Mathematics Monograph Series **36**

Generalized Inverses: Theory and Computations

(Second Edition)

Guorong Wang (王国荣) Yimin Wei (魏益民)
Sanzheng Qiao (乔三正)

（广义逆：理论与计算）

（第二版）

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Preface to Mathematics Monograph Series

Science Press asked me to write a preface for their series of books called "Mathematics Monograph Series". They told me that the Press had published nearly 30 mathematical monographs in this series since 2006. This reminded me that, also in 2006, I received an email message from the Editor in Chief of "Sugaku Tushin" ("Mathematical Communications", the membership magazine of the Mathematical Society of Japan). The Editor in Chief told me that they were planning to have a special section on "Recent development in Chinese mathematical community", and invited me to write an article for the special section. As a result, I published in their magazine an article called "Some Aspects of Mathematical Community in China". Among other things, in the article I demonstrated that, with the favorable environment, the Chinese mathematical community had made great progress since the late 1970s (when China started to implement "Reform and Opening-up Policy"): A large number of publications (including articles and monographs) have been written by Chinese mathematicians, there have been always Chinese mathematicians presenting their speeches at various international academic conferences or workshops, many Chinese mathematicians have served as editors of international academic journals, or as members in various academic organizations. All these reveal that the Chinese mathematical community which has been growing rapidly has exerted more and more influence in the world.

Indeed, the series of mathematical monographs published by Science Press reflects partly more and more influence of Chinese mathematical community in the world. Chinese people are good at mathematics. In the past, Science Press published many high level mathematical monographs and textbooks. Among them some were written in Chinese and some were written in English. Some monographs which appeared originally in Chinese have been purchased by international publishers who then re-published them abroad in English, and this has gained influence in corresponding areas of the international mathematical community.

In recent years most Chinese mathematicians have mastered good English. In accordance with this situation, Science Press has decided to publish "Mathematics Monograph Series" — a series in which high level mathematical monographs and textbooks are written directly in English. The goal of this series is to provide further good service for Chinese mathematicians and to enhance further the influence of the mathematics study in China in the international mathematical community.

I would like to conclude this short preface with the following wish which I expressed also at the end of the afore mentioned article "Some Aspects of Mathematical Community in China":

The Chinese mathematical community will continuously make its effort to work hard, and to strengthen its international exchanges and collaborations, so as to make more contributions to the study and development of mathematics in the world.

Zhi-Ming Ma
March 15, 2015

Preface to the Second Edition

Since the first publication of the book more than one decade ago, we have witnessed exciting developments in the study of the generalized inverses. We are encouraged by colleagues, Science Press, and Springer to update our book. This edition is the result of their encouragement. To include recent developments, this edition has two new chapters on the generalized inverses of special matrices and an updated bibliography. The new chapter six is about the generalized inverses of structured matrices, such as Toeplitz matrix and more general matrices of low displacement rank. It discusses the structure of the generalized inverses of structured matrices and presents efficient algorithms for computing the generalized inverses by exploiting the structure. The new chapter ten studies the generalized inverses of polynomial matrices, that is, matrices whose entries are polynomials. Remarks and references are updated to include recent publications. More than seventy publications are added to the bibliography.

To Science Press and Springer, we are grateful for their encouragement of the publication of this new edition. We would like to thank the reviewers for their constructive comments, which helped us improve the presentation and readability of the book.

Also, we would like to thank National Natural Science Foundation of China under grant 11171222 for supporting Wang Guorong, International Cooperation Project of Shanghai Municipal Science and Technology Commission under grant 16510711200 for supporting Yimin Wei and Sanzheng Qiao, National Natural Science Foundation of China under grant 11771099 and Key Laboratory of Mathematics for Nonlinear Science of Fudan University for supporting Yimin Wei, and Natural Science and Engineering Council of Canada under grant RGPIN-2014-04252 for supporting Sanzheng Qiao.

Shanghai, China — Guorong Wang
Shanghai, China — Yimin Wei
Hamilton, Canada — Sanzheng Qiao
November 2017 — Shanghai Normal University

Preface

The concept of the generalized inverses was first introduced by I. Fredholm [81] in 1903. He proposed a generalized inverse of an integral operator, called pseudoinverse. The generalized inverses of differential operators were brought up in D. Hilbert's [107] discussion of the generalized Green's functions in 1904. For a history of the generalized inverses of differential operators, the reader is referred to W. Reid's paper [189] in 1931.

The generalized inverse of a matrix was first introduced by E. H. Moore [166] in 1920, where a unique generalized inverse by means of projectors of matrices is defined. Little was done in the next 30 years until the mid-1950s, when the discoveries of the least-squares properties of certain generalized inverses and the relationship of the generalized inverses to solutions of linear systems brought new interests in the subject. In particular, R. Penrose [174] showed in 1955 that Moore's inverse is the unique matrix satisfying four matrix equations. This important discovery revived the study of the generalized inverses. In honor of Moore and Penrose's contribution, this unique generalized inverse is called the Moore–Penrose inverse.

The theory, applications, and computational methods for the generalized inverses have been developing rapidly during the last 50 years. One milestone is the publication of several books and monographs [9, 19, 92, 187] on the subject in 1970s. Particularly, the excellent volume by Ben-Israel and Greville [9] has made a long-lasting impact on the subject. The other milestone is the publications of the two volumes of proceedings. The first, edited by M. Z. Nashed, is the volume of the proceedings [167] of the Advanced Seminar on the Generalized Inverses and Applications held at the University of Wisconsin–Madison in 1973. It is an excellent and extensive survey book. It contains 14 survey papers on the theory, computations and applications of the generalized inverses, and a comprehensive bibliography that includes all related references up to 1975. The other, edited by S. L. Campbell, is the volume of the proceedings [18] of the AMS Regional Conference held in Columbia, South Carolina, in 1976. It is a new survey book consisting of 12 papers on the latest applications of the generalized inverses. The volume describes the developments in the research directions and the types of the generalized inverses since the mid-1970s. Prior to this period, due to the applications in statistics, research often centered in the generalized inverses for solving linear systems and the generalized inverses with the least-squares properties. Recent studies focus on such topics as: infinite-dimensional theory, numerical computation, matrices of special types (Boolean, integral), matrices over algebraic structures other than real or complex field, systems theory, and non-equation solving generalized inverses.

I have been teaching and conducting research in the generalized inverses of matrices since 1976. I gave a course "Generalized Inverses of Matrices" and held many seminars for graduate students majoring in Computational Mathematics in our department. Since 1979, my colleagues, graduated students, and I have obtained a number of results on the generalized inverses in the areas of perturbation theory, condition numbers, recursive algorithms, finite algorithms, embedding algorithms, parallel algorithms, the generalized inverses of rank-r modified matrices and Hessenberg matrices, extensions of the Cramer's rule, and the representation and approximation of the generalized inverses of linear operators. Dozens of papers have been published in refereed journals in China and other countries. They have drawn attention from researchers around the world. I have received letters from more than ten universities in eight countries, USA, Germany, Sweden, etc., requesting papers or seeking academic contacts. Colleagues in China show strong interests and support in our work and request a systematic presentation of our work. With the support of the Academia Sinica Publishing Foundation and the National Natural Science Foundation of China, Science Press published my book "Generalized Inverses of Matrices and Operators" [241] in Chinese in 1994. That book is noticed and well received by researchers and colleagues in China. It has been adopted by several universities as a textbook or reference book for graduate courses. The book was reprinted in 1998.

In order to improve graduate teaching and international academic exchange, I was encouraged to write this English version based on the Chinese version. This English version is not a direct translation of the Chinese version. In addition to the contents in the Chinese version, this book includes the contents from more than 100 papers since 1994. The final product is an entirely new book, while the spirit of the Chinese version still lives. For example, Sects. 2, 3, and 5 of Chap. 3; Sect. 1 of Chap. 6; Sects. 4 and 5 of Chap. 7; Sects. 1, 4, and 5 of Chap. 8, Chaps. 4, 10, and 11 are all new.

Yimin Wei of Fudan University in China and Qiao Sanzheng of McMaster University in Canada were two of my former excellent students. They have made many achievements in the area of the generalized inverses and are recognized internationally. I would not be able to finish this book without their cooperation.

We would like to thank A. Ben-Israel, Jianmin Miao of Rutgers University; R. E. Hartwig, S. L. Campbell, and C. D. Meyer, Jr. of North Carolina State University; and C. W. Groetsch of University of Cincinnati. The texts [9], [19], and [92] undoubtedly have had an influence on this book. We also thank Erxiong Jiang of Shanghai University, Zihao Cao of Fudan University, Musheng Wei of East China Normal University and Yonglin Chen of Nanjing Normal

University for their help and advice in the subject for many years, and my doctoral student Yaomin Yu for typing this book.

I would appreciate any comments and corrections from the readers.

Finally, I am indebted to the support by the Graduate Textbook Publishing Foundation of Shanghai Education Committee and Shanghai Normal University.

June 2003

Guorong Wang
Shanghai Normal University

Notations

Matrices: For the matrices A and B, and the indices α and β

I	Identity matrix
A^T	Transpose of A
A^*	Complex conjugate and transpose of A
$A^{\#}$	Weighted conjugate transpose of A
A^{-1}	Inverse of A
$A^{(1)}$	$\{1\}$-inverses of A
$A^{(1,3)}, A^{(1,3M)}$	$\{1,3\}$-, $\{1,3M\}$-inverses of A
$A^{(1,4)}, A^{(1,4N)}$	$\{1,4\}$-, $\{1,4N\}$-inverses of A
$A^{(1,2)}_{T,S}$	$\{1,2\}$-inverse of A with prescribed range T and null space S
$A^{(2)}_{T,S}$	$\{2\}$-inverse of A with prescribed range T and null space S
$A^{\dagger}$	Moore–Penrose inverse of A
$A^{\dagger}_{MN}$	Weighted Moore–Penrose inverse of A
A_d	Drazin inverse of A
A_g	Group inverse of A
$A_{d,W}$	W-weighted Drazin inverse of A
$A^{(-1)}_{(L)}$	Bott–Duffin inverse of A
$A^{(\dagger)}_{(L)}$	Generalized Bott–Duffin inverse of A
$A[\alpha,\beta]$ or $A_{\alpha\beta}$	Submatrix of A having row indices α and column indices β
$A[\alpha]$ or A_{α}	Submatrix $A_{\alpha\alpha}$ of A
$A[\alpha,*]$ or $A_{\alpha *}$	Submatrix of A consisting of the rows indexed by α
$A[*,\beta]$ or $A_{*\beta}$	Submatrix of A consisting of the columns indexed by β
$A[\alpha',\beta']$	Submatrix obtained from A by deleting the rows indexed by α and the columns indexed by β
$A[\alpha']$	Submatrix $A[\alpha',\alpha']$ of A
$\mathrm{adj}(A)$	Adjoint matrix of A

$C_k(A)$	kth compound matrix of A
$A(j \to \mathbf{b})$	Matrix obtained from A by replacing the jth column with the vector $\mathbf{b}$
$A(\mathbf{d}^{\mathrm{T}} \leftarrow i)$	Matrix obtained from A by replacing the ith row with the row vector $\mathbf{d}^{\mathrm{T}}$
$A \otimes B$	Kronecker product of A and B

Sets and Spaces: For the matrices A and B

$\mathcal{N}(A)$	Null space of A
$\mathcal{N}_c(A)$	Subspace complementary to $\mathcal{N}(A)$
$\mathcal{N}(A, B)$	Null space of (A, B)
$\mathcal{R}(A)$	Range of A
$\mathcal{R}_c(A)$	Subspace complementary to $R(A)$
$\mathcal{R}(A, B)$	Range of (A, B)
$\mathbb{R}$, $\mathbb{C}$	Fields of real, complex numbers
$\mathbb{R}^n$, $\mathbb{C}^n$	n-dimensional real, complex vector spaces
$\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$	Sets of $m \times n$ matrices over $\mathbb{R}$, $\mathbb{C}$
$\mathbb{R}_r^{m\times n}$, $\mathbb{C}_r^{m\times n}$	Sets of $m \times n$ matrices of rank r over $\mathbb{R}$, $\mathbb{C}$

Index sets: For $A \in \mathbb{R}_r^{m\times n}$, and the indices α and β

$Q_{k,n}$	$Q_{k,n} = \{\alpha :\ \alpha = (\alpha_1, \cdots, \alpha_k),\ 1 \le \alpha_1 < \cdots < \alpha_k \le n\}$
$\mathcal{I}(A)$	$\mathcal{I}(A) = \{I \in Q_{r,m} :\ \mathrm{rank}(A_{I*}) = r\}$
$\mathcal{J}(A)$	$\mathcal{J}(A) = \{J \in Q_{r,n} :\ \mathrm{rank}(A_{*J}) = r\}$
$\mathcal{B}(A)$	$\mathcal{B}(A) = \mathcal{I}(A) \times \mathcal{J}(A) = \{(I, J) \in Q_{r,m} \times Q_{r,n} :\ \mathrm{rank}(A_{IJ}) = r\}$
$\mathcal{I}(\alpha)$	$\mathcal{I}(\alpha) = \{I \in \mathcal{I}(A) :\ \alpha \subset J\}$
$\mathcal{J}(\beta)$	$\mathcal{J}(\beta) = \{J \in \mathcal{J}(A) :\ \beta \subset J\}$
$\mathcal{B}(\alpha, \beta)$	$\mathcal{B}(\alpha, \beta) = \mathcal{I}(\alpha) \times \mathcal{J}(\beta)$

Miscellaneous: For the matrix A

$\det(A)$	Determinant of A
$\frac{\partial}{\partial \lvert A_{\alpha\beta}\rvert} \lvert A\rvert$	The coefficient of $\det(A_{\alpha\beta})$ in the Laplace expansion of $\det(A)$
$\frac{\partial}{\partial a_{ij}} \lvert A\rvert$	Cofactor of a_{ij}
$\mathrm{Vol}(A)$	Volume of A, $\mathrm{Vol}(A) = \sqrt{\sum_{(I,J)\in\mathcal{N}(A)} det^2(A_{IJ})}$
$\mathrm{rank}(A)$	Rank of A
$\mathrm{null}(A)$	Nullity of A
$\mathrm{Ind}(A)$	Index of A
$\mathrm{tr}(A)$	Trace of A

$\lambda(A)$	Spectrum of A
$\sigma(A)$	Set of singular values of A
$\mu_{MN}(A)$	Set of weighted (M, N) singular values of A, where M and N are Hermitian positive definite matrices
$\rho(A)$	Spectral radius of A
$\kappa(A)$	Condition number with respect to the inverse of A
$\kappa_{MN}(A)$	Condition number with respect to the weighted Moore–Penrose inverse of A
$\kappa_2(A)$	Condition number with respect to the Moore–Penrose inverse of A
$\kappa_d(A)$	Condition number with respect to the Drazin inverse of A
$\dim(L)$	Dimension of a space L
$P_{L,M}$	Projector on a space L along a space M
P_L	Orthogonal projector on L along $L^\perp$
p.d.	Positive definite
L-p.d.	L-positive definite
p.s.d.	Positive semi-definite
L-p.s.d.	L-positive semi-definite
$\|\cdot\|_p$	ℓ_p-norm

Contents