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The Complete Works of Wu Wen-Tsun

Topology II

吴文俊全集 ◆拓扑学卷II

——A Theory of Imbedding, Immersion, and Isotopy of Polytopes in a Euclidean Space

吴文俊/著 李邦河/编订

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内 容 简 介

本卷收录了吴文俊的 *A Theory of Imbedding, Immersion, and Isotopy of Polytopes in a Euclidean Space* 一书. 一个空间嵌入另一空间(例如欧氏空间)是否可能以及这些嵌入所依据的同痕的分类问题, 已成为拓扑学中重要的中心问题之一, 也是许多拓扑学家从各种不同角度用各种不同方法研究的对象之一. 本书是作者从 1954 年以来在这方面研究工作的一个总结报告, 它的方法在于研究空间的去核 p 重积, 即将 p 重积除去对角以后所余的空间, 这一概念可追溯到 Van Kampen 早在 1932 年的一篇重要论文. 其次再应用 P. A. Smith 有关周期变换的理论以获得若干作为 Smith 特殊群中上类的不变量, 它们之为 0 是嵌入的必要条件而在某些极端情形又同时为充分条件. 关于嵌入的许多已知结果以及一些新的结果, 虽有着种种不同的来源, 都可用这一统一的方法得出. 浸入与同痕也可用同样办法处理并得出相应的类似结果.

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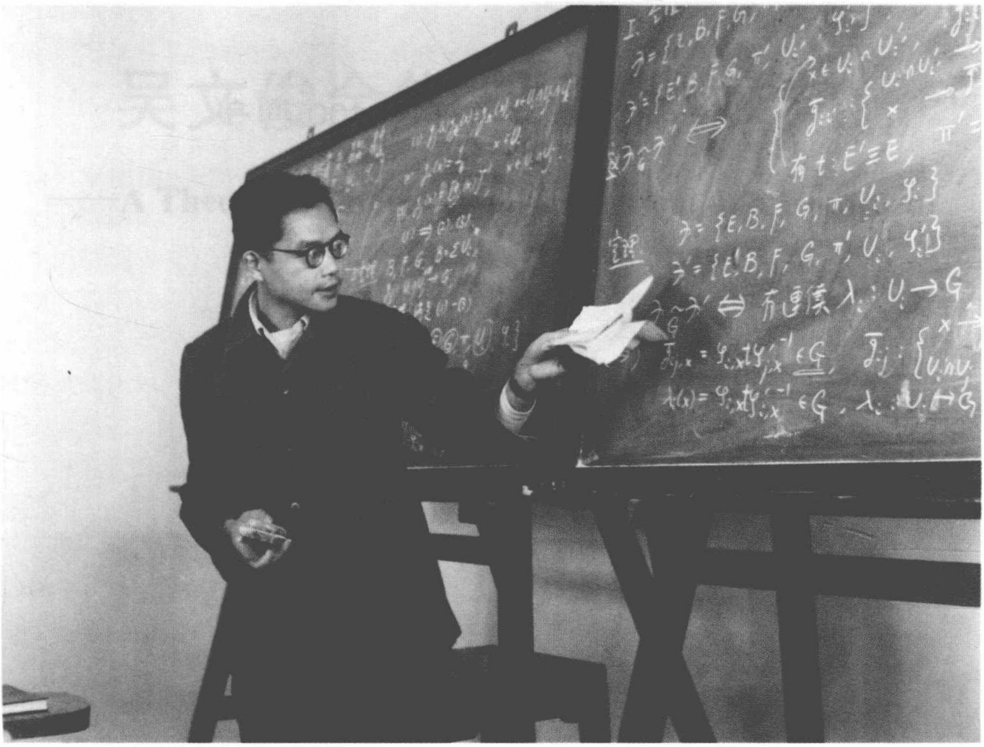


纪念吴文俊先生诞辰100周年

编　者　序

中国现代数学的崛起，开始于20世纪初，经历了几代人坚苦卓绝的努力．在这百年奋战中涌现出来的数学家中，吴文俊是最杰出的代表之一．他早年留学法国，留学期间就已在拓扑学方面做出了杰出贡献，提出了后来以他的名字命名的“吴公式”和“吴示性类”．回国后提出了“吴示嵌类”等拓扑不变量，发展了统一的嵌入理论．他关于示性类与示嵌类的研究，已成为20世纪拓扑学的经典，至今还在前沿研究中使用．20世纪70年代以来，吴文俊院士在汲取中国古代数学精髓的基础上，开创了崭新的现代数学领域——数学机械化．他发明的被国际上誉为“吴方法”的数学机械化方法，改变了国际自动推理的面貌，形成了自动推理的中国学派，已使中国在数学机械化领域处于国际领先地位．上述工作无疑属于 20 世纪中国数学赶超国际先进水平的标志性成果，而吴文俊院士博大精深的科学研究，除了拓扑学与数学机械化以外，还跨越了代数几何、博弈论、中国数学史、计算图论、人工智能等众多领域，并在每个领域都留下了这位多能数学家的重要贡献．

吴文俊先生是一位具有强烈爱国精神的数学家．自 1950 年谢绝法国师友的挽留回到祖国后，半个世纪如一日，为在他深爱的中华故土发展数学事业而鞠躬尽瘁．除了第一流的科研成果，吴文俊先生长期身处中国数学界领导地位，在团结带领整个中国数学界赶超世界先进水平方面，也做出了不可磨灭的贡献．特别是，吴文俊先生在担任中国数学会理事长期间，领导中国数学最终成功地加入了国际数学联盟，此举大大提高了我国数学界的国际地位，同时也为我国成功举办 2002 年国际数学家大会铺平了道路．

吴文俊治学严谨，学术思想活跃，无论获得多么高的声誉，他总是勤奋地在科研第一线工作，一生积极进取，锲而不舍，不断取得新的成就．在开始从事机器证明时，他已近花甲之年，从零开始学习编写计算机程序，每天十多个小时在机房连续工作，终于在几何定理机器证明这一难题上取得成功．

吴文俊先生为中国现代数学的发展建立了丰功伟绩，而他本人却始终淡泊、谦逊．他处事公正豁达，待人充满善意，受过他帮助的人可以说不计其数．正因如此，这位有着崇高国际声望而平易近人的学者，受到了每一个认识他的人格外的爱戴与尊敬．

2019 年 5 月 12 日是吴文俊先生百年诞辰．为了纪念这个特殊的日子，我们编辑出版了《吴文俊全集》，通过系统地收录、整理吴文俊先生的学术著作和论文，纪

念吴先生的学术思想及学术成就. 全集共计 13 卷, 包括拓扑学 4 卷、数学机械化 5 卷以及数学史、博弈论与代数几何、数学思想各 1 卷; 同时, 全集还设有附卷, 收录吴文俊先生的同事、学生和其他社会各界人士发表过的与吴先生有关的各类文献资料.

最后, 我们对在全集编辑中给予帮助的各位同事表示衷心感谢; 感谢国家出版基金对于全集出版的资助; 感谢科学出版社编辑人员在出版全集时认真细致的专业精神; 感谢相关出版与新闻机构在版权方面提供的帮助.

李邦河　高小山　李文林

2019 年 3 月

Preface

§ 1. The Problem of Realization or Imbedding

Geometry, as formulated by F. Klein, studies certain kinds of properties of certain kinds of figures and is splitted into various branches in accordance with the kinds of figures and properties to be studied. Now the geometric figures to be studied, whatever their origin may be, are at the last analysis, often "concrete" figures situated in a euclidean space, eventually one of infinite dimension or a Hilbert space when the origin is from analysis. On the other hand, in order to study the properties intrinsic to such figures themselves but independant of the environment space in which the figures are situated, it is necessary to define such figures in an "abstract" manner. Thus, the study of point sets by Cantor in a euclidean space gives rise gradually to the concept of topological spaces, the smooth curves and surfaces in a euclidean space of the classical differential geometry are abstracted to give rise to the concept of differential manifolds and Riemannian manifolds, etc. A natural problem then arises to identify the "abstract" concepts on the one hand and the "concrete" ones on the other hand, or more precisely, to decide whether any "abstract" object necessarily realize as a "concrete" one in a certain euclidean space of finite or infinite dimension. An affirmative answer to such a problem will then be implied by what we shall call a realization or an imbedding theorem. Many fundamental theorems in geometry are just of such a nature. Thus, we may cite some examples from topology as follows:

1° **Theorem of Urysohn.** Normal spaces with countable basis are just subspaces of a Hilbert space.

2° **Theorem of Menger-Nöbeling.** Compact spaces of finite dimension are just bounded subsets of a euclidean space of finite dimension.

These theorems furnish a whole view of certain category of spaces. The normality, compacity, of spaces are abstracted from analysis of concrete figures in a euclidean space. The two theorems above stated show that, among the innumerable topological properties of that category of figures in consideration, these properties have characteristic significance which justify the important position they occupy in topology.

For such realization or imbedding problems we may go even a step further to study figures in a euclidean space of a fixed dimension, e.g. linear graphs in a plane, Riemann surfaces of a 3-dimensional euclidean space, etc. From the point of view of topology, we are interested particularly in the following problems:

Problem 1 (Topological realization of spaces). Find the conditions for a topological space to be realizable as a subspace of a euclidean N-space.

Problem 2 (Semi-linear realization of complexes). Find the conditions for a complex to possess a subdivision realizable as a euclidean subcomplex of a euclidean N-space.

Problem 3 (Differential realization of differential manifolds). Find the conditions of a differential manifold to be realizable as a smooth manifold in a euclidean N-space.

A complete answer to these problems has surely little hope. Thus we may compare with the classical problem in differential geometry of the realization of a Riemannian manifold in a euclidean N-space for which very little are known even in the case of "local" realizations. We shall therefore content ourselves in getting partial results of the problems 1—3 (mainly problem 2) which is the aim of the present book.

§ 2. Analysis of Some Known Methods

We shall describe in what follows three methods for studying problems 1—3 occuring in the literature which are more or less general in character and have brought about important informations concerning possibility of realization in a euclidean N-space.

Method 1. Suppose that a compact space X has been realized as a subspace in a euclidean N-space R^N. The study of properties of $R^N - X$ in terms of those of X will then give conditions for X to be realizable in such an R^N. This method can be traced up to early works of Alexander. Thus, from the Alexander duality theorem we have for the Betti numbers $p^i(R^N - X) = p^{N-i-1}(X)$, $i \neq 0$, $p^0(R^N - X) = p^{N-1}(X) + 1$. It follows that $p^N(X) = p^{-1}(R^N - X) = 0$ which gives a necessary condition for X to be realizable in R^N. In particular, a closed n-manifold is not realizable in R^n. With this method Hantzsche has got conditions in terms of Betti numbers and torsion numbers for a closed n-manifold X to be realizable in an R^{n+1}, by studying duality between homology groups of X and $R^{n+1}-X$. Similarly, Hopf has proved that an n-dimensional projective space X cannot be realized in R^{n+1} by studying the duality between the cohomology rings of X and $R^{n+1}-X$. In the same manner R. Thom obtained conditions in terms of cohomology ring of X for a closed n-manifold X to be realizable in R^{n+1}. A recent work of Peterson studies the duality between cohomology operations in X and R^N-X (or more exactly S^N-X, where S^N is an N-sphere in which X is supposed to be realized) and has given rise to interesting realizability theorems.

Method 2. Instead of considering the relations between X supposed realized in R^N and the rest R^N-X of X as in Method 1, we may consider the relations between X and its neighborhoods in R^N. This method seems to be first introduced by Whitney to study differential realizations of differential manifolds in a euclidean space. Based on such considerations Whitney has created the theory of sphere bundles and introduced the so-called Stiefel-Whitney classes $W^i \in H^i(M, \bmod 2)$ and dual Whitney classes $\overline{W}^i \in H^i(M, \bmod 2)$ of a differential manifold M which have played so important a role in topology and differential geometry. So far as the realization problem is concerned, we may state for the moment the following classical theorem of Whitney:

Theorem 1. *For a differential manifold M^n of dimension n to be differentially realizable in a euclidean N-space R^N, it is necessary that*

$$\overline{W}^k(M^n)=0, \qquad k \geqslant N-n. \tag{1}$$

The notion of Stiefel-Whitney classes have been generalized by Pontrjagin who introduced characteristic classes of differential manifolds among which, besides the Stiefel-Whitney classes, the most important ones are the so-called Pontrjagin classes $P^{4k} \in H^{4k}(M)$ and their dual classes $\overline{P}^{4k} \in H^{4k}(M)$. Just as the above theorem of Whitney we have also.

Theorem 2. *For a differential manifold M^n of dimension n to be differentially realizable in a euclidean N-space R^N, it is necessary that*

$$\bar{P}^{4k}(M^n) = 0, \qquad 2k > N - n. \tag{2}$$

Based on the same principle but applied to topological realization of topological spaces, Thom has derived from the study of Steenrod squares in X and those in its neighborhoods in R^N in which X is realized, the following

Theorem 3. *For a locally contractible compact Hausdorff space X to be topologically realizable in a euclidean N-space R^N it is necessary that*

$$Sm^i H^r(X, \bmod 2) = 0, \qquad 2i + r \geqslant N, \tag{3}$$

in which Sm^i: $H^r(X, \bmod 2) \to H^{r+i}(X, \bmod 2)$ are certain homomorphisms determined from the Steenrod squares by the relations

$$\sum_{i+j=k} Sm^i\ Sq^j = \begin{cases} 0, & k > 0, \\ \text{ident.}, & k = 0. \end{cases} \tag{4}$$

The same principle has also been applied to the differential realizability of differential manifolds e.g. by Massey who studies the cohomology ring of the tube about a manifold realized in a euclidean space and also by Atiyah who studies the above tube considered as defining an element in the Grothendieck ring of the manifold. It seems that much more can be expected along this line of thought.

Method 3. In 1932 van Kampen has published a highly interesting paper about linear realizations of complexes in a euclidean space based on principles entirely different from those of the preceding methods. Owing to its importance we shall describe it in some details.

It is well-known for some time that any simplicial n-complex K^n is linearly realizable in a euclidean $(2n+1)$-space. Van Kampen took up the problem of the linear realizability of K^n in a euclidean $2n$-space R^{2n} and proceeded as follows.

Let us try to realize linearly a finite simplicial n-complex K^n in R^{2n} as far as possible. In general we cannot get a true realization but we will do get a nearly true realization in the sense that it is one up to certain "singularities", finite in number, which are the self-intersections of two n-simplexes in K^n originally disjoint from each other. The problem becomes now: (i) From the study of such singularities can we derive certain invariants independent of the chosen nearly true realizations, and (ii) under what conditions can we remove such singularities to get a true realization?

For this purpose let us denote the k-dimensional simplexes of the given complex K^n by S_i^k. Any two simplexes of K^n with no vertices in common will be said to be disjoint. Let A be the set of all unordered index pairs (i, j), corresponding to pairs of disjoint n-dimensional simplexes S_i^n and S_j^n. Construct a vector space $\mathscr{L}$ on the ring of integers with dimension equal to the number of elements in A. Any vector of $\mathscr{L}$ may then be represented by a system of integers (a_{ij}) where $(i, j) \in A$. To each pair of disjoint simplexes S_a^{n-1} and S_l^n in K^n a certain vector $V_{la} = (a_{ij})$ of $\mathscr{L}$ may be determined in the following manner. If both $i, j \neq l$ or one of them, say $j = l$, but S_a^{n-1} is not a face of S_i^n, then we put $a_{ij} = 0$. Otherwise we put $a_{il} = \pm 1$ (with sign conventiently chosen). Two vectors P, P' of $\mathscr{L}$ will then be said to be equivalent if $P - P'$ is a certain linear combination with integer coefficients of vectors of form V_{la} above defined. The vectors of $\mathscr{L}$ are thus distributed in such equivalence classes. Any nearly true realization of K^n in R^{2n} as described above will determine now a vector $P = (a_{ij})$ of $\mathscr{L}$ for which a_{ij} is the duly signed intersection number of S_i^n and S_j^n in R^{2n} with a fixed orientation. Van Kampen's work shows that, whatever the nearly true realization of K^n in R^{2n} may be, the equivalence class of the corresponding vectors P will be one and the same and is therefore an invariant of the complex K^n, answering thus the question (i) above. Denote this invariant by $V(K)$, then it is clear that we have the following

Theorem 4. *For finite simplicial n-complex K^n to be linearly (or even semi-linearly) realizable in R^{2n} it is necessary that*

$$V(K^n) = 0. \tag{5}$$

(i.e., the equivalence class $V(K^n)$ contains the zero-vector of $\mathscr{L}$)

From the definition of $V(K^n)$ we see that the vector P associated to nearly true realization f of K^n in R^{2n} serves somewhat as a measure of the deviation of f from a true realization and it is natural to suspicion that the nullity of $V(K^n)$ would imply the possibility of removing the singularities so that condition (5) will be not only necessary, but also sufficient for K^n to be linearly or semi-linearly realizable in R^{2n}. Van Kampen does publish a proof in the case $n > 2$ but a fatal mistake in it destroys the proof and the question remains open until much later which we shall discuss below.

While van Kampen considered the problem of linear or semi-linear realization of complexes in a euclidean space, Whitney and Pontrjagin considered the problem of differential realization of differential manifolds in a euclidean space, based on somewhat the same method of "obstruction". Thus, by a theorem of Whitney early in 1936 a differential manifold M^n of dimension n is always differentially realizable in a euclidean $(2n+1)$-space. If we try to realize differentiably as far as possible M^n in a euclidean N-space of dimension $N \leqslant 2n$, then singularities will in general occur which, as shown by Pontrjagin, will carry certain cycles whose dual cohomology classes are independent of the nearly true realization reasonably chosen and have henceforth called *characteristic classes* of the manifolds, among which we have the Stiefel-Whitney classes and Pontrjagin classes already mentioned in Method 2. From this Theorems 1 and 2 follow also trivially. Moreover, in the case of $N = 2n$, the singularities of a reasonable nearly true realization of M^n in R^{2n} will consist of isolated points and device to remove such singular points has led Whitney to the proof of the following

Theorem 5. *Any differential n-manifold is differentiably realizable in a euclidean 2n-space.*

The device used above by Whitney, as pointed out by Shapiro and the author, may also be used to remove the singularities occured in a nearly true semi-linear realization of an n-complex in a R^{2n} for $n > 2$ under certain conditions, thus filling the gaps in the original proof of van Kampen and giving an answer to question (ii) cited above in the case considered:

Theorem 6. *A finite simplicial n-complex* K^n *of dimension* $n > 2$ *is semi-linearly realizable in* R^{2n} *if*

$$V(K^n) = 0.$$

It is to be remarked that the basic ideas originated by van Kampen and Whitney as described above has been greatly extended by Haefliger in the case of differential realizations of differential manifolds. It would be very interesting to know to what extent this theory of Haefliger of manifolds may be generalized to realization of polytopes or complexes.

§ 3. Method of this book

The method of treatment of the realization problem used in this book is based on the following observation. The (topological) realization of a space X in a space Y, say, requires the existence of a 1-1-map f of X in Y so that $f(X)$ is homeomorphic to X under f. Such a realization-map f is thus a "topological" continuous map of which a principal characteristic is being one-to-one. The problem is thus in fact a "topological" one in contrast to most problems current in algebraic topology which are mainly of a "homotopic" character so that the strong methods of algebraic topology, mainly of a "homotopic" character, may be more or less directly applied. To overcome the difficulties inherited in the realization problem we seek therefore to make apparent the 1-1-character of the map f and then apply the usual homotopic methods.

For this purpose we introduce the space $\tilde{X}^*$ (resp. X^*) consisting of all ordered (resp. unordered) pairs of points (x_1, x_2) (resp. $(x_1, x_2)^* = (x_2, x_1)^*$) of X, $x_1 \neq x_2 \in X$. Consider spaces $\tilde{Y}^*$ and Y^* derived from Y in a similar manner. A realization map $f: X \to Y$ induces then naturally realization-maps $\tilde{F}: \tilde{X}^* \to \tilde{Y}^*$ and $F: X^* \to Y^*$, where $\tilde{F}(x_1, x_2) = (f(x_1), f(x_2))$ and $F(x_1, x_2)^* = (f(x_1), f(x_2))^*$. Taking into account that the one-to-one-ness of the map f has already been applied in defining the very maps $\tilde{F}$ and F, we are naturally led to conjecture that the very *continuity* of the maps $\tilde{F}$ and F would imply already something more than the mere continuity of the map f and the "homotopic" methods applied to $\tilde{F}$ and F would give essential informations about the realizability of X in Y. It turns out that this is really the case.

In fact, denote the natural projections of $\tilde{X}^*$ onto X^* (resp. $\tilde{Y}^*$ onto Y^*) by π_X (resp. π_Y), then we have the following commutative diagram of *continuous* maps (the one-to-one-ness of maps $\tilde{F}$ and F being disregarded)

$$(*) \qquad \begin{array}{ccc} \tilde{X}^* & \xrightarrow{\tilde{F}} & \tilde{Y}^* \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X^* & \xrightarrow{F} & Y^* \end{array}$$

The problem is now: what consequences can we draw from such a commutative diagram?

For such a problem there are tools ready at hand. Clearly $\widetilde{X}^*$ is a 2-sheeted covering space over X^* with the covering transformation operating in $\widetilde{X}^*$ with no fixed points and with a period $= 2$. The well known theory of P. A. Smith shows that there may be associated to such a pair of spaces $(\widetilde{X}^*, X^*)$ a system of classes $A^m(\widetilde{X}^*, X^*) \in H^m(X^*, I_{(m)})$, $m \geqslant 0$, in which $I_{(m)}$ = group of integers or group of integers mod 2 according as m is even or odd. As $\widetilde{X}^*$, X^*, and π_X depend wholly on the space X, the classes $A^m(\widetilde{X}^*, X^*)$ are topological invariants of X itself and it is thus legitimate to set

$$\Phi^m(X) = A^m(\widetilde{X}^*, X^*) \in H^m(X^*, I_{(m)}).$$

It is easy to show that $\Phi^m(X) = 0$ would imply $\Phi^{m+i}(X) = 0$ for all $i \geqslant 0$ and we may introduce therefore $I(X)$ as the least integer n (eventually $+\infty$) for which $\Phi^n(X) = 0$. Define in a similar manner $\Phi^m(Y)$ and $I(Y)$. Then the theory of P. A. Smith shows that from the commutative diagram (*) we would get

$$F^* \Phi^m(Y) = \Phi^m(X).$$

We have therefore

Theorem I. *For a space X to be topologically realizable in a space Y, it is necessary that*

$$I(X) \leqslant I(Y).$$

In particular, for Y = a euclidean N-space R^N, it is easy to show that $I(R^N) = N$. We have therefore the following theorem as a corollary:

Theorem II. *For a space X to be topologically realizable in a euclidean N-space R^N, it is necessary that*

$$I(X) \leqslant N,$$

or equivalently,

$$\Phi^N(X) = 0.$$

These two theorems, which are immediate consequences of the theory of P. A. Smith form in fact the germ of our whole theory. What is remarkable is that, these theorems, trivial as they are, contain as particular cases the main results of Whitney, Thom, van Kampen, etc. as described in §2. More precisely, we have the following theorems (ρ_2

means reduction mod 2).

Theorem III. *If X is a locally contractible compact Hausdorff space with $\rho_2 \Phi^N(X) = 0$, then*

$$Sm^i H^r(X, \bmod 2) = 0, \qquad 2i + r \geqslant N.$$

Theorem IV. *If M^n is a differential n-manifold with $\rho_2\Phi^N(M^n)=0$, then*

$$\overline{W}^k(M^n) = 0, \qquad k \geqslant N - n.$$

Theorem V. *The van Kampen invariant $V(K)$ of a finite n-dimensional simplicial complex K^n, duly interpreted, is the class $\Phi^{2n}(|K^n|)$.*

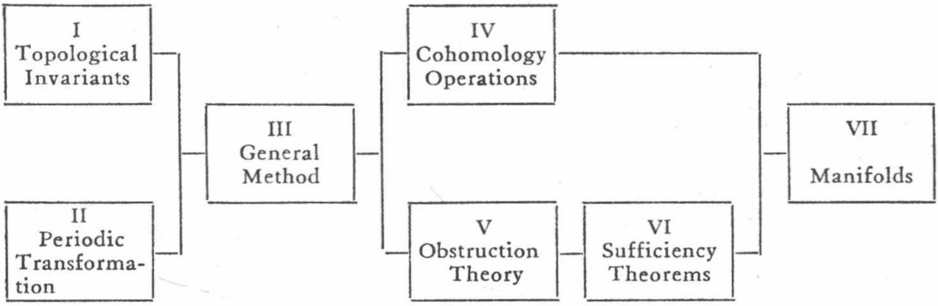
The last theorem shows that the class $\Phi^{2n}(|K^n|)$ of an n-complex K^n can be interpreted as certain obstructions to a linear or semi-linear realization of K^n in R^{2n}. The same is true for other classes $\Phi^m(|K^n|)$ and this gives rise to an obstruction theory of these Φ-classes. Such a theory has been developed by myself and also by Shapiro. In particular, the theorem 6 of §2 which is proved by van Kampen, with a certain gap filled by means of a method due to Whitney described at the end of §2, combined with theorem 4 of §2, becomes:

Theorem VI. *For a finite complex K^n of dimension $n > 2$ to be semi-linearly realizable in R^{2n} it is necessary and sufficient that $\Phi^{2n}(|K^n|) = 0$.*

It is to be remarked that the theory outlined above can be extended in various manners. First, not only realizations or imbeddings, but also local realizations or immersions and isotopies can be studied in the same manner for which results for example analogous to Theorem VI may be obtained. Secondly, the space $\widetilde{X}^*$ associated to a space X can be interpreted as the space $T(Z, X)$ of topological maps of a standard space Z consisting of two points only into the space X. Instead of Z and topological maps of Z in X we can use other spaces as standard spaces and other kinds of maps of Z in X more flexible than the topological maps. Thus, in the present book the case of $Z =$ a space consisting of p points with p any prime and the case of $Z =$ a disk will also be considered. Finally, a theory of second obstruction when certain $\Phi^m(X) = 0$ can be developed which has shown to be quite fruitful at least in the case of $X =$ differential manifolds. However, we are obliged to omit it in the present book since this theory is still in the course of developement cf. [71,71a,74,75].

§4. Structure of the book

The book is divided into seven chapters of which the logical connections may be seen from the following diagram:



Chapter III ocuppies a central position in the whole book. It describes the general method to be used, explains the fundamental concepts connected therewith, and presents the fundamental invariants, for the study of various problems, namely the $\tilde{\Phi}_p$-classes for imbeddings, the $\tilde{\Psi}_p$-classes for immersions, and the $\tilde{\Theta}$-classes for isotopies. Chapters I and II are of a preparatory character: the former shows the peculiar character, namely topological but non-homotopic one, of the fundamental invariants to be introduced in Chap. III and describes a general method in forming such kind of invariants; the latter presents the theory of P. A. Smith about periodic transformations without fixed points which furnishes the tools in forming these fundamental invariants in the case we are interested.

The last four chapters give mainly applications of the general method described in Chap. III. Chapter IV introduces the cohomology operations based again on the general theory of P. A. Smith about periodic transformations and gives conditions of imbedding and immersion in terms of such operations, proving thus in particular Th. III of §3 and Th. 3 of Thom in §2. Chapter V consists of an extension of van Kampen's obstruction theory to imbeddings, immersions as well as isotopies. Based on this obstruction theory it is proved in Chap. VI that the necessary conditions in the general case in terms of our funda-