美国数学会经典影印系列

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AMERICAN MATHEMATICAL SOCIETY

Analytic and Algebraic Geometry:

Common Problems, Different Methods

解析几何与代数几何：相同问题，不同方法

Jeffery McNeal
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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于1888年，是国际上极具影响力的专业学术组织，目前拥有近30000会员和580余个机构成员，出版图书3500多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

高等教育出版社

2016年12月

Preface

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the "Regional Geometry Institute" initiative of the National Science Foundation. In mid 1993 the program found an institutional home at the Institute for Advanced Study (IAS) in Princeton, New Jersey.

The IAS/Park City Mathematics Institute encourages both research and education in mathematics and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school teachers, undergraduate faculty, and researchers in mathematics education. One of PCMI's main goals is to make all of the participants aware of the total spectrum of activities that occur in mathematics education and research: we wish to involve professional mathematicians in education and to bring modern concepts in mathematics to the attention of educators. To that end the summer institute features general sessions designed to encourage interaction among the various groups. In-year activities at the sites around the country form an integral part of the High School Teachers Program.

Each summer a different topic is chosen as the focus of the Research Program and Graduate Summer School. Activities in the Undergraduate Summer School deal with this topic as well. Lecture notes from the Graduate Summer School are being published each year in this series. The first seventeen volumes are:

- Volume 1: *Geometry and Quantum Field Theory* (1991)
- Volume 2: *Nonlinear Partial Differential Equations in Differential Geometry* (1992)
- Volume 3: *Complex Algebraic Geometry* (1993)
- Volume 4: *Gauge Theory and the Topology of Four-Manifolds* (1994)
- Volume 5: *Hyperbolic Equations and Frequency Interactions* (1995)
- Volume 6: *Probability Theory and Applications* (1996)
- Volume 7: *Symplectic Geometry and Topology* (1997)
- Volume 8: *Representation Theory of Lie Groups* (1998)
- Volume 9: *Arithmetic Algebraic Geometry* (1999)
- Volume 10: *Computational Complexity Theory* (2000)
- Volume 11: *Quantum Field Theory, Supersymmetry, and Enumerative Geometry* (2001)
- Volume 12: *Automorphic Forms and their Applications* (2002)
- Volume 13: *Geometric Combinatorics* (2004)
- Volume 14: *Mathematical Biology* (2005)
- Volume 15: *Low Dimensional Topology* (2006)
- Volume 16: *Statistical Mechanics* (2007)
- Volume 17: *Analytic and Algebraic Geometry: Common Problems, Different Methods* (2008)

Volumes are in preparation for subsequent years.

Some material from the Undergraduate Summer School is published as part of the Student Mathematical Library series of the American Mathematical Society. We hope to publish material from other parts of the IAS/PCMI in the future.

This will include material from the High School Teachers Program and publications documenting the interactive activities which are a primary focus of the PCMI. At the summer institute late afternoons are devoted to seminars of common interest to all participants. Many deal with current issues in education: others treat mathematical topics at a level which encourages broad participation. The PCMI has also spawned interactions between universities and high schools at a local level. We hope to share these activities with a wider audience in future volumes.

John C. Polking
Series Editor
July 2010

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Jeffery D. McNeal and Mircea Mustaţă

IAS/Park City Mathematics Series
Volume 17, 2008

Introduction

Jeffery D. McNeal and Mircea Mustaţă

1. The subject

Interactions between algebraic and analytic techniques in the study of complex algebraic varieties go back to the beginnings of the twentieth century. However, over the past twenty years these connections have particularly flourished. Analytic methods and ideas around multiplier ideals have been successfully introduced in the study of higher-dimensional algebraic varieties, and tools developed as part of the Minimal Model Program have had growing impact. Taken together these new methods achieved great success, culminating in the proof of a fundamental problem in the field, the finite generation of the canonical ring of an algebraic variety. While some of the analytic techniques have already found an algebraic counterpart and vice versa, much is left to be done, and it is expected that we will see more of this interaction in the future.

The 2008 PCMI Summer School was centered around these exciting new developments at the crossroads of analytic and algebraic geometry, and in particular, on the two existing approaches to the finite generation of the canonical ring. The program had several components. There was a Graduate Program, consisting of eight mini-courses, half of which were devoted to analytic, and respectively, algebraic topics. The lecturers were Bo Berndtsson, John D'Angelo, Jean-Pierre Demailly, Christopher Hacon, János Kollár, Robert Lazarsfeld, Mircea Mustaţă, and Dror Varolin. In parallel with this there was an active Research Program, organized around one or two daily seminar talks. The research environment was enhanced by the presence of the Clay Senior Scholar Robert Lazarsfeld, and of the Program Principal Yum-Tong Siu. They each gave a public lecture, introducing some of the main questions in analytic and algebraic geometry.

This volume consists of the contributions of the eight lecturers in the Graduate Program. In addition, it contains two expository presentations introducing the finite generation of the canonical ring, from the two different perspectives.

2. Content of the volume

The mini-courses consisted of five lectures each, and three additional problem sessions run by TA's. Each sequence of four mini-courses, on the algebraic and on the analytic side, was organized as to give a gradual introduction to the recent developments. We have decided to keep the same order for the contributions in this

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volume. At the end of each of the two sequences, we have included one expository paper that was not based on the lectures in the PCMI program. We hope that in terms of both topic and presentation, these additions fit well with the rest of the volume.

As expected, there was a lot of interaction between the two sequences of courses, and this is transparent in the lecture notes in this volume. Some key concepts (most notably, multiplier ideals) and results (such as vanishing and lifting theorems) have played an important role in the cross-fertilization of analytic and algebraic geometry, and especially in the recent applications to birational geometry. We hope that seeing these notions and results developed in both contexts, will give the reader a sense of the close interaction between these two fields.

The contributions on the analytic side are the following.

An Introduction to Things $\bar{\partial}$. These are the lecture notes from Bo Berndtsson's course at PCMI 2008. The aim of this course was to give an introduction, appropriate for a beginning student in complex analysis, to the basic analytic techniques behind weighted L^2 estimates for the $\bar{\partial}$-equation. The notes here reflect this aim by starting with the $\bar{\partial}$-equation in one-variable, carefully examining the fundamental issues in this special case, and then seeing how these issues are handled in ever greater generality, i.e. for domains in several variables, on Stein manifolds, and for bundle-valued forms over general complex manifolds. The emphasis throughout the notes is on how the basic L^2 estimates for $\bar{\partial}$ change as additional structure (primarily metrics on both the base manifold and on the bundles) is added. Several applications of L^2 estimates on $\bar{\partial}$ are also presented, including vanishing theorems of Kodaira! type and various extension results connected to the Ohsawa-Takegoshi theorem.

Real and Complex Geometry Meet the Cauchy-Riemann Equations. In his course at the Summer School, John D'Angelo discussed several analytic topics that involve an interplay between real and complex geometry. The order of contact between a complex analytic variety and a real hypersurface inside a complex manifold, and its relation to estimates for the $\bar{\partial}$ operator, forms the basis for the first half of these notes. An expanded discussion of Kohn's subelliptic multipliers for the $\bar{\partial}$-Neumann problem, which inspired the general notion of multiplier ideals studied in algebraic and analytic geometry, is also given. A detailed presentation of a version of Hilbert's 17th problem, and a sketch of its relationship with special metrics over the complex projective space concludes these notes.

Three Variations on a Theme in Complex Analytic Geometry. This chapter is based on Dror Varolin's lecture series. These notes show how L^2 methods are used to obtain three foundational results in analytic geometry: Kodaira's embedding theorem, the L^2 holomorphic extension theorem of Ohsawa-Takegoshi, and Skoda's division theorem. The connection between curvature, in several forms, and the estimates on $\bar{\partial}$ used to obtain these results is emphasized throughout the notes. Multiplier ideal sheaves are also discussed, from the analytic perspective, and proofs of Nadel's vanishing theorem and Siu's theorem on the global generation of these ideals are presented.

Structure Theorems for Projective and Kähler Varieties. These are lecture notes based on Jean-Pierre Demailly's course. They examine vanishing results for $\bar{\partial}$ cohomology on compact Kähler manifolds (or more particularly, projective manifolds) that carry line bundles of various types, e.g. numerically effective, pseudo-effective,