Cambridge
Emma Previato, Boston

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Contents

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Hitchin's Projectively Flat Connection, Toeplitz Operators and the Asymptotic Expansion of TQFT Curve Operators

Jørgen Ellegaard Andersen and Niels Leth Gammelgaard

ABSTRACT. In this paper, we will provide a review of the geometric construction, proposed by Witten, of the SU(n) quantum representations of the mapping class groups which are part of the Reshetikhin-Turaev TQFT for the quantum group $U_q(sl(n, \mathbb{C}))$. In particular, we recall the differential geometric construction of Hitchin's projectively flat connection in the bundle over Teichmüller space obtained by push-forward of the determinant line bundle over the moduli space of rank n, fixed determinant, semi-stable bundles fibering over Teichmüller space. We recall the relation between the Hitchin connection and Toeplitz operators which was first used by the first named author to prove the asymptotic faithfulness of the SU(n) quantum representations of the mapping class groups. We further review the construction of the formal Hitchin connection, and we discuss its relation to the full asymptotic expansion of the curve operators of Topological Quantum Field Theory. We then go on to identify the first terms in the formal parallel transport of the Hitchin connection explicitly. This allows us to identify the first terms in the resulting star product on functions on the moduli space. This is seen to agree with the first term in the star-product on holonomy functions on these moduli spaces defined by Andersen, Mattes and Reshetikhin.

1. Introduction

Witten constructed, via path integral techniques, a quantization of Chern-Simons theory in $2+1$ dimensions, and he argued in **[Wi]** that this produced a TQFT, indexed by a compact simple Lie group and an integer level k. For the group SU(n) and level k, let us denote this TQFT by $Z_k^{(n)}$. Witten argues in **[Wi]** that the theory $Z_k^{(2)}$ determines the Jones polynomial of a knot in S^3. Combinatorially, this theory was first constructed by Reshetikhin and Turaev, using the representation theory of $U_q(\mathrm{sl}(n, C))$ at $q = e^{(2\pi i)/(k+n)}$, in **[RT1]** and **[RT2]**. Subsequently, the TQFT's $Z_k^{(n)}$ were constructed using skein theory by Blanchet, Habegger, Masbaum and Vogel in **[BHMV1]**, **[BHMV2]** and **[B1]**.

The two-dimensional part of the TQFT $Z_k^{(n)}$ is a modular functor with a certain label set. For this TQFT, the label set $\Lambda_k^{(n)}$ is a finite subset (depending on k) of the set of finite dimensional irreducible representations of SU(n). We use the

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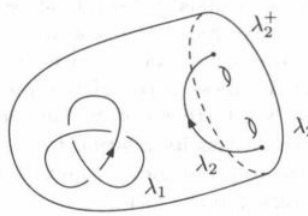
usual labeling of irreducible representations by Young diagrams, so in particular $\square \in \Lambda_k^{(n)}$ is the defining representation of $\mathrm{SU}(n)$. Let further $\lambda_0^{(d)} \in \Lambda_k^{(n)}$ be the Young diagram consisting of d columns of length k. The label set is also equipped with an involution, which is simply induced by taking the dual representation. The trivial representation is a special element in the label set which is clearly preserved by the involution.

$$Z_k^{(n)}: \quad \left\{\begin{array}{l}\text{Category of (ex-}\\ \text{tended) closed}\\ \text{oriented surfaces}\\ \text{with } \Lambda_k^{(n)}\text{-labeled}\\ \text{marked points with}\\ \text{projective tangent}\\ \text{vectors}\end{array}\right\} \to \left\{\begin{array}{l}\text{Category of finite}\\ \text{dimensional vector}\\ \text{spaces over } \mathbb{C}\end{array}\right\}$$

The three-dimensional part of $Z_k^{(n)}$ is an association of a vector,

$$Z_k^{(n)}(M, L, \lambda) \in Z_k^{(n)}(\partial M, \partial L, \partial\lambda),$$

to any compact, oriented, framed 3–manifold M together with an oriented, framed link $(L, \partial L) \subseteq (M, \partial M)$ and a $\Lambda_k^{(n)}$-labeling $\lambda : \pi_0(L) \to \Lambda_k^{(n)}$.



This association has to satisfy the Atiyah-Segal-Witten TQFT axioms (see e.g. [**At**], [**Se**] and [**Wi**]). For a more comprehensive presentation of the axioms, see Turaev's book [**T**].

The geometric construction of these TQFTs was proposed by Witten in [**Wi**] where he derived, via the Hamiltonian approach to quantum Chern-Simons theory, that the geometric quantization of the moduli spaces of flat connections should give the two-dimensional part of the theory. Further, he proposed an alternative construction of the two-dimensional part of the theory via WZW-conformal field theory. This theory has been studied intensively. In particular, the work of Tsuchiya, Ueno and Yamada in [**TUY**] provided the major geometric constructions and results needed. In [**BK**], their results were used to show that the category of integrable highest weight modules of level k for the affine Lie algebra associated to any simple Lie algebra is a modular tensor category. Further, in [**BK**], this result is combined with the work of Kazhdan and Lusztig [**KL**] and the work of Finkelberg [**Fi**] to argue that this category is isomorphic to the modular tensor category associated to the corresponding quantum group, from which Reshetikhin and Turaev constructed their TQFT. Unfortunately, these results do not allow one to conclude the validity of the geometric constructions of the two-dimensional part of the TQFT proposed by Witten. However, in joint work with Ueno, [**AU1**], [**AU2**], [**AU3**] and [**AU4**], we have given a proof, based mainly on the results of [**TUY**], that the TUY-construction of the WZW-conformal field theory, after twist by a fractional power of an abelian theory, satisfies all the axioms of a modular functor. Furthermore, we have proved that the full $(2+1)$-dimensional TQFT resulting from this is isomorphic to the aforementioned one, constructed by BHMV via skein theory. Combining this with the theorem of Laszlo [**La1**], which identifies (projectively) the representations of the mapping class groups obtained from the geometric quantization of the moduli space of flat connections with the ones obtained from the TUY-constructions, one gets a proof of the validity of the construction proposed by Witten in [**Wi**].

Part of this TQFT is the quantum $\mathrm{SU}(n)$ representations of the mapping class groups. Namely, if Σ is a closed oriented surfaces of genus g, Γ is the mapping class group of Σ, and p is a point on Σ, then the modular functor induces a representation

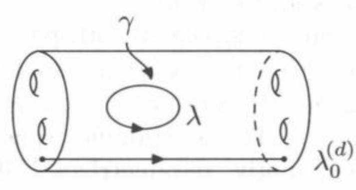
$$Z_k^{(n,d)} : \Gamma \to \mathbb{P}\,\mathrm{Aut}\big(Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})\big). \tag{1}$$

For a general label of p, we would need to choose a projective tangent vector $v_p \in T_p\Sigma/\mathbb{R}_+$, and we would get a representation of the mapping class group of (Σ, p, v_p). But for the special labels $\lambda_0^{(d)}$, the dependence on v_p is trivial and in fact we get a representation of Γ. Furthermore, the curve operators are also part of any TQFT: For $\gamma \subseteq \Sigma - \{p\}$ an oriented simple closed curve and any $\lambda \in \Lambda_k^{(n)}$, we have the operators

$$Z_k^{(n,d)}(\gamma,\lambda) : Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \to Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}), \tag{2}$$

defined as

$$Z_k^{(n,d)}(\gamma,\lambda) = Z_k^{(n,d)}(\Sigma \times I, \gamma \times \{\tfrac{1}{2}\} \coprod \{p\} \times I, \{\lambda, \lambda_0^{(d)}\}).$$



The curve operators are natural under the action of the mapping class group, meaning that the following diagram,

$$\begin{array}{ccc} Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) & \xrightarrow{Z_k^{(n,d)}(\gamma,\lambda)} & Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \\ {\scriptstyle Z_k^{(n,d)}(\phi)}\downarrow & & \downarrow{\scriptstyle Z_k^{(n,d)}(\phi)} \\ Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) & \xrightarrow{Z_k^{(n,d)}(\phi(\gamma),\lambda)} & Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}), \end{array}$$

is commutative for all $\phi \in \Gamma$ and all labeled simple closed curves $(\gamma, \lambda) \subset \Sigma - \{p\}$.

For the curve operators, we can derive an explicit formula using factorization: Let Σ' be the surface obtained from cutting Σ along γ and identifying the two boundary components to two points, say $\{p_+, p_-\}$. Here p_+ is the point corresponding to the "left" side of γ. For any label $\mu \in \Lambda_k^{(n)}$, we get a labeling of the ordered points (p_+, p_-) by the ordered pair of labels $(\mu, \mu^\dagger)$.

Since $Z_k^{(n)}$ is also a modular functor, one can factor the space $Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})$ as a direct sum, 'along' γ, over $\Lambda_k^{(n)}$. That is, we get an isomorphism

$$Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \cong \bigoplus_{\mu \in \Lambda_k^{(n)}} Z^{(k)}(\Sigma', p_+, p_-, p, \mu, \mu^\dagger, \lambda_0^{(d)}). \tag{3}$$

Strictly speaking, we need a base point on γ to induce tangent directions at $p_\pm$. However, the corresponding subspaces of $Z^{(k)}(\Sigma, p, \lambda_0^{(d)})$ do not depend on the choice of base point. The isomorphism (3) induces an isomorphism

$$\mathrm{End}(Z^{(k)}(\Sigma, p, \lambda_0^{(d)})) \cong \bigoplus_{\mu \in \Lambda_k^{(n)}} \mathrm{End}(Z^{(k)}(\Sigma', p_+, p_-, p, \mu, \mu^\dagger, \lambda_0^{(d)})),$$

which also induces a direct sum decomposition of $\mathrm{End}(Z^{(k)}(\Sigma, p, \lambda_0^{(d)}))$, independent of the base point.

The TQFT axioms imply that the curve operator $Z^{(k)}(\gamma, \lambda)$ is diagonal with respect to this direct sum decomposition along γ. One has the formula

$$Z^{(k)}(\gamma, \lambda) = \bigoplus_{\mu \in \Lambda_k^{(n)}} S_{\lambda,\mu}(S_{0,\mu})^{-1} \, \mathrm{Id}_{Z^{(k)}(\Sigma', p_+, p_-, p, \mu, \mu^\dagger, \lambda_0^{(d)})}.$$

Here $S_{\lambda,\mu}$ is the S-matrix[1] of the theory $Z_k^{(n)}$. See e.g. [**B1**] for a derivation of this formula.

Let us now briefly recall the geometric construction of the representations $Z_k^{(n,d)}$ of the mapping class group, as proposed by Witten, using geometric quantization of moduli spaces.

We assume from now on that the genus of the closed oriented surface Σ is at least two. Let M be the moduli space of flat $\mathrm{SU}(n)$ connections on $\Sigma - p$ with holonomy around p equal to $\exp(2\pi i d/n)\,\mathrm{Id} \in \mathrm{SU}(n)$. When (n, d) are coprime, the moduli space is smooth. In all cases, the smooth part of the moduli space has a natural symplectic structure ω. There is a natural smooth symplectic action of the mapping class group Γ of Σ on M. Moreover, there is a unique prequantum line bundle $(\mathcal{L}, \nabla, (\cdot,\cdot))$ over (M, ω). The Teichmüller space $\mathcal{T}$ of complex structures on Σ naturally, and Γ-equivariantly, parametrizes Kähler structures on (M, ω). For $\sigma \in \mathcal{T}$, we denote by M_σ the manifold (M, ω) with its corresponding Kähler structure. The complex structure on M_σ and the connection ∇ in $\mathcal{L}$ induce the structure of a holomorphic line bundle on $\mathcal{L}$. This holomorphic line bundle is simply the determinant line bundle over the moduli space, and it is an ample generator of the Picard group [**DN**].

By applying geometric quantization to the moduli space M, one gets, for any positive integer k, a certain finite rank bundle over Teichmüller space $\mathcal{T}$ which we will call the *Verlinde bundle* $\mathcal{V}_k$ at level k. The fiber of this bundle over a point $\sigma \in \mathcal{T}$ is $\mathcal{V}_{k,\sigma} = H^0(M_\sigma, \mathcal{L}^k)$. We observe that there is a natural Hermitian structure $\langle\cdot,\cdot\rangle$ on $H^0(M_\sigma, \mathcal{L}^k)$ by restricting the L_2-inner product on global L_2 sections of $\mathcal{L}^k$ to $H^0(M_\sigma, \mathcal{L}^k)$.

The main result pertaining to this bundle is:

THEOREM 1 (Axelrod, Della Pietra and Witten; Hitchin). *The projectivization of the bundle $\mathcal{V}_k$ supports a natural flat Γ-invariant connection $\hat{\nabla}$.*

---

[1] The S-matrix is determined by the isomorphism that a modular functor induces from two different ways of glueing an annulus to obtain a torus. For its definition, see e.g. [**MS**], [**Se**], [**Wa**] or [**BK**] and references therein. It is also discussed in [**AU3**].

This is a result proved independently by Axelrod, Della Pietra and Witten [**ADW**] and by Hitchin [**H**]. In section 2, we review our differential geometric construction of the connection $\hat{\nabla}$ in the general setting discussed in [**A6**]. We obtain as a corollary that the connection constructed by Axelrod, Della Pietra and Witten projectively agrees with Hitchin's.

DEFINITION 1. *We denote by* $Z_k^{(n,d)}$ *the representation*

$$Z_k^{(n,d)} : \Gamma \to \mathbb{P}\operatorname{Aut}\big(Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})\big),$$

obtained from the action of the mapping class group on the covariant constant sections of $\mathbb{P}(\mathcal{V}_k)$ *over* $\mathcal{T}$.

The projectively flat connection $\hat{\nabla}$ induces a *flat* connection $\hat{\nabla}^e$ in $\operatorname{End}(\mathcal{V}_k)$. Let $\operatorname{End}_0(\mathcal{V}_k)$ be the subbundle consisting of traceless endomorphisms. The connection $\hat{\nabla}^e$ also induces a connection in $\operatorname{End}_0(\mathcal{V}_k)$, which is invariant under the action of Γ.

In [**A3**], we proved

THEOREM 2 (Andersen). *Assume that* n *and* d *are coprime or that* $(n,d) = (2,0)$ *when* $g = 2$. *Then, we have that*

$$\bigcap_{k=1}^{\infty} \ker(Z_k^{(n,d)}) = \begin{cases} \{1, H\} & g = 2,\ n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise,} \end{cases}$$

where H *is the hyperelliptic involution.*

The main ingredient in the proof of this Theorem is the *Toeplitz operators* associated to smooth functions on M. For each $f \in C^\infty(M)$ and each point $\sigma \in \mathcal{T}$, we have the Toeplitz operator,

$$T_{f,\sigma}^{(k)} : H^0(M_\sigma, \mathcal{L}_\sigma^k) \to H^0(M_\sigma, \mathcal{L}_\sigma^k),$$

which is given by

$$T_{f,\sigma}^{(k)} = \pi_\sigma^{(k)}(fs)$$

for all $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$. Here $\pi_\sigma^{(k)}$ is the orthogonal projection onto $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ induced from the L_2-inner product on $C^\infty(M, \mathcal{L}^k)$. We get a smooth section of $\operatorname{End}(\mathcal{V}^{(k)})$,

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \operatorname{End}(\mathcal{V}^{(k)})),$$

by letting $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$ (see [**A3**]). See section 3 for further discussion of the Toeplitz operators and their connection to deformation quantization.

The sections $T_f^{(k)}$ of $\operatorname{End}(\mathcal{V}^{(k)})$ over $\mathcal{T}$ are not covariant constant with respect to Hitchin's connection $\hat{\nabla}^e$. However, they are asymptotically so as k goes to infinity. This will be made precise when we discuss the formal Hitchin connection below.

As a further application of TQFT and the theory of Toeplitz operators together with the theory of coherent states, we recall the first author's solution to a problem in geometric group theory, which has been around for quite some time (see e.g. Problem (7.2) in Chapter 7, "A short list of open questions", of [**BHV**]): In [**A8**], Andersen proved that

THEOREM 3 (Andersen). *The mapping class group of a closed oriented surface, of genus at least two, does not have Kazhdan's property (T).*

Returning to the geometric construction of the Reshetikhin-Turaev TQFT, let us recall the geometric construction of the curve operators. First of all, the decomposition (3) is geometrically obtained as follows (see [**A7**] for the details):

One considers a one parameter family of complex structures $\sigma_t \in \mathcal{T}$, $t \in \mathbb{R}_+$, such that the corresponding family in the moduli space of curves converges in the Mumford-Deligne boundary to a nodal curve, which topologically corresponds to shrinking γ to a point. By the results of [**A1**], the corresponding sequence of complex structures on the moduli space M converges to a non-negative polarization on M whose isotropic foliation is spanned by the Hamiltonian vector fields associated to the holonomy functions of γ. The main result of [**A7**] is that the covariant constant sections of $\mathcal{V}^{(k)}$ along the family σ_t converge to distributions supported on the Bohr-Sommerfeld leaves of the limiting non-negative polarization as t goes to infinity. The direct sum of the geometric quantization of the level k Bohr-Sommerfeld levels of this non-negative polarization is precisely the left-hand side of (3). A sewing construction, inspired by conformal field theory (see [**TUY**]), is then applied to show that the resulting linear map from the right-hand side of (3) to the left-hand side is an isomorphism. This is described in detail in [**A7**].

In [**A7**], we further prove the following important asymptotic result. Let $h_{\gamma,\lambda} \in C^\infty(M)$ be the holonomy function obtained by taking the trace in the representation λ of the holonomy around γ.

THEOREM 4 (Andersen). *For any one-dimensional oriented submanifold γ and any labeling λ of the components of γ, we have that*

$$\lim_{k\to\infty} \|Z_k^{(n,d)}(\gamma,\lambda) - T_{h_{\gamma,\lambda}}^{(k)}\| = 0.$$

Let us here give the main idea behind the proof of Theorem 4 and refer to [**A7**] for the details. One considers the explicit expression for the S-matrix, as given in formula (13.8.9) in Kac's book [**Kac**],

$$S_{\lambda,\mu}/S_{0,\mu} = \lambda(e^{-2\pi i \frac{\breve{\mu}+\breve{\rho}}{k+n}}), \tag{4}$$

where ρ is half the sum of the positive roots and $\breve{\nu}$ (ν any element of Λ) is the unique element of the Cartan subalgebra of the Lie algebra of SU(n) which is dual to ν with respect to the Cartan-Killing form $(\cdot,\cdot)$.

From the expression (4), one sees that under the isomorphism $\breve{\mu} \mapsto \mu$, the expression $S_{\lambda,\mu}/S_{0,\mu}$ makes sense for any $\breve{\mu}$ in the Cartan subalgebra of the Lie algebra of SU(n). Furthermore, one finds that the values of this sequence of functions (depending on k) are asymptotic to the set of values of the holonomy function $h_{\gamma,\lambda}$ at the level k Bohr-Sommerfeld sets of the limiting non-negative polarizations discussed above (see [**A1**]). From this, one can deduce Theorem 4. See again [**A7**] for details.

Let us now consider the general setting treated in [**A6**]. Thus, we consider, as opposed to only considering the moduli spaces, a general prequantizable symplectic manifold (M,ω) with a prequantum line bundle $(L,(\cdot,\cdot),\nabla)$. We assume that $\mathcal{T}$ is a complex manifold which holomorphically and rigidly (see Definition 5) parameterizes Kähler structures on (M,ω). Then, the following theorem, proved in [**A6**], establishes the existence of the Hitchin connection under a mild cohomological condition.

THEOREM 5 (Andersen). *Suppose that I is a rigid family of Kähler structures on the compact, prequantizable symplectic manifold (M,ω) which satisfies that there* *exists an* $n \in \mathbb{Z}$ *such that the first Chern class of* (M, ω) *is* $n[\frac{\omega}{2\pi}] \in H^2(M, \mathbb{Z})$ *and* $H^1(M, \mathbb{R}) = 0$. *Then, the Hitchin connection* $\hat{\nabla}$ *in the trivial bundle* $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ *preserves the subbundle* $H^{(k)}$ *with fibers* $H^0(M_\sigma, \mathcal{L}^k)$. *It is given by*

$$\hat{\nabla}_V = \hat{\nabla}^t_V + \frac{1}{4k+2n}\left\{\Delta_{G(V)} + 2\nabla_{G(V)dF} + 4kV'[F]\right\},$$

where $\hat{\nabla}^t$ *is the trivial connection in* $\mathcal{H}^{(k)}$, *and* V *is any smooth vector field on* $\mathcal{T}$.

In section 4, we study the formal Hitchin connection which was introduced in [**A6**]. Let $\mathcal{D}(M)$ be the space of smooth differential operators on M acting on smooth functions on M. Let C_h be the trivial $C_h^\infty(M)$-bundle over $\mathcal{T}$.

DEFINITION 2. *A formal connection* D *is a connection in* C_h *over* $\mathcal{T}$ *of the form*

$$D_V f = V[f] + \tilde{D}(V)(f),$$

where $\tilde{D}$ *is a smooth one-form on* $\mathcal{T}$ *with values in* $\mathcal{D}_h(M) = \mathcal{D}(M)[[h]]$, f *is any smooth section of* C_h, V *is any smooth vector field on* $\mathcal{T}$ *and* $V[f]$ *is the derivative of* f *in the direction of* V.

Thus, a formal connection is given by a formal series of differential operators

$$\tilde{D}(V) = \sum_{l=0}^{\infty} \tilde{D}^{(l)}(V)h^l.$$

From Hitchin's connection in $H^{(k)}$, we get an induced connection $\hat{\nabla}^e$ in the endomorphism bundle $\mathrm{End}(H^{(k)})$. As previously mentioned, the Teoplitz operators are not covariant constant sections with respect to $\hat{\nabla}^e$, but asymptotically in k they are. This follows from the properties of the formal Hitchin connection, which is the formal connection D defined through the following theorem (proved in [**A6**]).

THEOREM 6. *(Andersen) There is a unique formal connection* D *which satisfies that*

$$\hat{\nabla}^e_V T^{(k)}_f \sim T^{(k)}_{(D_V f)(1/(k+n/2))} \tag{5}$$

for all smooth sections f *of* C_h *and all smooth vector fields on* $\mathcal{T}$. *Moreover,*

$$\tilde{D} = 0 \mod h.$$

Here $\sim$ *means the following: For all* $L \in \mathbb{Z}_+$ *we have that*

$$\left\| \hat{\nabla}^e_V T^{(k)}_f - \left(T^{(k)}_{V[f]} + \sum_{l=1}^{L} T^{(k)}_{\tilde{D}^{(l)}_V f} \frac{1}{(k+n/2)^l} \right) \right\| = O(k^{-(L+1)}),$$

uniformly over compact subsets of $\mathcal{T}$, *for all smooth maps* $f : \mathcal{T} \to C^\infty(M)$.

Now fix an $f \in C^\infty(M)$ which does not depend on $\sigma \in \mathcal{T}$, and notice how the fact that $\tilde{D} = 0 \bmod h$ implies that

$$\left\| \hat{\nabla}^e_V T^{(k)}_f \right\| = O(k^{-1}).$$

This expresses the fact that the Toeplitz operators are asymptotically flat with respect to the Hitchin connection.

We define a mapping class group equivariant formal trivialization of D as follows.

DEFINITION 3. *A formal trivialization of a formal connection D is a smooth map $P : \mathcal{T} \to \mathcal{D}_h(M)$ which modulo h is the identity, for all $\sigma \in \mathcal{T}$, and which satisfies*

$$D_V(P(f)) = 0,$$

for all vector fields V on $\mathcal{T}$ and all $f \in C_h^\infty(M)$. Such a formal trivialization is mapping class group equivariant if $P(\phi(\sigma)) = \phi^* P(\sigma)$ for all $\sigma \in \mathcal{T}$ and $\phi \in \Gamma$.

Since the only mapping class group invariant functions on the moduli space are the constant ones (see [**Go1**]), we see that, in the case where M is the moduli space, such a P, if it exists, must be unique up to multiplication by a formal constant.

Clearly, if D is not flat, such a formal trivialization cannot exist even locally on $\mathcal{T}$. However, if D is flat and its zero-order term is just given by the trivial connection in C_h, then a local formal trivialization exists, as proved in [**A6**].

Furthermore, it is proved in [**A6**] that flatness of the formal Hitchin connection is implied by projective flatness of the Hitchin connection. As was proved by Hitchin in [**H**], and stated above in Theorem 1, this is the case when M is the moduli space. Furthermore, the existence of a formal trivialization implies the existence of a unique (up to formal scale) mapping class group equivariant formal trivialization, provided that $H^1_\Gamma(\mathcal{T}, D(M)) = 0$. The first steps towards proving that this cohomology group vanishes have been taken in [**AV1, AV2, AV3, Vi**]. In this paper, we prove that

THEOREM 7. *The mapping class group equivariant formal trivialization of the formal Hitchin connection exists to first order, and we have the following explicit formula for the first order term of P;*

$$P_\sigma^{(1)}(f) = \frac{1}{4}\Delta_\sigma(f) + i\nabla_{X_F''}(f),$$

where X_F'' denotes the (0,1)-part of the Hamiltonian vector field for the Ricci potential.

For the proof of the theorem, see section 4. We will make the following conjecture.

CONJECTURE 1. *The mapping class group equivariant formal trivialization of the formal Hitchin connection exists, and for any one-dimensional oriented submanifold γ and any labeling λ of the components of γ, we have the full asymptotic expansion*

$$Z_k^{(n,d)}(\gamma, \lambda) \sim T^{(k)}_{P(h_{\gamma,\lambda})},$$

which means that for all L and all $\sigma \in \mathcal{T}$, we have that

$$\|Z_k^{(n,d)}(\gamma, \lambda) - \sum_{l=0}^{L} T^{(k)}_{P_\sigma^{(l)}(h_{\gamma,\lambda})} \frac{1}{(k+n/2)^l}\| = O(k^{L+1}).$$

It is very likely that the techniques used in [**A7**] to prove Theorem 4 can be used to prove this conjecture.

When we combine this conjecture with the asymptotics of the product of two Toeplitz operators (see Theorem 11), we get the full asymptotic expansion of the product of two curve operators:

$$Z_k^{(n,d)}(\gamma_1, \lambda_1) Z_k^{(n,d)}(\gamma_2, \lambda_2) \sim T^{(k)}_{P(h_{\gamma_1,\lambda_1}) \tilde{\star}^{BT}_\sigma P(h_{\gamma_2,\lambda_2})},$$

where $\tilde{\star}_{\sigma}^{BT}$ is very closely related to the Berezin-Toeplitz star product for the Kähler manifold (M_σ, ω), as first defined in [**BMS**]. See section 3 for further details regarding this.

Suppose that we have a mapping class group equivariant formal trivialization P of the formal Hitchin connection D. We can then define a new smooth family of star products parametrized by $\mathcal{T}$ as follows:

$$f \star_\sigma g = P_\sigma^{-1}(P_\sigma(f) \tilde{\star}_\sigma^{\mathrm{BT}} P_\sigma(g))$$

for all $f, g \in C^\infty(M)$ and all $\sigma \in \mathcal{T}$. Using the fact that P is a trivialization, it is not hard to prove that $\star_\sigma$ is independent of σ, and we simply denote it $\star$. The following theorem is proved in section 4.

THEOREM 8. *if The star product $\star$ has the form*

$$f \star g = fg - \frac{i}{2}\{f,g\}h + O(h^2).$$

We observe that this formula for the first-order term of $\star$ agrees with the first-order term of the star product constructed by Andersen, Mattes and Reshetikhin in [**AMR2**], when we apply the formula in Theorem 8 to two holonomy functions h_{γ_1,λ_1} and h_{γ_2,λ_2}:

$$h_{\gamma_1,\lambda_1} \star h_{\gamma_2,\lambda_2} = h_{\gamma_1\gamma_2,\lambda_1\cup\lambda_2} - \frac{i}{2} h_{\{\gamma_1,\gamma_2\},\lambda_1\cup\lambda_2} + O(h^2).$$

We recall that $\{\gamma_1, \gamma_2\}$ is the Goldman bracket (see [**Go2**]) of the two simple closed curves γ_1 and γ_2.

A similar result was obtained for the abelian case, i.e. in the case where M is the moduli space of flat $U(1)$-connections, by the first author in [**A2**], where the agreement between the star product defined in differential geometric terms and the star product of Andersen, Mattes and Reshetikhin was proved to all orders. We conjecture that the two star products also *agree*[2] in the non-abelian case.

We also remark that the constructions presented here seems to explicitly realized, to first order, some of the constructions contemplated by Gukov and Witten in [**GW**] in the part concerned with Chern-Simons theory.

We would finally also like to recall that the first named author has shown that the Nielsen-Thurston classification of mapping classes is determined by the Reshetikhin-Turaev TQFTs. We refer to [**A5**] for the full details of this.

Warm thanks are due to the editor of this volume for her persistent encouragements towards the completion of this contribution.

2. The Hitchin connection

In this section, we review our construction of the Hitchin connection using the global differential geometric setting of [**A6**]. This approach is close in spirit to Axelrod, Della Pietra and Witten's in [**ADW**], however we do not use any infinite dimensional gauge theory. In fact, the setting is more general than the gauge theory setting in which Hitchin in [**H**] constructed his original connection. But when applied to the gauge theory situation, we get the corollary that Hitchin's connection agrees with Axelrod, Della Pietra and Witten's.

[2]By agree we don't just mean agree up to equivalence, but that the two star products of any two functions exactly agree.