Solvability of Abstract Evolution Equations with Nonlocal Conditions and Applications

（抽象发展方程非局部问题的可解性及其应用）

Chen Pengyu（陈鹏玉） Li Yongxiang（李永祥） Zhang Xuping（张旭萍）

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Preface

Nonlocal Cauchy problems of abstract evolution equations were motivated by physical problems. As a matter of fact, it is demonstrated that the abstract evolution equations with nonlocal initial conditions have a wide range of applications in real world applications and have better effects in applications than the classical Cauchy problems. For example, nonlocal problems are used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion. This book gives a systematic study to the existence, uniqueness, regularity and global asymptotic stability of solutions for several classes of nonlocal Cauchy problem of abstract evolution equations. All of these results are obtained by the authors in recent years.

This book is divided into seven chapters.

Chapter 1 gives a survey to the background, history and research development for nonlocal problem of abstract evolution equations, monotone iterative method based on lower and upper solutions, impulsive differential equations, fractional differential equations, non-autonomous evolution equations and nonlocal evolution equations with delay.

Chapter 2 briefly introduces some basic definitions, properties and theorems about operator semigroups, measure of noncompactness, fixed point theorems, cone and partial order as well as fractional integral and derivative.

In chapter 3, we use analytic semigroups theory, the gradually regularization method and relevant fixed point theorem to systematically study the existence, uniqueness and global asymptotic stability of strong solutions for evolution equations with nonlocal initial conditions in the frame of Hilbert space. In § 3.1, the existence and uniqueness of strong solutions are studied to a class of evolution equations with nonlocal initial conditions on the real compact interval $[0, a]$ for constant $a > 0$. Moreover, two examples are given to illustrate the application of the main results. § 3.2 is concerned with the existence and uniqueness of global strong solutions for a class of semilinear evolution equations with nonlocal initial conditions on infinite interval $[0, +\infty)$. The maximal regularity results of linear evolution equations with positive definite operator in Hilbert spaces and Ascoli-Arzela theorem on infinite interval play an important role in the proof of the main result. An example about parabolic partial differential equations with nonlocal conditions is also given to illustrate the feasibility of abstract result. § 3.3 deals with the global asymptotic stability of strong solutions for semilinear evolution equations with nonlocal initial conditions on infinite interval $[0, +\infty)$.

Chapter 4 builds monotone iterative methods for abstract evolution equations with nonlocal initial conditions as well as nonlocal Cauchy problems with instantaneous impulses and non-instantaneous impulses on ordered Banach spaces. § 4.1 constructs a monotone iterative method based on lower and upper solutions for abstract semilinear evolution equations with nonlocal initial conditions, and obtain the existence and uniqueness of mild solutions by applying the theory of semigroups of linear operators and the method of lower and upper solutions. Particularly, an existing result without using noncompactness measure condition is obtained in ordered and weakly sequentially complete Banach spaces, which is very convenient for application. § 4.2 deals with the existence of mild solutions for a class of semilinear nonlocal impulsive evolution equations on ordered Banach spaces. The existence and uniqueness theorem of mild solution for the associated linear nonlocal impulsive evolution equation is established. With the aid of this theorem, the existence of mild solutions for nonlinear nonlocal impulsive evolution equation is obtained by using perturbation method and monotone iterative technique. The theorems proved in this section improve and extend some related results in ordinary differential equations and partial differential equations. Meanwhile, two examples are presented to illustrate the feasibility of our abstract results. § 4.3 is concerned with the existence of mild solutions for a new class of abstract evolution equations with non-instantaneous impulses on ordered Banach spaces. The existence and uniqueness theorem of mild solution for the associated linear evolution equation with non-instantaneous impulses is established. With the aid of this theorem, the existence of mild solutions for nonlinear evolution equation with non-instantaneous impulses is obtained by using perturbation technique and iterative method under the situation that the corresponding solution semigroup $T(\cdot)$ and non-instantaneous impulsive function g_k are compact, $T(\cdot)$ is not compact and g_k is compact, $T(\cdot)$ and g_k are not compact, respectively. The results obtained in this chapter essentially improve and extend some related conclusions on this topic. Two concrete examples about parabolic partial differential equations with non-instantaneous impulses are given to illustrate that the abstract results are valuable.

Chapter 5 is devoted to study existence of mild solutions and positive mild solutions for semilinear fractional evolution equations with nonlocal initial conditions by using fixed point theory and monotone iterative method. § 5.1 investigates the existence of mild solutions for a class of semilinear fractional evolution equations with nonlocal initial conditions and noncompact semigroup in an arbitrary Banach space. We assume that the linear part generates an equicontinuous semigroup, and the nonlinear part satisfies noncompactness measure condition and growth condition. An example to illustrate the applications of the abstract result is also given. In § 5.2, a general class of semilinear fractional evolution equations of mixed type with nonlocal initial conditions on infinite dimensional Banach spaces is concerned. Under more general conditions, the existence of mild solutions and positive mild solutions are obtained by utilizing a new estimation technique of the measure of noncompactness and a new fixed point theorem with respect to convex-power condensing operator. §5.3 deals with the existence of mild solutions for a class of fractional evolution equations with mixed monotone nonlocal conditions. Under a new concept of coupled lower and upper L-quasi-solutions, we construct a new monotone iterative method for nonlocal problem of fractional evolution equations with mixed monotone nonlocal term and obtain the existence of coupled extremal mild L-quasi-solutions and the mild solution between them. The results obtained generalize the recent conclusions on this topic. Furthermore, two applications are given to illustrate the feasibility of the abstract results.

In chapter 6, we shall investigate the existence of mild solutions for a class of fractional non-autonomous integro-differential evolution equation of mixed type with nonlocal initial conditions and measure of noncompactness in infinite-dimensional Banach spaces. § 6.1 investigates the existence of mild solutions for a class of nonlinear time fractional non-autonomous integro-differential evolution equation of mixed type with nonlocal initial conditions and measure of noncompactness in Banach space. Combining the theory of fractional calculus and evolution families, the fixed point theorem with respect to convex-power condensing operator and a new estimation technique of the measure of noncompactness, we obtained the existence of mild solutions under the situation that the nonlinear term and nonlocal function satisfy some appropriate local growth conditions and a noncompactness measure condition. Our results generalize and improve some previous results on this topic, since the condition of uniformly continuity of the nonlinearity is not required, and also the strong restriction on the constants in the condition of noncompactness measure is completely deleted. As samples of applications for the abstract results obtained in §6.1, we consider a time fractional non-autonomous partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial condition in §6.2.

Chapter 7 is devoted to study existence and regularity results of mild solutions for several class of semilinear evolution equations subjected to nonlocal initial conditions by using tools involving the Kuratowski measure of noncompactness, the fraction power theory and α-norm as well as fixed point theory. In § 7.1, we establishes some existence results of mild solutions for semilinear evolution equations with nonlocal initial condition $u_0 = \phi + g(u)$ without the assumption of compactness on the associated semigroup. Moreover, we present an example to illustrate the application of the main results. In § 7.2, we establishes the existence and regularity results of mild solutions in the α-norm for the neutral delay evolution equations

$$\frac{d}{dt}[u(t) + G(t, u(t), u_t)] + Au(t) = F(t, u(t), u_t), \quad t \in [0, a]$$

subjected to a general mixed nonlocal plus local initial condition of the form $u(t) = g(u)(t) + \phi(t)$, $t \in [-r, 0]$. The results obtained can be applied to equations with terms involving spatial derivatives. Furthermore, an example to illustrate the application of the main results is also given. Under more general conditions, the existence of mild solutions and positive mild solutions for a class of fractional retarded evolution equations

$${}^C D_t^q u(t) + Au(t) = f\Big(t, u_t, \int_0^t h(t, s, u_s)ds\Big), \quad t \in [0, a]$$

subjected to a general mixed nonlocal plus local initial condition of the form $u(t) = g(u)(t) + \phi(t)$, $t \in [-r, 0]$ are obtained by means of fractional calculus and fixed point theory for condensing maps in § 7.3. Moreover, we present an example to illustrate the feasibility of the main results.

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Chen Pengyu Li Yongxiang Zhang Xuping

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Contents

Chapter 1

Introduction and Background

This Chapter gives a survey to the background, history and research development for nonlocal problem of abstract evolution equations, monotone iterative method based on lower and upper solutions, impulsive differential equations, fractional differential equations, non-autonomous evolution equations and nonlocal evolution equations with delay.

1.1 Nonlocal problem of abstract evolution equations

In 1991, Byszewski [1] firstly studied the existence of mild and strong solutions for a class of semilinear evolution equations with nonlocal initial condition in Banach space E

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in (t_0, t_0 + a], \tag{1.1.1}$$

$$u(t_0) + g(t_1, \cdots, t_p, u(\cdot)) = u_0, \tag{1.1.2}$$

where $a > 0$ is a constant, $0 \leqslant t_0 < t_1 < \cdots < t_p \leqslant t_0 + a$, $p \in \mathbb{N}$, $g \in C([t_0, t_0 + a] \times E, E)$ is a nonlocal function.

A strong motivation for investigating the nonlocal Cauchy problem (1.1.1)-(1.1.2) comes from physics. It is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. Nonlocal problem is used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion [2].

In 1993, Deng [3] studied the existence and uniqueness as well as exponential decay of the solutions for a class of reaction diffusion equations with nonlocal initial conditions, in which the nonlocal term is defined by

$$g(u) = \sum_{k=1}^{p} c_k u(t_k), \tag{1.1.3}$$

where $0 < t_1 < t_2 < \cdots < t_p < a$, c_k is a given constant. Deng use the studied problem to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, condition (1.1.3) allows the additional measurements at t_k, $k = 1, 2, \cdots, p$, which is more precise than the measurement just at $t = 0$.

In 1999, Byszewski [4] obtained the existence and uniqueness of classical solution to a class of abstract functional differential equations with nonlocal conditions of the form

$$u'(t) = f(t, u(t), u(\xi(t))), \quad t \in I = [t_0, t_0 + b], \tag{1.1.4}$$

$$u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = u_0, \tag{1.1.5}$$

where $b > 0$ is a constant, $t_0 < t_1 < \cdots < t_p \leqslant t_0 + b$, $f : I \times E \times E \to E$ and $\xi : I \to I$ are given functions satisfying some assumptions, E is a Banach space, $u_0 \in E$, $c_k \neq 0$ $(k = 1, 2, \cdots, p)$ and $p \in \mathbb{N}$. The author pointed out that if $c_k \neq 0$, $k = 1, 2, \cdots, p$, then the results of the paper can be applied to kinematics to determine the location evolution $t \to u(t)$ of a physical object for which we do not know the positions $u(0), u(t_1), \cdots, u(t_p)$, but we know that the nonlocal condition (1.1.5) holds. Consequently, to describe some physical phenomena, the nonlocal condition can be more useful than the standard initial condition $u(0) = u_0$.

Nonlocal initial condition $u(0) = g(u)$ contains the following situations:

(i) $g(u) = u(\omega)$ (periodicity condition);

(ii) $g(u) = -u(\omega)$ (anti-periodicity condition);

(iii) $g(u) = \int_0^a h(s, u(s))ds$ (integral condition);

(iv) $g(u) = \int_0^a k(s)u(s)ds$, where $k \in L([0, a], [0, +\infty))$, $\int_0^a k(s)ds = 1$ (mean value condition);

(v) $g(u) = \sum\limits_{k=1}^{p} c_k u^{\frac{1}{3}}(t_k)$ where $\sum\limits_{k=1}^{p} |c_k| \leqslant 1$ (multi-point discrete mean condition).

In recent years, the existence, uniqueness and stability for abstract evolution equations with nonlocal initial conditions have been studied extensively, see [1,3-45] and the references therein for more comments and citations. In 1997, Ntouyas and Tsamatos [5] studied the existence of mild solutions for the following nonlocal evolution equations

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & t \in [0, a], \\ u(0) + g(u) = u_0, \end{cases} \tag{1.1.6}$$

where $-A$ generates a compact C_0-semigroup $T(t)(t \geqslant 0)$.

However, as it is shown by Ezzinbi and Liu [25], Liu [32] and Aizicovici and McKibben [45], the proof of the main result in [5] did not work because the most important place $t = 0$ was neglected when checking the compactness of the solution operator. In fact, the solution operator of nonlocal evolution equation (1.1.6) is not compact in the point $t = 0$ due to the influence of nonlocal function g. In order to fill this gap, some authors added a condition on the compactness to the nonlocal function g. But referring to possible application to physics, this condition is too strong. Noticed that in some articles, the nonlocal term is given by

$$g(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)). \tag{1.1.7}$$

Especially, in many works, the nonlocal function g is defined by (1.1.3). In above two cases, the nonlocal function g is completely determined by the value in $[t_1, a]$, at this time, $t = 0$ can be neglected. Therefore, Liang, Liu and Xiao [28] introduced the following condition

(F1) There exists a constant $\delta > 0$ such that $g(u) = g(v)$ for any u, $v \in Y_r$ with $u(s) = v(s)$, $s \in [\delta, a]$, where $Y_r = \{u \in C([0, a], E) : \|u(t)\| \leqslant r,\ t \in [0, a]\}$.

The condition (F1) contains the situations (1.1.3) and (1.1.7). (F1) means that the nonlocal function g depends only on the value of u in the interval $[\delta, a]$ for some constant $\delta > 0$. In this way, the point $t = 0$ is not needed to be considered. Therefore, a nature problem is:

Q 1.1.1 If nonlocal function g is not completely continuous and g depends on the value of u in the whole interval $[0, a]$, then does the problem (1.1.6) have a solution?

We observe that in most of the existing articles, the existence of mild solutions for evolution equations with nonlocal initial conditions have been studied extensively, but there are very few paper studied the regularity for nonlocal evolution equations. In 1998, under the situation that suppose the nonlinear term f is continuously differentiable and the nonlocal operator satisfy some very strong regularity conditions, Byszewski [10] obtained the existence of classical solutions for a class of semilinear evolution equations with nonlocal initial conditions of the form

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), \quad t \in (0, a], \\ u(0) = \displaystyle\sum_{k=1}^{p} c_k u(t_k) \end{cases} \tag{1.1.8}$$

in reflexive Banach space E, where $-A$ is the infinitesimal generator of a C_0-semigroup of operators on E, $a > 0$ is a constant, $0 < t_1 < \cdots < t_p \leqslant a$, $c_k \neq 0$ for $k = 1, 2, \cdots, p$ and $p \in \mathbb{N}$.

Recently, Xiao and Liang [42] studied a class of nonautonomous evolution equations with nonlocal initial conditions of the form

$$\begin{cases} u'(t) = A(t)u(t) + f(t, u(t)), & t \in (s, T], \\ u(s) + g(u) = u_0 \end{cases} \tag{1.1.9}$$

in Banach space E, where $T > s \geqslant 0$, $A(t)$ is a sectorial operator in E for each $t \in [0, T]$. The authors obtained the existence of classical solutions by using Banach's contraction principle and Schauder's fixed point theorem, as well as the theory of evolution families and interpolation spaces.

In [4, 10, 42], the regularity of solutions for nonlocal evolution equations have been studied under some strong assumptions. Moreover, to the best of the authors' knowledge, all the existing articles are only devoted to study the local regularity of solutions for evolution equations with nonlocal conditions on compact interval, but so far there is no result yet existing on the global regularity of solutions for evolution equations with nonlocal initial conditions on infinite interval. Therefore, our problems are:

Q 1.1.2 Dose the nonlocal problem (1.1.8) have a strong solution on compact interval under more weaker conditions?

Q 1.1.3 Can we obtain the global existence and uniqueness of strong solutions for a class semilinear evolution equations with nonlocal initial value conditions on infinite interval $[0, +\infty)$?

The dynamical characteristics (including stable, unstable, attract, oscillatory and chaotic behavior) of differential equations have become a subject of intense research activities. For the details of this field, we refer the reader to the monographs of Burton [46], Hale [47] and the papers of Caicedo et al. [48], Chen and Guo [49], Li and Wang [50], Wang, Liu and Liu [51], Zhu, Liu and Li [52]. As far as we know, no work has been done for the asymptotic stability of strong solutions for nonlocal evolution equations. Motivated by the above-mentioned aspects, an interesting problem is:

Q 1.1.4 Can we obtain the uniqueness and global asymptotic stability of strong solutions for a class of semilinear evolution equations with nonlocal initial conditions on half line $\mathbb{R}_+ = [0, +\infty)$?

1.2 Monotone iterative method based on lower and upper solutions

The monotone iterative method based on lower and upper solutions is an effective and flexible mechanism. The most advantage by using the iterative method based on lower and upper solutions is that it not only provides a method to obtain the existence of extremal solutions, but also yields iterative sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. The iterative sequences are very useful in numerical calculation, which provide a computing rule in computer simulation. As early as 1976, Amann [53] established the lower and upper solutions theorem for operator equation in ordered Banach spaces. In 1982, Du and Lakshmikantham [54] built a monotone iterative method for the initial value problem of the ordinary differential equation in Banach space E

$$\begin{cases} u'(t) = f(t, u(t)), & t \in J, \\ u(0) = x_0, \end{cases} \tag{1.2.1}$$

they proved that if IVP (1.2.1) has a lower solution v_0 and an upper solution w_0 with $v_0 \leqslant w_0$, and nonlinear term f satisfies the monotonicity condition

$$f(t, u_2) - f(t, u_1) \geqslant -M(u_2 - u_1), \quad \forall\, t \in J,\ v_0(t) \leqslant u_1 \leqslant u_2 \leqslant w_0(t) \tag{1.2.2}$$

with a positive constant M, and the noncompactness measure condition

$$\alpha(f(t, D)) \leqslant L\alpha(D), \quad \forall\, t \in J, \quad \text{bounded } D \subset E, \tag{1.2.3}$$

where $L > 0$ is a constant, $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness in E, then IVP (1.2.1) has minimal and maximal solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively. Later, when E is weakly sequentially completed, Sun [55] improved the result of Du and Lakshmikantham [54], and deleted the noncompactness measure condition (1.2.3).

In 1993, Guo and Liu [56] obtained the existence of extremal solutions for the initial value problem (IVP) of impulsive differential equations

$$\begin{cases} u'(t) = f(t, u(t)), & t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \cdots, m, \\ u(0) = u_0, \end{cases} \tag{1.2.4}$$

under condition (1.2.2) for the nonlinear term f and monotoneity condition

$$I_k(u_1) \leqslant I_k(u_2), \quad k = 1, 2, \cdots, m, \ \forall\, t \in J, \ v_0(t) \leqslant u_1 \leqslant u_2 \leqslant w_0(t) \tag{1.2.5}$$

for impulsive function I_k. They also demand that the nonlinear term f and impulsive function I_k satisfy the noncompactness condition (1.2.3) and the following condition

$$\alpha(I_k(U)) \leqslant L_k \alpha(U), \qquad \text{bounded } U \subset E, \tag{1.2.6}$$

where L_k are positive constants, and satisfy

$$2a(M + L) + \sum_{k=1}^{m} L_k < 1. \tag{1.2.7}$$

One can easily see that the inequality (1.2.7) is a strongly restricted condition, and it is not easy to satisfy the condition (1.2.7) in applications. Latter, Li and Liu [57], Chen, Zhang and Li [61] developed the iterative method for ordinary differential equations with instantaneous impulses in Banach spaces, and deleted the restriction condition (1.2.7) by adopting a method of piecewise argument, and then largely improved the results in [56].

In recent years, the monotone iterative method has been extended to evolution equations in ordered Banach spaces. In [58] and [59], Li established a monotone iterative method for initial value problem of evolution equations without impulse in an ordered Banach space E

$$\begin{cases} u'(t) + Au(t) = \ f(t, u(t)), \quad t \in [0, T], \\ u(0) = x_0. \end{cases} \tag{1.2.8}$$

By applying the operators semigroups theory and the method of upper and lower solutions, the author obtained the existence of mild solution for IVP (1.2.8). When many partial differential equations involving time-variable t turn to evolution equations in Banach spaces, they always generate an unbounded closed operator term A, see [60]. A is corresponding to linear partial differential operator with certain boundary conditions. In this case, the results in [54, 55] are not suitable, and the method for ordinary differential equations can not be applied to the evolution equations. Therefore, the work of Li [58, 59] is applicable. We refer to the papers by Wang and Wang [62] and EI-Gebeily, O'Regan and Nieto [63] for evolution equations with classical initial vlaue conditions, and to the papers by Chen and Li [65], Chen, Li and Yang [16] and Chen and Li [18] for evolution equations with impulses in ordered Banach spaces.