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Walter Rudin

Function Theory  
in the  
Unit Ball of  $C^n$

$C^n$ 单位球上的函数理论

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Walter Rudin

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Walter Rudin  
University of Wisconsin  
Department of Mathematics  
Madison, WI 53706  
USA

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# Preface

Around 1970, an abrupt change occurred in the study of holomorphic functions of several complex variables. Sheaves vanished into the background, and attention was focused on integral formulas and on the "hard analysis" problems that could be attacked with them: boundary behavior, complex-tangential phenomena, solutions of the  $\bar{\partial}$ -problem with control over growth and smoothness, quantitative theorems about zero-varieties, and so on. The present book describes some of these developments in the simple setting of the unit ball of  $\mathbb{C}^n$ .

There are several reasons for choosing the ball for our principal stage. The ball is the prototype of two important classes of regions that have been studied in depth, namely the strictly pseudoconvex domains and the bounded symmetric ones. The presence of the second structure (i.e., the existence of a transitive group of automorphisms) makes it possible to develop the basic machinery with a minimum of fuss and bother. The principal ideas can be presented quite concretely and explicitly in the ball, and one can quickly arrive at specific theorems of obvious interest. Once one has seen these in this simple context, it should be much easier to learn the more complicated machinery (developed largely by Henkin and his co-workers) that extends them to arbitrary strictly pseudoconvex domains.

In some parts of the book (for instance, in Chapters 14–16) it would, however, have been unnatural to confine our attention exclusively to the ball, and no significant simplifications would have resulted from such a restriction.

Since the Contents lists the topics that are covered, this may be the place to mention some that might have been included but were not:

The fact that the automorphisms of the ball form a Lie group has been totally ignored.

There is no discussion of concepts such as curvature or geodesics with respect to the geometry that has these automorphisms as isometries.

The Heisenberg group is only mentioned in passing, although it is an active field of investigation in which harmonic analysis interacts with several complex variables.

Most of the refined estimates that allow one to control solutions of the  $\bar{\partial}$ -problem have been omitted. I have included what was needed to present the

Henkin-Skoda theorem that characterizes the zeros of functions of the Nevanlinna class.

Functions of bounded mean oscillation are not mentioned, although they have entered the field of several complex variables and will certainly play an important role there in the future.

To some extent, these omissions are due to considerations of space—I wanted to write a book of reasonable size—but primarily they are of course a matter of personal choice.

As regards prerequisites, they consist of advanced calculus, the basic facts about holomorphic functions of one complex variable, the Lebesgue theory of measure and integration, and a little functional analysis. The existence of Haar measure on the group of unitary matrices is the most sophisticated fact assumed from harmonic analysis. Everything that refers specifically to several complex variables is proved.

I have included a collection of open problems, in the hope that this may be one way to get them solved. Some of these look very simple. The fact that they are still unsolved shows quite clearly that we have barely begun to understand what really goes on in this area of analysis, in spite of the considerable progress that has been made.

I have tried to be as accurate as possible with regard to credits and priorities. The literature grows so rapidly, however, that I may have overlooked some important contributions. If this happened, I offer my sincere apologies to their authors.

Several friends have helped me to learn the material that is presented here—in conversations, by correspondence, and in writing joint papers. Among these, I especially thank Pat Ahern, Frank Forelli, John Fornaess, Alex Nagel, and Lee Stout.

Finally, I take this opportunity to express my appreciation to the National Science Foundation for supporting my work over a period of many years, to the William F. Vilas Trust Estate for one of its Research Professorships, and to the Mathematics Department of the University of Wisconsin for being such a friendly and stimulating place to work in.

*Madison, Wisconsin*  
*March 1980*

Walter Rudin



Walter Rudin  
Department of Mathematics  
University of Wisconsin  
480 Lincoln Drive  
Madison, WI 53706  
USA

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# List of Symbols and Notations

The numbers that follow the symbols indicate the paragraphs in which their meanings are explained. For example, 10.4.2 means Chapter 10, Section 4, paragraph 2.

## Sets

$\mathbb{C}, \mathbb{C}^n$	1.1.1	$(Z), (P)$	10.1.1
$B_n, B$	1.1.2	$(I), (PI)$	10.1.1
$S = \partial B_n$	1.1.2	$(N), (TN)$	10.1.1
$U, T$	1.1.2	$V(\zeta, \delta)$	10.4.2
$D(a; r)$	1.1.5	$(D)$	10.6.1
$U^n, T^n$	1.1.5	$E_1(f), \dots, E_3(f)$	11.4.2
$E(a, \varepsilon)$	2.2.7	$\mathcal{Q}$	12.3.1
$Q(\zeta, \delta)$	5.1.1	$D_k$	12.4.3
$D_\alpha(\zeta)$	5.4.1	$\Sigma(\Omega)$	12.4.3
$\Omega(E, \alpha)$	5.5.1	$\Delta, \Delta'$	14.1.1
$Z(f)$	7.3.1	$D_z$	15.3.1
$E_c$	8.5.3	$\Delta$	16.6.1

## Function Spaces

$L^p, C^k, C(X)$	1.1.1	$M_z$	9.1.2
$H(\Omega)$	1.1.4	$A^\perp$	9.1.4
$(L^p \cap H)(B)$	3.1.1	$A^*$	9.2.1
$A(B)$	3.2.3	$\operatorname{Re} A$	9.5.2
$X_\lambda$	4.2.1	$C_R(X)$	9.5.2
$C_0(B)$	4.2.6	$HM, TS$	9.8.1
$\mathbf{RP}(\Omega)$	4.4.1	$H$	10.6.4
$C^\infty(\{0\})$	4.4.3	$A(\Omega)$	10.6.7
$H_\phi(B), H^p(B)$	5.6.1	$A^n(B)$	10.7.1
$N(B)$	5.6.1	$A^\infty(B)$	10.7.1
$A(S), H^p(S)$	5.6.7	$\mathcal{P}_k, \mathcal{H}_k$	12.1.1
$L \log L$	6.3.2	$H(p, q)$	12.2.1
$H_E^\infty(B)$	6.6.2	$E_\Omega, X_\Omega$	12.3.1
$A(B, E, \{\alpha\})$	6.6.2	$\operatorname{conj} A(S)$	13.1.3

$(LH)^p(\Omega)$	7.4.1	$\text{plh}(S)$	13.1.3
$l^\infty, c_0$	7.4.5	$P(B)$	13.3.1
$C_0(C)$	7.5.2	$\text{plh}(B)$	13.3.1
$C(X)^*$	9.1.2	$\text{conj } A(B)$	13.3.1
$M(X)$	9.1.2	$W, \bar{W}$	19.1.6
		$N_*(B)$	19.1.11

## Maximal Functions

$M\mu$	5.5.2		
$M_\alpha F$	5.4.4	$M_{\text{rad}} F$	5.4.11

## Kernels and Transforms

$K(z, w)$	3.1.1	$K_s(z, w)$	7.1.1
$K[f]$	3.1.1	$T_s f$	7.1.1
$C(z, \zeta)$	3.2.1	$K_z(w)$	12.2.5
$C[f], C[\mu]$	3.2.1	$K_s(z, \zeta)$	16.5.1
$P(z, \zeta)$	3.3.1	$K_b(z, \zeta)$	16.5.2
$P[f], P[\mu]$	3.3.1	$Tf$	16.7.2

## Derivatives

$D_j, \bar{D}_j$	1.2.2	$\mathcal{R}f$	6.4.4
$D^2$	1.2.2	$d$	16.1.3
$\partial/\partial z_j, \partial/\partial \bar{z}_j$	1.3.1	$\partial, \bar{\partial}$	16.2.2
$\Delta$	1.3.4	$\Delta_{\text{rad}}$	17.2.2
$F'$	1.3.6	$\Delta_{\text{tan}}$	17.2.2
$\bar{\Delta}$	4.1.1	$L_{ij}, \bar{L}_{ij}$	18.3.1
$\mathcal{D}\mu$	5.3.3		

## Differential Forms

$\wedge$	16.1.1	$dz_i, d\bar{z}_i, dz_I, d\bar{z}_J$	16.2.1
$dx_I$	16.1.1	$\omega(z), \omega_f(z), \omega'(z)$	16.4.1
$\alpha_T$	16.1.4		

## Measures

$\nu$	1.4.1	$ \mu , \ \mu\ $	5.2.1
$\sigma$	1.4.1	$\mu \leq \sigma, \mu \perp \sigma$	5.2.1
$\tau$	2.7.6	$\mu_a, \mu_s$	12.2.4

## Other Symbols

$\langle z, w \rangle$	1.1.2	$\ f\ _p$	5.6.1
$ z $	1.1.2	$\Delta(\zeta, \omega, \alpha, \delta)$	6.1.2
$ \alpha , \alpha!$	1.1.6	$T_\varphi$	6.5.1
$z^x$	1.1.6	$\omega_\varphi(t)$	6.5.1
$f_\zeta$	1.2.5	$V_\varphi$	6.5.4
$JF, J_R F$	1.3.6	$\rho f$	7.2.3
$O(2n)$	1.4.1	$Eg$	7.2.3
$\mathcal{U}$	1.4.6	$n_f, N_f$	7.3.2
$I$	2.1.1	$   f   _p$	7.4.3
$\varphi_a$	2.2.1	$F_x, F^y$	9.4.1
$\ f\ _\infty$	3.2.3	$\pi_{pq}$	12.2.4
$u_r$	3.3.4	$[f, g]$	12.2.4
$\mathcal{H}$	3.3.6	$\mu(p, q; r, s)$	12.4.3
$f^*$	4.2.1	$g_a(z, w)$	12.5.1
$g_a(z)$	4.2.2	$\#(w)$	15.1.3
$d(a, b)$	5.1.1	$\rho$	15.5.1
$A_3$	5.2.2	$N(w)$	15.5.1
$z \cdot w$	5.4.2	$H_w, P_w, Q_w$	15.5.1
$T_\zeta, T_\zeta^c$	5.4.2	$\partial\Phi$	16.1.5
$K\text{-lim}$	5.4.6	$M(u)$	17.2.3
$g^*$	5.5.8	$A(E)$	17.3.1
$f_r$	5.6.1	$A(V)$	17.3.3
		$(\#_i f)(w)$	17.3.3

# Contents

List of Symbols and Notations	xi
Chapter 1	
Preliminaries	1
1.1 Some Terminology	1
1.2 The Cauchy Formula in Polydiscs	3
1.3 Differentiation	7
1.4 Integrals over Spheres	12
1.5 Homogeneous Expansions	19
Chapter 2	
The Automorphisms of $B$	23
2.1 Cartan's Uniqueness Theorem	23
2.2 The Automorphisms	25
2.3 The Cayley Transform	31
2.4 Fixed Points and Affine Sets	32
Chapter 3	
Integral Representations	36
3.1 The Bergman Integral in $B$	36
3.2 The Cauchy Integral in $B$	38
3.3 The Invariant Poisson Integral in $B$	50
Chapter 4	
The Invariant Laplacian	47
4.1 The Operator $\tilde{\Delta}$	47
4.2 Eigenfunctions of $\tilde{\Delta}$	49
4.3 $\mathcal{M}$ -Harmonic Functions	55
4.4 Pluriharmonic Functions	59

Chapter 5	
Boundary Behavior of Poisson Integrals	65
5.1 A Nonisotropic Metric on $S$	65
5.2 The Maximal Function of a Measure on $S$	67
5.3 Differentiation of Measures on $S$	70
5.4 $K$ -Limits of Poisson Integrals	72
5.5 Theorems of Calderón, Privalov, Plessner	79
5.6 The Spaces $N(B)$ and $H^p(B)$	83
5.7 Appendix: Marcinkiewicz Interpolation	88
Chapter 6	
Boundary Behavior of Cauchy Integrals	91
6.1 An Inequality	92
6.2 Cauchy Integrals of Measures	94
6.3 Cauchy Integrals of $L^p$ -Functions	99
6.4 Cauchy Integrals of Lipschitz Functions	101
6.5 Toeplitz Operators	110
6.6 Gleason's Problem	114
Chapter 7	
Some $L^p$ -Topics	120
7.1 Projections of Bergman Type	120
7.2 Relations between $H^p$ and $L^p \cap H$	126
7.3 Zero-Varieties	133
7.4 Pluriharmonic Majorants	145
7.5 The Isometries of $H^p(B)$	152
Chapter 8	
Consequences of the Schwarz Lemma	161
8.1 The Schwarz Lemma in $B$	161
8.2 Fixed-Point Sets in $B$	165
8.3 An Extension Problem	166
8.4 The Lindelöf-Čirka Theorem	168
8.5 The Julia-Carathéodory Theorem	174
Chapter 9	
Measures Related to the Ball Algebra	185
9.1 Introduction	185
9.2 Valskii's Decomposition	187
9.3 Henkin's Theorem	189
9.4 A General Lebesgue Decomposition	191
9.5 A General F. and M. Riesz Theorem	195

9.6 The Cole-Range Theorem	198
9.7 Pluriharmonic Majorants	198
9.8 The Dual Space of $A(B)$	202

## Chapter 10

Interpolation Sets for the Ball Algebra	204
10.1 Some Equivalences	204
10.2 A Theorem of Varopoulos	207
10.3 A Theorem of Bishop	209
10.4 The Davie-Øksendal Theorem	211
10.5 Smooth Interpolation Sets	214
10.6 Determining Sets	222
10.7 Peak Sets for Smooth Functions	229

## Chapter 11

Boundary Behavior of $H^\infty$ -Functions	234
11.1 A Fatou Theorem in One Variable	234
11.2 Boundary Values on Curves in $S$	237
11.3 Weak*-Convergence	244
11.4 A Problem on Extreme Values	247

## Chapter 12

Unitarily Invariant Function Spaces	253
12.1 Spherical Harmonics	253
12.2 The Spaces $H(p, q)$	255
12.3 $\mathcal{H}$ -Invariant Spaces on $S$	259
12.4 $\mathcal{H}$ -Invariant Subalgebras of $C(S)$	264
12.5 The Case $n = 2$	270

## Chapter 13

Moebius-Invariant Function Spaces	278
13.1 $\mathcal{H}$ -Invariant Spaces on $S$	278
13.2 $\mathcal{H}$ -Invariant Subalgebras of $C_0(B)$	280
13.3 $\mathcal{H}$ -Invariant Subspaces of $C(\bar{B})$	283
13.4 Some Applications	285

## Chapter 14

Analytic Varieties	288
14.1 The Weierstrass Preparation Theorem	288
14.2 Projections of Varieties	291
14.3 Compact Varieties in $\mathbb{C}^n$	294
14.4 Hausdorff Measures	295

Chapter 15	
<b>Proper Holomorphic Maps</b>	300
15.1 The Structure of Proper Maps	300
15.2 Balls vs. Polydiscs	305
15.3 Local Theorems	309
15.4 Proper Maps from $B$ to $B$	314
15.5 A Characterization of $B$	319
 Chapter 16	
<b>The <math>\bar{\partial}</math>-Problem</b>	330
16.1 Differential Forms	330
16.2 Differential Forms in $\mathbb{C}^n$	335
16.3 The $\bar{\partial}$ -Problem with Compact Support	338
16.4 Some Computations	341
16.5 Koppelman's Cauchy Formula	346
16.6 The $\bar{\partial}$ -Problem in Convex Regions	350
16.7 An Explicit Solution in $B$	357
 Chapter 17	
<b>The Zeros of Nevanlinna Functions</b>	364
17.1 The Henkin-Skoda Theorem	364
17.2 Plurisubharmonic Functions	366
17.3 Areas of Zero-Varieties	381
 Chapter 18	
<b>Tangential Cauchy-Riemann Operators</b>	387
18.1 Extensions from the Boundary	387
18.2 Unsolvable Differential Equations	395
18.3 Boundary Values of Pluriharmonic Functions	397
 Chapter 19	
<b>Open Problems</b>	403
19.1 The Inner Function Conjecture	403
19.2 RP-Measures	409
19.3 Miscellaneous Problems	413
 Bibliography	419
 Index	431



## Preliminaries

## 1.1. Some Terminology

**1.1.1.** Throughout this book,  $\mathbb{C}$  will denote the complex field, and  $\mathbb{C}^n$  will be the cartesian product of  $n$  copies of  $\mathbb{C}$ ; here  $n$  is any positive integer. The points of  $\mathbb{C}^n$  are thus ordered  $n$ -tuples  $z = (z_1, \dots, z_n)$ , where each  $z_i \in \mathbb{C}$ . Algebraically,  $\mathbb{C}^n$  is an  $n$ -dimensional vector space over  $\mathbb{C}$ . Topologically,  $\mathbb{C}^n$  is the euclidean space  $\mathbb{R}^{2n}$  of real dimension  $2n$ .

The usual vector space notations

$$(1) \quad \lambda A = \{\lambda a : \lambda \in \mathbb{C}, a \in A\},$$

$$(2) \quad A + B = \{a + b : a \in A, b \in B\}$$

will be freely used (for  $A \subset \mathbb{C}^n, B \subset \mathbb{C}^n, \lambda \in \mathbb{C}$ ), as will the customary symbols for the Lebesgue spaces  $L^p(\mu)$  (consisting of measurable complex functions  $f$  such that  $|f|^p$  is integrable with respect to whatever measure  $\mu$  is under consideration) and for the spaces  $C^k$  (consisting of complex functions whose  $k$ th-order partial derivatives are continuous).

The symbol

$$(3) \quad f: X \rightarrow Y$$

means that  $f$  is a map with domain  $X$ , whose range lies in  $Y$ .

As usual,  $C(X)$  is the space of all continuous functions  $f: X \rightarrow \mathbb{C}$ , where  $X$  is any topological space.

**1.1.2. The inner product**

$$(1) \quad \langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j \quad (z, w \in \mathbb{C}^n)$$

and the associated norm

$$(2) \quad |z| = \langle z, z \rangle^{1/2} \quad (z \in \mathbb{C}^n)$$