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A Primer of Real Analytic Functions

Second Edition

Steven G. Krantz
Harold R. Parks

实解析函数入门 第2版

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To the memory of Frederick J. Almgren, Jr. (1933–1997),
teacher and friend

Preface to the Second Edition

It is a pleasure and a privilege to write this new edition of *A Primer of Real Analytic Functions*. The theory of real analytic functions is the wellspring of mathematical analysis. It is remarkable that this is the first book on the subject, and we want to keep it up to date and as correct as possible.

With these thoughts in mind, we have utilized helpful remarks and criticisms from many readers and have thereby made numerous emendations. We have also added material. There is a now a treatment of the Weierstrass preparation theorem, a new argument to establish Hensel's lemma and Puiseux's theorem, a new treatment of Faà di Bruno's formula, a thorough discussion of topologies on spaces of real analytic functions, and a second independent argument for the implicit function theorem. We trust that these new topics will make the book more complete, and hence a more useful reference.

It is a pleasure to thank our editor, Ann Kostant of Birkhäuser Boston, for making the publishing process as smooth and trouble-free as possible. We are grateful for useful communications from the readers of our first edition, and we look forward to further constructive feedback.

Steven G. Krantz
Harold R. Parks
May, 2002

Preface to the First Edition

The subject of real analytic functions is one of the oldest in mathematical analysis. Today it is encountered early in one's mathematical training: the first taste usually comes in calculus. While most working mathematicians use real analytic functions from time to time in their work, the vast lore of real analytic functions remains obscure and buried in the literature. It is remarkable that the most accessible treatment of Puiseux's theorem is in Lefschetz's quite old *Algebraic Geometry*, that the clearest discussion of resolution of singularities for real analytic manifolds is in a book review by Michael Atiyah, that there is no comprehensive discussion in print of the embedding problem for real analytic manifolds.

We have had occasion in our collaborative research to become acquainted with both the history and the scope of the theory of real analytic functions. It seems both appropriate and timely for us to gather together this information in a single volume. The material presented here is of three kinds. The elementary topics, covered in Chapter 1, are presented in great detail. Even results like a real analytic inverse function theorem are difficult to find in the literature, and we take pains here to present such topics carefully. Topics of middling difficulty, such as separate real analyticity, Puiseux series, the FBI transform, and related ideas (Chapters 2–4), are covered thoroughly but rather more briskly. Finally there are some truly deep and difficult topics: embedding of real analytic manifolds, sub- and semi-analytic sets, the structure theorem for real analytic varieties, and resolution of singularities are discussed and described. But thorough proofs in these areas could not possibly be provided in a volume of modest length.

Our aim, therefore, has been to provide an introduction to and a map (a primer if you will) of the subject of real analytic functions. Perhaps this monograph will help to bring to light a diverse and important literature.

It is a pleasure to thank Richard Beals, Edward Bierstone, Brian Blank, Harold Boas, Ralph Boas, Joséf Siciak, Kennan T. Smith, David Tartakoff, and Michael E. Taylor for many useful comments and suggestions. Of course the responsibility for all remaining errors remains the province of the authors.

Steven G. Krantz
Harold R. Parks
1992

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Contents

1

Elementary Properties

1.1 Basic Properties of Power Series

We begin with power series on the real line $\mathbb{R}$. A formal expression

$$\sum_{j=0}^{\infty} a_j (x-\alpha)^j ,$$

with the a_j's being either real or complex constants and with $\alpha \in \mathbb{R}$, is called a *power series*. It is usually convenient to take the coefficients a_j to all be real; there is no loss of generality in doing so. Our first task is to determine the nature of the set on which a power series converges.

Proposition 1.1.1 *Assume that the power series*

$$\sum_{j=0}^{\infty} a_j (x-\alpha)^j$$

converges at the value $x = c$. Let $r = |c-\alpha|$. Then the series converges uniformly and absolutely on compact subsets of $\mathcal{I} = \{x : |x-\alpha| < r\}$.

Proof. We may take the compact subset of $\mathcal{I}$ to be $K = [\alpha - s, \alpha + s]$ for some number $0 < s < r$. It holds that

$$\sum_{j=0}^{\infty} \left| a_j (x-\alpha)^j \right| = \sum_{j=0}^{\infty} \left| a_j (c-\alpha)^j \right| \cdot \left| \frac{x-\alpha}{c-\alpha} \right|^j .$$

In the sum on the right, the first expression in absolute values is bounded by some constant C (by the convergence hypothesis). For $x \in K$, the quotient in absolute values is majorized by $L = s/r < 1$. The series on the right is thus dominated by

$$\sum_{j=0}^{\infty} C \cdot L^j .$$

This geometric series converges. By the Weierstrass M-Test, the original series converges absolutely and uniformly on K. □

An immediate consequence of the proposition is that the set on which the power series

$$\sum_{j=0}^{\infty} a_j (x-\alpha)^j$$

converges is an interval centered about α. This interval is termed the *interval of convergence*. The series will converge absolutely and uniformly on compact subsets of the interval of convergence. The *radius* of the interval of convergence is defined to be half its length. Whether convergence holds at the endpoints of the interval will depend on the particular series. Let us use the notation C to denote the open interval of convergence. While we have seen that a power series is uniformly convergent on compact subintervals of C, it is an interesting and nontrivial fact that if the series converges at either of the endpoints, then the convergence is uniform up to that endpoint. This fact is a consequence of the following lemma due to Abel (see [AN 26]).

Lemma 1.1.2 *Let $u_0, u_1, \ldots$ be a sequence of reals, and set*

$$s_n = \sum_{j=0}^{n} u_j, \qquad \text{for } n = 0, 1, \ldots .$$

If

$$a \le s_n \le A, \qquad \text{for } n = 0, 1, \ldots$$

and if

$$\epsilon_0 \ge \epsilon_1 \ge \cdots \ge \epsilon_n \ge 0,$$

then

$$\epsilon_0 a \le \sum_{j=0}^{n} \epsilon_j u_j \le \epsilon_0 A, \qquad \text{for } n = 0, 1, \ldots .$$

Proof. One can write

$$u_0 = s_0, \; u_1 = s_1 - s_0, \; \ldots, u_n = s_n - s_{n-1}, \; \ldots .$$

Hence

$$\begin{aligned}\epsilon_0 u_0 + \epsilon_1 u_1 + \cdots + \epsilon_n u_n &= \epsilon_0 s_0 + \epsilon_1 (s_1 - s_0) + \cdots + \epsilon_n (s_n - s_{n-1}) \\ &= (\epsilon_0 - \epsilon_1) s_0 + \cdots + (\epsilon_{n-1} - \epsilon_n) s_{n-1} + \epsilon_n s_n \quad (1.1)\end{aligned}$$

We also have

$$(\epsilon_j - \epsilon_{j+1}) a \le (\epsilon_j - \epsilon_{j+1}) s_j \le (\epsilon_j - \epsilon_{j+1}) A ,$$

for $j = 0, 1, \ldots,$ and

$$\epsilon_n a \le \epsilon_n s_n \le \epsilon_n A .$$

Adding up these inequalities and using (1.1), we obtain the result. □

Proposition 1.1.3 *Assume that the power series*

$$\sum_{j=0}^{\infty} a_j (x - \alpha)^j$$

has the bounded interval of convergence C. If p is an endpoint of C and if the power series converges at the value $x = p$, then the series converges uniformly on the closed interval bounded by α and p.

Proof. We may assume that $C = (-1, 1)$ and that the series converges at $x = 1$. We take $\epsilon_j = x^j$, $u_j = a_j$ and consider summation from $j = m$ to $j = m + n$, with m large. The assertion is then immediate from Lemma 1.1.2.

□

Remark 1.1.4 The procedure exhibited in Lemma 1.1.2 and its proof is often referred to as "summation by parts." Indeed, the usual integration by parts procedure in calculus may be verified by applying summation by parts to the Riemann sums for the integral. □

On the interval of convergence C, the power series defines a function f. Such a function is said to be real analytic at α. More precisely, we have the following definition.

Definition 1.1.5 A function f, with domain an open set $U \subseteq \mathbb{R}$ and range either the real or the complex numbers, is said to be *real analytic* at α if the function f may be represented by a convergent power series on some interval of positive radius centered at α:

$$f(x) = \sum_{j=0}^{\infty} a_j (x - \alpha)^j .$$

The function is said to be *real analytic* on $V \subseteq U$ if it is real analytic at each $\alpha \in V$.

Remark 1.1.6 It is true, but not obvious, that the function which a convergent power series defines is real analytic on the open interval of convergence. This assertion will be proved in the next section. A consequence is that the set V in the preceding definition may as well always be chosen to be open.

We need to know both the algebraic and the calculus properties of a real analytic function: is it continuous? differentiable? How does one add/subtract/multiply/divide two such functions?

Proposition 1.1.7 *Let*

$$\sum_{j=0}^{\infty} a_j(x-\alpha)^j \text{ and } \sum_{j=0}^{\infty} b_j(x-\alpha)^j$$

be two power series with open intervals of convergence C_1 and C_2. Let $f(x)$ be the function defined by the first series on C_1 and $g(x)$ the function defined by the second series on C_2. Then, on their common domain $C = C_1 \cap C_2$, it holds that

(1) $f(x) \pm g(x) = \sum_{j=0}^{\infty}(a_j \pm b_j)(x-\alpha)^j$;

(2) $f(x) \cdot g(x) = \sum_{m=0}^{\infty} \sum_{j+k=m}(a_j \cdot b_k)(x-\alpha)^m$;

Proof. Let

$$A_N = \sum_{j=0}^{N} a_j(x-\alpha)^j \text{ and } B_N = \sum_{j=0}^{N} b_j(x-\alpha)^j$$

be, respectively, the N^{th} partial sums of the power series that define f and g. If C_N is the N^{th} partial sum of the series

$$\sum_{j=0}^{\infty}(a_j \pm b_j)(x-\alpha)^j,$$

then

$$f(x) \pm g(x) = \lim_{N\to\infty} A_N \pm \lim_{N\to\infty} B_N = \lim_{N\to\infty} [A_N \pm B_N] = \sum_{j=0}^{\infty}(a_j \pm b_j)(x-\alpha)^j,$$

proving (1).

For (2), let

$$D_N = \sum_{m=0}^{N} \sum_{j+k=m} (a_j \cdot b_k)(x-\alpha)^m \text{ and } R_N = \sum_{j=N+1}^{\infty} b_j(x-\alpha)^j.$$

We have

$$\begin{aligned} D_N &= a_0 B_N + a_1(x-\alpha)B_{N-1} + \cdots + a_N(x-\alpha)^N B_0 \\ &= a_0(g(x) - R_N) + a_1(x-\alpha)(g(x) - R_{N-1}) \\ &\quad + \cdots + a_N(x-\alpha)^N(g(x) - R_0) \\ &= g(x)\sum_{j=0}^{N} a_j(x-\alpha)^j \\ &\quad -[a_0R_N + a_1(x-\alpha)R_{N-1} + \cdots + a_N(x-\alpha)^N R_0]. \end{aligned}$$

Clearly

$$g(x)\sum_{j=0}^{N} a_j(x-\alpha)^j$$

converges to $g(x)f(x)$ as N approaches ∞. It will thus suffice to show that

$$|a_0R_N + a_1(x-\alpha)R_{N-1} + \cdots + a_N(x-\alpha)^N R_0|$$

converges to 0 as N approaches ∞.

Consider $x \in C$ to be fixed. We know that

$$\sum_{j=0}^{\infty} a_j(x-\alpha)^j$$

is absolutely convergent so we may set

$$A = \sum_{j=0}^{\infty} |a_j||x-\alpha|^j.$$

Given $\epsilon > 0$, we can find N_0 so that $N \geq N_0$ implies $|R_N| \leq \epsilon$. So we have

$$\begin{aligned} &|a_0R_N + a_1(x-\alpha)R_{N-1} + \cdots + a_N(x-\alpha)^N R_0| \\ &\quad\leq |a_0R_N + \cdots + a_{N-N_0}(x-\alpha)^{N-N_0}R_{N_0}| \\ &\quad+ |a_{N-N_0+1}(x-\alpha)^{N-N_0+1}R_{N_0-1} + \cdots + a_N(x-\alpha)^N R_0| \\ &\quad\leq \epsilon A + |a_{N-N_0+1}(x-\alpha)^{N-N_0+1}R_{N_0-1} + \cdots + a_N(x-\alpha)^N R_0|. \end{aligned}$$

By holding N_0 fixed and letting N approach ∞ we obtain the result. □

Next we turn to division of real analytic functions. Proving the analyticity of the quotient of analytic functions is more delicate than doing so for the sum or the product. This endeavor will be facilitated by the following lemma and its corollary.

Lemma 1.1.8 *For the power series*

$$\sum_{j=0}^{\infty} a_j (x-\alpha)^j ,$$

define A and ρ by

$$A = \limsup_{n\to\infty} |a_n|^{1/n}$$

and

$$\rho = \begin{cases} 0 & \text{if } A = \infty, \\ 1/A & \text{if } 0 < A < \infty, \\ \infty & \text{if } A = 0. \end{cases} \tag{1.2}$$

Then ρ is the radius of convergence of the power series about α.

Remark 1.1.9 Equation (1.2) is called the *Hadamard formula* for the radius of convergence of a power series.

Proof. Observing that

$$\limsup_{n\to\infty} |a_n(x-\alpha)^n|^{1/n} = A|x-\alpha|,$$

we see the lemma is an immediate consequence of the root test. □

Corollary 1.1.10 *The power series*

$$\sum_{j=0}^{\infty} a_j (x-\alpha)^j$$

has radius of convergence ρ if and only if, for each $0 < R < \rho$, there exists a constant $0 < C = C_R$ such that

$$|a_n| \le \frac{C}{R^n} . \tag{1.3}$$

We will need the following elementary lemma.

Lemma 1.1.11 *For $-\infty < M < \infty$ and $j = 1, 2, \ldots$, it holds that*

$$1 + M \sum_{\ell=0}^{j-1} (1+M)^\ell = (1+M)^j .$$

Proof. The result is easily proved by induction on j. □

If f and g are real analytic functions at a point α and if g does not vanish on an open interval containing α, then we would like to show that f/g is real analytic at α (it certainly is a well-defined *function*) and we would like to be able to calculate its power series expansion at α by formal long division. These facts are what the next result tells us.

Proposition 1.1.12 *Let*

$$\sum_{j=0}^{\infty} a_j(x-\alpha)^j \quad \text{and} \quad \sum_{j=0}^{\infty} b_j(x-\alpha)^j$$

be two power series with open intervals of convergence C_1 and C_2. Let $f(x)$ be the function defined by the first series on C_1 and $g(x)$ the function defined by the second series on C_2. If $\alpha \in C_1 \cap C_2$, and g does not vanish at α, then the function

$$h(x) = \frac{f(x)}{g(x)}$$

is real analytic at α. Moreover the power series expansion of h at α may be obtained by formal long division of the series for g into the series for f. That is, the zeroth coefficient c_0 of the series for h is

$$c_0 = a_0/b_0,$$

and the higher order coefficients c_j, $j = 1, 2, \ldots$, are given recursively by

$$c_j = \frac{1}{b_0}\left(a_j - \sum_{\ell=1}^{j} b_\ell c_{j-\ell}\right). \tag{1.4}$$

Proof. By (1.4), the coefficient of x^n in

$$g(x)\cdot h(x) = \left(\sum_{k=0}^{\infty} b_k(x-\alpha)^k\right)\left(\sum_{j=0}^{\infty} c_j(x-\alpha)^j\right)$$

is

$$\begin{aligned}\sum_{\ell=0}^{n} b_\ell c_{n-\ell} &= b_0 c_n - \sum_{\ell=1}^{n} b_\ell c_{n-\ell}\\ &= a_n,\end{aligned}$$

so the equation $g \cdot h = f$ holds at the level of formal power series. Thus if we can show that the power series

$$\sum_{j=0}^{\infty} c_j(x-\alpha)^j$$

converges on an open interval about α, then the result on multiplication of series in Proposition 1.1.7 yields this proposition.

There is no loss of generality in assuming that $\alpha = 0$ and $b_0 = 1$. Further, by dilation or contraction, it is also no loss of generality to assume that

$$(-1-\epsilon,\ 1+\epsilon) \subseteq C_1 \cap C_2,$$

for some $\epsilon > 0$.