普通高等教育“十三五”规划教材
普通高等院校数学精品教材

Linear Algebra

毛纲源　马迎秋　梁　敏　编著

华中科技大学出版社
中国·武汉

丛书编委会

主　编　米洪海　毛纲源

副主编　孙洪波　侯　健　高　洁

编　委　（按姓氏笔画排序）

马迎秋　毛纲源　米洪海

孙洪波　周海婴　侯　健

徐丽莉　高　洁　梁　敏

Preface

The authors are pleased to see the text of Linear Algebra in English version for Chinese students at the university level. This book not only shows and explains the useful and beautiful knowledge of mathematics, but also presents the structure and arrangement of linear algebra.

1. The Significance of this Book

"One sows a seed in the spring, thousands of grains autumn to him brings." All the Chinese students had strict training step by step in the study of mathematics before they become a university student. Intuitive and experimental methods are basic and important study patterns, but the target of mathematical education is to form and improve the deductive ability. So far, Chinese students have distinct and excellent achievement in international comparison of mathematics all over the world. As the improvement in educational exchange internationally, more and more Chinese students choose to study abroad at their university level or higher level. Therefore, mathematical textbook on the basis of Chinese students' mathematical study in English version is urgent needed and essential. This book provides the strong support for the students who will study Economics, Finance, Management, Social Science and so on in local country or abroad.

2. The Difference between Linear Algebra and Calculus

Calculus is mostly about symmetric and beautiful things. One is differentiation, another is its inverse—integration. Calculus can help us to solve the problems in continuous and analog situation in our life. How about other discrete and digital things? Linear algebra can give us help, and vector and matrix are the second type of language we need to study and understand. Study to read a matrix is the most meaningful and key goal in linear algebra, and it gives wide variety for this mathematical area. There are three examples given here:

Triangular Matrix	Symmetric Matrix	Orthogonal Matrix
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -5 & 6 \\ 3 & -5 & 1 & -7 \\ 4 & 6 & -7 & 1 \end{bmatrix}$	$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

3. The Structure of this Book

This book organizes the content basis on the logical relationship among number, matrix and vector. It lists the structure from determinant, to matrix, to solve system of linear equations, to vector, to structure of solutions, to eigenvalue and eigenvector, to quadratic form finally.

Here is the structure of this book:

Chapter 1 starts with determinant. There are three important points about the determinant. The first is the definition, the second is property, and the last is its expansion. The Cramer's rule is given basis on these three points.

Chapter 2 gives all the varieties of matrix. After the study of concept of matrix, it begins with algebra operations, and shows some special matrices. It is following with how to partition matrix, and how to find the inverse of matrix. After given the elementary operations and elementary matrix, this chapter is ended by rank of matrix.

Chapter 3 shows the relationship between matrix and the system of linear equations. Certainly, it is the most important that using matrix to solve the system of linear equations. Gaussian Elimination Method is the most helpful technique.

Chapter 4 begins studying vector. Definition and operation are two basic study points. Linear dependence and rank of vector are two new knowledge structures.

Chapter 5 is mainly basis on chapter 3 and chapter 4. Here is similar framework for giving the structure of solutions of homogeneous and nonhomogeneous system of linear equations. Both these two parts discuss the corresponding property firstly, and give the details of their structure respectively.

Chapter 6 is mostly in eigenvalue and eigenvector. Besides the definition of them, there are three points of matrix using both two of them which are diagonalization, similar matrix and real symmetric matrix.

Chapter 7 is quadratic form which has three points. The first point is about the definition. The first is the basic, almost, which is the principal and organization order of studying mathematical knowledge. The second is the classification of quadratic form and positive definite matrix. The last is criterion of congruent matrix.

4. Help with this Book

"Not knowing that flower close to the water earlier blow, I wonder if it's last winter's unmelted snow." This textbook is emerged with the strong support from Applied Mathematical Department of BNUZ firstly, and the cooperation of senior professor and junior lecture in warm, selfless and enthusiastic environment. Certainly, it has very close relationship with the developing and open international education in BNUZ. Thank you all.

Authors

October 10,2017

Contents

Chapter 1 Determinant

Linear algebra is an important branch of mathematics, in which system of linear equations and their solutions constitute its major topics. In linear algebra, the notion of determinants is fundamental. Theory of determinant is established to satisfy the need for solving system of linear equations. In mathematics and many practical area determinants are very useful (for economics, physics, etc.). By using of determinant we can deduce the well-known Cramer's rule, moreover many other important concepts can be established and introduced.

The present chapter will mainly deal with the following three problems:

(1) The definition of determinant.

(2) Some important properties of determinant and basic methods of evaluating determinant.

(3) Solving a particular system of linear equations by using determinant as application of determinant.

1.1 Definition of Determinant

1.1.1 Determinant arising from the solution of linear system

In high school, we have already learned how to solve a system of linear equations in 2 or 3 unknowns by using determinant of order 2 or 3 respectively. We recall the method studied in high school.

First of all, let us solve the following system of linear equations in 2 unknowns x_1 and x_2:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, & ① \\ a_{21}x_1 + a_{22}x_2 = b_2, & ② \end{cases} \tag{1.1.1}$$

where b_1 and b_2 are constant terms and $a_{ij}(i,j=1,2)$ is called the coefficient of $x_j(j=1,2)$ in the ith equation. In order to eliminate x_2 from the system (1.1.1) of linear equations. Multiplying the equation ① by a_{22} at the two sides of the equation ①, multiplying the equation ② by a_{12} at the two sides of the equation ②, and then subtracting the latter from the former, we obtain

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - a_{12}b_2.$$

Similarly, eliminating x_1, we get

$$(a_{11}a_{22}-a_{12}a_{21})x_2=b_2a_{11}-b_1a_{21}.$$

Therefore, when $D=a_{11}a_{22}-a_{12}a_{21}\neq 0$, we have

$$x_1=\frac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}},\quad x_2=\frac{b_2a_{11}-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}. \qquad ③$$

That is to say, if $D=a_{11}a_{22}-a_{12}a_{21}\neq 0$, then the system (1. 1. 1) of linear equations must have a solution and its unique solution must be the equation ③.

To facilitate memorization we introduce the notation

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}, \qquad ④$$

and call it a determinant of order 2. It consists of two (horizontal) rows and two (vertical) columns.

The numbers appearing in the determinant D of order 2 are called its elements or its entries.

From the above formula ④, we easily see that the value of a determinant of order 2 is the algebraic sum of 2 terms: by forming the product of the entries on the diagonal from the left upper corner to the right lower corner (the line is called the principal diagonal of the determinant D) and subtracting from this number the product of the entries on the line from right upper corner to left lower corner:

Thus, if $D=\begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$, then

$$D=\begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}=2\times 5-(-3)\times 4=22.$$

We can now represent the solution by a fashion of determinants:

$$x_1=\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}\Big/\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\quad x_2=\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}\Big/\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Denoting

$$D=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\quad D_1=\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix},\quad D_2=\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix},$$

we have $x_1=\frac{D_1}{D}$, $x_2=\frac{D_2}{D}$, where D is called the coefficient determinant of the system (1. 1. 1). The solution is valid only if $D\neq 0$, so D determines whether the system (1. 1. 1) have a unique solution. This is the reason of D being called a determinant.

Let us then solve the following system of linear equations in 3 unknowns:

$$\begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1, \\ a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2, \\ a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3. \end{cases} \tag{1.1.2}$$

Analogous to the above, eliminating x_3 from the first two equations and from the last two equations gives rise to a new system of two linear equations in x_1, x_2, the solution of which may be obtained as before, and x_3 is then obtained by substitution in any of the given equations. By some rather lengthy calculations, we finally get

$$\begin{cases} x_1=\frac{1}{D}(b_1a_{22}a_{33}+a_{12}a_{23}b_3+a_{13}b_2a_{32}-b_1a_{23}a_{32}-a_{12}b_2a_{33}-a_{13}a_{22}b_3), \\ x_2=\frac{1}{D}(a_{11}b_2a_{33}+b_1a_{23}a_{31}+a_{13}a_{21}b_3-a_{11}a_{23}b_3-b_1a_{21}a_{33}-a_{13}b_2a_{31}), \\ x_3=\frac{1}{D}(a_{11}a_{22}b_3+a_{12}b_2a_{31}+b_1a_{21}a_{32}-a_{11}b_2a_{32}-a_{12}a_{21}b_3-b_1a_{21}a_{32}). \end{cases} \quad ⑤$$

Let

$$D=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31},$$

if $D\neq 0$, then ⑤ is the actually unique solution of system (1.1.2).

Analogous to the case $n=2$, we define a determinant of order 3:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}. \tag{1.1.3}$$

Which is the algebraic sum of 6 terms. Each term is a product of three entries which are in different rows and different columns of the determinant D.

The above diagram will help us to remember how to evaluate the value of a determinant of 3 order: As in the above figure, we add up the products of the three elements situated on each solid line with a positive sign and those on each dotted line with a negative sign.

Example 1.1.1 Let $D=\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix}$, evaluate D.

Solution Substitute in the formula (1.1.3), we find that

$$D=1\cdot 1\cdot 2+2\cdot 3\cdot 3+3\cdot 2\cdot 1-1\cdot 3\cdot 1-2\cdot 2\cdot 2-3\cdot 1\cdot 3=6.$$

In the expression ⑤, the common denominator of x_1, x_2, x_3 is D, while the numerators are respectively the determinants:

$$D_1=\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},\quad D_2=\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},\quad D_3=\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

which may be obtained by replacing the first, the second and the third column in D by column formed by b_1, b_2, b_3 respectively.

Thus, the expression ⑤ may be simply written as

$$x_1=\frac{D_1}{D},\quad x_2=\frac{D_2}{D},\quad x_3=\frac{D_3}{D}.$$

The structure of which is quite similar to the case of $n=2$.

The above two examples illustrate that determinant arose first in the solution of linear systems.

It is worthy pointing out that the methods given for evaluating determinant D do not apply for $n\geqslant 4$.

We have rather simple expressions for the solutions of system (1. 1. 1) in terms of determinants of order 2 respectively under the assumption $D\neq 0$. It is natural to ask whether we could express the solution of a system of linear equations in n unknowns in terms of determinants of order n using a certain appropriate definition for such determinants. We easily know that it is very easy to eliminate one unknown in a system of 2 equations in 2 unknowns, but it is rather lengthy to eliminate $n-1$ unknowns in a system of n equations in n unknowns and then to define determinant of order n, then the procedure would be very complicated and tedious, and even impossible for larger n. In order to give such a definition, we are forced to find an alternative way. However, we may get some in separations from the structures of determinants of order 2 and 3, which be useful in the formulation of a definition of determinant of an arbitrary order.

The next result gives us a similar formula of the solution for the following system of n linear equations in n unknown numbers:

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1, \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2, \\ \qquad\qquad\vdots \\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=b_n. \end{cases}$$

Its coefficient determinant is

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Let

$$D_1=\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix},\cdots,D_j=\begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \quad (j=1,2,\cdots,n),$$

then

$$x_1=\frac{D_1}{D}, \quad \cdots, \quad x_j=\frac{D_j}{D}, \quad \cdots, \quad x_n=\frac{D_n}{D}.$$

1.1.2 The definition of determinant of order n

Definition 1.1.1 Let $S=\{1,2,\cdots,n\}$ be the set of integers from 1 to n, arranged in ascending order. A rearrangement $i_1, i_2, \cdots, i_n$ of the elements of S is called a permutation of S.

For example, let $S=\{1,2,3,4,5\}$, then $4,1,2,3,5$ is a permutation of S. It correspond to the function $f: S \to S$ defined by

$$f(1)=4, \quad f(2)=1, \quad f(3)=2, \quad f(4)=3, \quad f(5)=5.$$

We can put any one of the n elements of S in the first position. Any one of the remaining $n-1$ elements in the second position, any one of the remaining $n-2$ elements in third position, and so on, until the nth position can only be filled by the last remaining element. Thus there are

$$n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1=n!$$

permutations of S, we denote the set of all permutations of S by S_n. Therefore S_n consists of $n!$ permutations of the set $S=\{1,2,\cdots,n\}$.

For example, S_1 consists of only $1!=1$ permutation of the set $\{1\}$, namely, 1;

S_2 consists of $2!=1\cdot 2=2$ permutations of the set $\{1,2\}$, namely, 12 and 21;

S_3 consists of $3!=1\cdot 2\cdot 3=6$ permutations of the set $\{1,2,3\}$, namely, 123, 132, 213, 231, 312 and 321.

Definition 1.1.2 A permutation $i_1 i_2 \cdots i_n$ of $S=\{1,2,\cdots,n\}$ is said to have an inversion if a larger integer i_t precedes a smaller one i_s ($i_t > i_s$). We call the two elements i_t and i_s constituting a permutation.

Definition 1.1.3 The total numbers of inversion in a permutation $i_1 i_2 \cdots i_n$ is called the number of inversion in the permutation, denoting by $N(i_1\, i_2 \cdots i_n)$. The permutation is called even or odd permutation according to whether the total number of inversion in it is even or odd.

Example 1.1.2 In the following permutations of $S_3=\{1,2,3\}$, which is even permutation? which is odd permutation?

$$123,\quad 231,\quad 312,\quad 132,\quad 213,\quad 321.$$

Solution The even permutations of S_3 are 123 (no inversion, $N(123)=0$); 231 (two inversions: 21 and 31; $N(231)=2$); 312 (two inversions: 31 and 32; $N(312)=2$).

The odd permutations of S_3 are 132 (one inversion: 32; $N(132)=1$); 213 (one inversion: 21; $N(213)=1$); and 321 (three inversions: 32, 31 and 21; $N(321)=3$).

In general, we have the following proposition.

Proposition 1.1.1 If there are r pairs of (j_p, j_q) such that $j_p<j_q$ whenever $p>q$, then

$$N(j_1j_2\cdots j_n)=r.$$

Definition 1.1.4 If we exchange the position of two elements (numbers) in the permutation $(j_1j_2\cdots j_n)$ and keep the other numbers fixed, we get another permutation. This procedure of the change is called a transposition.

We have the following basic result.

Theorem 1.1.1 A transposition changes the sign of a permutation.

Proof Firstly, we consider a special case that two numbers exchanged are adjacent. For instance,

$$Aj_kj_lB\rightarrow Aj_lj_kB,$$

where A, B are all the other elements except the elements j_k and j_l, comparing the numbers of inversions in the two permutations. Obviously, only one order of elements j_kj_l is changed to j_lj_k and all the other order of elements keep unchanged. Therefore the number of inversion in new permutation increase one number than original permutation $(j_l>j_k)$, or decrease one number than original permutation $(j_l<j_k)$. Hence, $(-1)^{N(j_1j_2\cdots j_kj_l\cdots j_n)}=-(-1)^{N(j_1j_2\cdots j_lj_k\cdots j_n)}$.

If j_k and j_l are not adjacent, suppose that there are m numbers between them. We may move j_k to the right by m times of adjacent transposition so that j_k and j_l become adjacent. Then we exchange j_k and j_l. Lastly we move j_l to the left by m times of adjacent transposition. Therefore exchange of j_k and j_l may be accomplished by $2m+1$ adjacent transpositions. It follows that

$$(-1)^{N(j_1j_2\cdots j_k\cdots j_l\cdots j_n)}=(-1)^{2m+1}(-1)^{N(j_1j_2\cdots j_l\cdots j_k\cdots j_n)}=-(-1)^{N(j_1j_2\cdots j_l\cdots j_k\cdots j_n)}.$$

Example 1.1.3 Find the number of inversions in the permutation 54132 and 52134.

Solution The number of inversions in the permutation 54132 is 8. The number of inversions in the permutation 52134 is 5. The permutation 52134 is obtained from 54132 by interchanging 2 and 4. The number of inversion differs by 3, and odd number. Therefore we have

$$(-1)^{N(54132)}=(-1)(-1)^{N(52134)}.$$

Definition 1.1.5 Given n^2 numbers $a_{ij}(i,j=1,2,\cdots,n)$. Arrange them as an array with n rows and n columns:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

calling it a determinant of order n, where a_{ij} is called its element or number situated at ith row and jth column, and defined as its value of the algebraic sum

$$\sum(-1)^{N(j_1j_2\cdots j_n)}a_{1j_1}a_{2j_2}\cdots a_{nj_n}=\sum(\pm 1)a_{1j_1}a_{2j_2}\cdots a_{nj_n},$$

where the summation ranges over all permutations $j_1j_2\cdots j_n$ of the set $S=\{1,2,\cdots,n\}$. The sign is taken as $+$ or $-$ according to whether the permutation $j_1j_2\cdots j_n$ is even or odd. The sign $+$ or $-$ is also determined according to the chess-board pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ - & + & - & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}.$$

In each term $(\pm)a_{1j_1}a_{2j_2}\cdots a_{nj_n}$ of the above determinant of order n, the row subscripts are in their natural order, whereas the column subscripts are in the order $j_1j_2\cdots j_n$. Since we sum over all the permutations of the set $S=\{1,2,\cdots,n\}$, this determinant of order n has $n!$ terms in the sum.

Example 1.1.4 Compute the following determinant of order n:

$$D=\begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix},$$

where $a_{ii}\neq 0\ (i=1,2,\cdots,n)$.

Solution Let $D=|a_{ij}|_{n\times n}$, then $a_{ij}=0$ for $i<j$. Therefore a term $a_{1j_1}a_{2j_2}\cdots a_{nj_n}$ in the expression for $D=|a_{ij}|_{n\times n}$ can be nonzero only for $j_1\leqslant 1, j_2\leqslant 2, j_3\leqslant 3,\cdots, j_n\leqslant n$. Since element a_{1j_1} in D is taken at the first row, whereas in the first row in D

nonzero element only has a_{11}, so $j_1=1$. Again since $j_1 j_2 \cdots j_n$ must be a permutation of $\{1,2,\cdots,n\}$, hence j_2 in a_{2j_2} is equal to 2.

Similarly, $j_3=3,\cdots,j_n=n$. Thus the only term of D that can be nonzero is the product of the elements on the main diagonal of D. Since the permutation $12\cdots n$ has no inversion, the sign associated with it is $+$, therefore $D=a_{11}a_{22}\cdots a_{nn}$. We call the determined D lower triangular determinant. In general, we have

Definition 1.1.6 (1) A determinant $D=|a_{ij}|_{n\times n}$ is called upper triangular determinant if $a_{ij}=0$ for $i>j$ (the elements below the main diagonal are zeros).

(2) It is called lower triangular determinant if $a_{ij}=0$ for $i<j$ (the elements above the main diagonal are zeros).

(3) Specially, it is called diagonal determinant if $a_{ij}=0$ for $i\neq j$ (the elements below and upper the main diagonal are zeros).

The values of determinants of the above three types are equal to the product of elements on the main diagonal, i,e. ,$D=a_{11}a_{22}\cdots a_{nn}$.

1.1.3 Determine the sign of each term in a determinant

In general we determine the sign of each term in $D=|a_{ij}|_{n\times n}$ by the following three methods.

(Ⅰ) Using the definition of determinant to determine the sign of each term in D, i. e. , the sign is determined by the odevity of the numbers of inversions.

Example 1.1.5 Evaluate determinant of order n $D_n=\begin{vmatrix} a & & 1 \\ & \ddots & \\ 1 & & a \end{vmatrix}$ (the unwritten elements being all zeros).

Solution By the definition of determinant, in D_n where are two nonzero terms. We have

$$\begin{aligned} D_n &= (-1)^{N(123\cdots n)} a_{11}a_{22}\cdots a_{nn} + (-1)^{N(n23\cdots n-11)} a_{1n}a_{22}a_{33}\cdots a_{n-1,n-1}a_{n1} \\ &= (-1)^0 a\cdot a\cdots a + (-1)^{2n-3} 1\cdot a\cdot a\cdots a\cdot 1 \\ &= a^n - a^{n-2} = a^{n-2}(a^2-1). \end{aligned}$$

(Ⅱ) Using the following result to determine its sign.

Theorem 1.1.2 Suppose that any term in determinant $D=|a_{ij}|$ of order n is $a_{i_1 j_1}a_{i_2 j_2}\cdots a_{i_n j_n}$, where $i_1 i_2 \cdots i_n$ and $j_1 j_2 \cdots j_n$ are two permutations of order n, then the sign in front of this term is

$$(-1)^{N(i_1 i_2 \cdots i_n)+N(j_1 j_2 \cdots j_n)}.$$

Example 1.1.6 If $(-1)^{N(i432k)+N(52j14)} a_{i5}a_{42}a_{3j}a_{21}a_{k4}$ is a term in $D_5=|a_{ij}|$, then what are values of i,j and k? What is the sign of this term?

Solution By the definition of determinant, we know that each term in D_5 is