Kendall Atkinson
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TEXTS IN APPLIED MATHEMATICS 39

Theoretical Numerical Analysis

A Functional Analysis Framework

Third Edition

理论数值分析 第3版

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by Kendall Atkinson, Weimin Han

Dedicated to

DAISY AND CLYDE ATKINSON
HAZEL AND WRAY FLEMING

and

DAQING HAN, SUZHEN QIN
HUIDI TANG, ELIZABETH AND MICHAEL

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics* (*TAM*).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences* (*AMS*) series, which will focus on advanced textbooks and research-level monographs.

Pasadena, California	J.E. Marsden
Providence, Rhode Island	L. Sirovich
College Park, Maryland	S.S. Antman

Preface

This textbook has grown out of a course which we teach periodically at the University of Iowa. We have beginning graduate students in mathematics who wish to work in numerical analysis from a theoretical perspective, and they need a background in those "tools of the trade" which we cover in this text. In the past, such students would ordinarily begin with a one-year course in *real and complex analysis*, followed by a one or two semester course in *functional analysis* and possibly a graduate level course in *ordinary differential equations, partial differential equations*, or *integral equations*. We still expect our students to take most of these standard courses. The course based on this book allows these students to move more rapidly into a research program.

The textbook covers basic results of functional analysis, approximation theory, Fourier analysis and wavelets, calculus and iteration methods for nonlinear equations, finite difference methods, Sobolev spaces and weak formulations of boundary value problems, finite element methods, elliptic variational inequalities and their numerical solution, numerical methods for solving integral equations of the second kind, boundary integral equations for planar regions with a smooth boundary curve, and multivariable polynomial approximations. The presentation of each topic is meant to be an introduction with a certain degree of depth. Comprehensive references on a particular topic are listed at the end of each chapter for further reading and study. For this third edition, we add a chapter on multivariable polynomial approximation and we revise numerous sections from the second edition to varying degrees. A good number of new exercises are included.

The material in the text is presented in a mixed manner. Some topics are treated with complete rigour, whereas others are simply presented without proof and perhaps illustrated (e.g. the principle of uniform boundedness). We have chosen to avoid introducing a formalized framework for *Lebesgue measure and integration* and also for *distribution theory*. Instead we use standard results on the completion of normed spaces and the unique extension of densely defined bounded linear operators. This permits us to introduce the Lebesgue spaces formally and without their concrete realization using measure theory. We describe some of the standard material on measure theory and distribution theory in an intuitive manner, believing this is sufficient for much of the subsequent mathematical development. In addition, we give a number of deeper results without proof, citing the existing literature. Examples of this are the *open mapping theorem, Hahn-Banach theorem, the principle of uniform boundedness*, and a number of the results on *Sobolev spaces.*

The choice of topics has been shaped by our research program and interests at the University of Iowa. These topics are important elsewhere, and we believe this text will be useful to students at other universities as well.

The book is divided into chapters, sections, and subsections as appropriate. Mathematical relations (equalities and inequalities) are numbered by chapter, section and their order of occurrence. For example, (1.2.3) is the third numbered mathematical relation in Section 1.2 of Chapter 1. Definitions, examples, theorems, lemmas, propositions, corollaries and remarks are numbered consecutively within each section, by chapter and section. For example, in Section 1.1, Definition 1.1.1 is followed by an example labeled as Example 1.1.2.

We give exercises at the end of most sections. The exercises are numbered consecutively by chapter and section. At the end of each chapter, we provide some short discussions of the literature, including recommendations for additional reading.

During the preparation of the book, we received helpful suggestions from numerous colleagues and friends. We particularly thank P.G. Ciarlet, William A. Kirk, Wenbin Liu, and David Stewart for the first edition, B. Bialecki, R. Glowinski, and A.J. Meir for the second edition, and Yuan Xu for the third edition. It is a pleasure to acknowledge the skillful editorial assistance from the Series Editor, Achi Dosanjh.

Contents

1
Linear Spaces

Linear (or vector) spaces are the standard setting for studying and solving a large proportion of the problems in differential and integral equations, approximation theory, optimization theory, and other topics in applied mathematics. In this chapter, we gather together some concepts and results concerning various aspects of linear spaces, especially some of the more important linear spaces such as Banach spaces, Hilbert spaces, and certain function spaces that are used frequently in this work and in applied mathematics generally.

1.1 Linear spaces

A linear space is a set of elements equipped with two binary operations, called vector addition and scalar multiplication, in such a way that the operations behave linearly.

Definition 1.1.1 *Let V be a set of objects, to be called* vectors; *and let $\mathbb{K}$ be a set of scalars, either $\mathbb{R}$, the set of real numbers, or $\mathbb{C}$, the set of complex numbers. Assume there are two operations: $(u, v) \mapsto u + v \in V$ and $(\alpha, v) \mapsto \alpha v \in V$, called* addition *and* scalar multiplication *respectively, defined for any $u, v \in V$ and any $\alpha \in \mathbb{K}$. These operations are to satisfy the following rules.*

1. *$u + v = v + u$ for any $u, v \in V$ (commutative law);*
2. *$(u + v) + w = u + (v + w)$ for any $u, v, w \in V$ (associative law);*

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3. *there is an element* $0 \in V$ *such that* $0+v=v$ *for any* $v \in V$ *(existence of the zero element);*

4. *for any* $v \in V$*, there is an element* $-v \in V$ *such that* $v+(-v)=0$ *(existence of negative elements);*

5. $1v = v$ *for any* $v \in V$*;*

6. $\alpha(\beta v) = (\alpha\beta)v$ *for any* $v \in V$*, any* $\alpha, \beta \in \mathbb{K}$ *(associative law for scalar multiplication);*

7. $\alpha(u+v) = \alpha u + \alpha v$ *and* $(\alpha+\beta)v = \alpha v + \beta v$ *for any* $u, v \in V$*, and any* $\alpha, \beta \in \mathbb{K}$ *(distributive laws).*

Then V *is called a* linear space*, or a* vector space.

When $\mathbb{K}$ is the set of the real numbers, V is a real linear space; and when $\mathbb{K}$ is the set of the complex numbers, V becomes a complex linear space. In this work, most of the time we only deal with real linear spaces. So when we say V is a linear space, the reader should usually assume V is a real linear space, unless explicitly stated otherwise.

Some remarks are in order concerning the definition of a linear space. From the commutative law and the associative law, we observe that to add several elements, the order of summation does not matter, and it does not cause any ambiguity to write expressions such as $u+v+w$ or $\sum_{i=1}^{n} u_i$. By using the commutative law and the associative law, it is not difficult to verify that the zero element and the negative element $(-v)$ of a given element $v \in V$ are unique, and they can be equivalently defined through the relations $v+0=v$ for any $v \in V$, and $(-v)+v=0$. Below, we write $u-v$ for $u+(-v)$. This defines the subtraction of two vectors. Sometimes, we will also refer to a vector as a point.

Example 1.1.2 (a) The set $\mathbb{R}$ of the real numbers is a real linear space when the addition and scalar multiplication are the usual addition and multiplication. Similarly, the set $\mathbb{C}$ of the complex numbers is a complex linear space.

(b) Let d be a positive integer. The letter d is used generally in this work for the spatial dimension. The set of all vectors with d real components, with the usual vector addition and scalar multiplication, forms a linear space $\mathbb{R}^d$. A typical element in $\mathbb{R}^d$ can be expressed as $\boldsymbol{x} = (x_1, \dots, x_d)^T$, where $x_1, \dots, x_d \in \mathbb{R}$. Similarly, $\mathbb{C}^d$ is a complex linear space.

(c) Let $\Omega \subset \mathbb{R}^d$ be an open set of $\mathbb{R}^d$. In this work, the symbol Ω always stands for an open subset of $\mathbb{R}^d$. The set of all the continuous functions on Ω forms a linear space $C(\Omega)$, under the usual addition and scalar multiplication of functions: For $f, g \in C(\Omega)$, the function $f+g$ defined by

$$(f+g)(\boldsymbol{x}) = f(\boldsymbol{x}) + g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$