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N. Jacobson

Lectures in Abrstract Algerbra II

Linear Algebra

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Nathan Jacobson

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PREFACE

The present volume is the second in the author's series of three dealing with abstract algebra. For an understanding of this volume a certain familiarity with the basic concepts treated in Volume I: groups, rings, fields, homomorphisms, is presupposed. However, we have tried to make this account of linear algebra independent of a detailed knowledge of our first volume. References to specific results are given occasionally but some of the fundamental concepts needed have been treated again. In short, it is hoped that this volume can be read with complete understanding by any student who is mathematically sufficiently mature and who has a familiarity with the standard notions of modern algebra.

Our point of view in the present volume is basically the abstract conceptual one. However, from time to time we have deviated somewhat from this. Occasionally formal calculational methods yield sharper results. Moreover, the results of linear algebra are not an end in themselves but are essential tools for use in other branches of mathematics and its applications. It is therefore useful to have at hand methods which are constructive and which can be applied in numerical problems. These methods sometimes necessitate a somewhat lengthier discussion but we have felt that their presentation is justified on the grounds indicated. A student well versed in abstract algebra will undoubtedly observe short cuts. Some of these have been indicated in footnotes.

We have included a large number of exercises in the text. Many of these are simple numerical illustrations of the theory. Others should be difficult enough to test the better students. At any rate a diligent study of these is essential for a thorough understanding of the text.

At various stages in the writing of this book I have benefited from the advice and criticism of many friends. Thanks are particularly due to A. H. Clifford, to G. Hochschild, and to I. Kaplansky for suggestions on earlier versions of the text. Also I am greatly indebted to W. H. Mills, Jr. for painstaking help with the proofs and for last minute suggestions for improvements of the text.

N. J.

New Haven, Conn.
September, 1952

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CHAPTER IV: SETS OF LINEAR TRANSFORMATIONS

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Chapter I

FINITE DIMENSIONAL VECTOR SPACES

In three-dimensional analytic geometry, vectors are defined geometrically. The definition need not be recalled here. The important fact from the algebraic point of view is that a vector v is completely determined by its three coordinates (ξ, η, ζ) (relative to a definite coordinate system). It is customary to indicate this by writing $v = (\xi, \eta, \zeta)$, meaning thereby that v is the vector whose x-, y-, and z-coordinates are, respectively, ξ, η, and ζ. Conversely, any ordered triple of real numbers (ξ, η, ζ) determines a definite vector. Thus there is a 1–1 correspondence between vectors in 3-space and ordered triples of real numbers.

There are three fundamental operations on vectors in geometry: addition of vectors, multiplication of vectors by scalars (numbers) and the scalar product of vectors. Again, we need not recall the geometric definitions of these compositions. It will suffice for our purposes to describe the algebraic processes on the triples that correspond to these geometric operations. If $v = (\xi, \eta, \zeta)$ and $v' = (\xi', \eta', \zeta')$, then the *sum*

$$v + v' = (\xi + \xi', \eta + \eta', \zeta + \zeta').$$

The *product* ρv of the vector v by the real number ρ is the vector

$$\rho v = (\rho\xi, \rho\eta, \rho\zeta)$$

and the *scalar product* (v, v') of v and v' is the real number

$$(v, v') = \xi\xi' + \eta\eta' + \zeta\zeta'.$$

A substantial part of analytic geometry—the theory of linear dependence and of linear transformations—depends only on the first two of these concepts. It is this part (in a generalized form) which constitutes the main topic of discussion in these Lectures. The concept of scalar product is a metric one, and this will be relegated to a relatively minor role in our discussion.

The study of vectors relative to addition and multiplication by numbers can be generalized in two directions. First, it is not necessary to restrict oneself to the consideration of triples; instead, one may consider n-tuples for any positive integer n. Second, it is not necessary to assume that the coordinates ξ, η, $\cdots$ are real numbers. To insure the validity of the theory of linear dependence we need suppose only that it is possible to perform rational operations. Thus any field can be used in place of the field of real numbers. It is fairly easy to go one step further, namely, to drop the assumption of commutativity of the basic number system.

We therefore begin our discussion with a given division ring Δ. For example, Δ may be taken to be any one of the following systems: 1) the field of real numbers, 2) the field of complex numbers, 3) the field of rational numbers, 4) the field of residues modulo p, or 5) the division ring of real quaternions.

Let n be a fixed positive integer and let $\Delta^{(n)}$ denote the totality of n-tuples $(\xi_1, \xi_2, \cdots, \xi_n)$ with the ξ_i in Δ. We call these n-tuples *vectors*, and we call $\Delta^{(n)}$ *the vector space of n-tuples over* Δ. If $y = (\eta_1, \eta_2, \cdots, \eta_n)$, we regard $x = y$ if and only if $\xi_i = \eta_i$ for $i = 1, 2, \cdots, n$. Following the pattern of the three-dimensional real case, we introduce two compositions in $\Delta^{(n)}$: addition of vectors and multiplication of vectors by elements of Δ. First, if x and y are arbitrary vectors, we define their *sum* $x + y$ to be the vector

$$x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \cdots, \xi_n + \eta_n).$$

As regards to multiplication by elements of Δ there are two possibilities: *left multiplication* defined by

$$\rho x = (\rho\xi_1, \rho\xi_2, \cdots, \rho\xi_n)$$

and right multiplication defined by

$$x\rho = (\xi_1\rho, \xi_2\rho, \cdots, \xi_n\rho).$$

Either of these can be used. Parallel theories will result from the two choices. In the sequel we give preference to left multiplication. It goes without saying that all of our results may be transferred to results on right multiplication.

The first eight chapters of this volume will be devoted to the study of the systems $\Delta^{(n)}$ relative to the compositions we have just defined. The treatment which we shall give will be an axiomatic one in the sense that our results will all be derived from a list of simple properties of the systems $\Delta^{(n)}$ that will serve as axioms. These axioms define the concept of a *finite dimensional (abstract) vector space* and the systems $\Delta^{(n)}$ are instances of such spaces. Moreover, as we shall see, any other instance of a finite dimensional vector space is essentially equivalent to one of the systems $\Delta^{(n)}$.

Thus the shift to the axiomatic point of view is not motivated by the desire to gain generality. Its purposes are rather to clarify the discussion by focusing attention on the essential properties of our systems, and to make it easier to apply the results to other concrete instances. Finally, the broadening of the point of view leads naturally to the consideration of other, more general, concepts which will be useful in studying vector spaces. The most important of these is the concept of a module which will be our main tool in the theory of a single linear transformation (Chapter III). In order to prepare the ground for this application we shall consider this concept from the beginning of our discussion.

The present chapter will be devoted to laying the foundations of the theory of vector spaces. The principal concepts that we shall consider are those of basis, linear dependence, subspace, factor space and the lattice of subspaces.

1. Abstract vector spaces. We now list the properties of the compositions in $\Delta^{(n)}$ from which the whole theory of these systems will be derived. These are as follows:

A1 $$(x + y) + z = x + (y + z).$$

A2 $$x + y = y + x.$$

A3 There exists an element 0 such that $x + 0 = x$ for all x.

A4 For any vector x there exists a vector $-x$ such that $x + (-x) = 0$.

S1 $$\alpha(x + y) = \alpha x + \alpha y.$$

S2 $$(\alpha + \beta)x = \alpha x + \beta x.$$

S3 $$(\alpha\beta)x = \alpha(\beta x).$$

S4 $$1x = x.$$

F There exist a finite number of vectors $e_1, e_2, \cdots, e_n$ such that every vector can be written in one and only one way in the form $\xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n$.

The verifications of A1, A2, S1–S4 are immediate. We can prove A3 by observing that $(0, 0, \cdots, 0)$ has the required property and A4 by noting that, if $x = (\xi_1, \cdots, \xi_n)$, then we can take $-x = (-\xi_1, \cdots, -\xi_n)$. To prove F we choose for e_i,

$$(1) \qquad e_i = (0, 0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0), \quad i = 1, 2, \cdots, n.$$

Then $\xi_i e_i$ has ξ_i in its ith place, 0's elsewhere. Hence $\sum_1^n \xi_i e_i = (\xi_1, \xi_2, \cdots, \xi_n)$. Hence if $x = (\xi_1, \xi_2, \cdots, \xi_n)$, then x can be written as the "linear combination" $\Sigma \xi_i e_i$ of the vectors e_i. Also our relation shows that, if $\Sigma \xi_i e_i = \Sigma \eta_i e_i$, then $(\xi_1, \xi_2, \cdots, \xi_n) = (\eta_1, \eta_2, \cdots, \eta_n)$ so that $\xi_i = \eta_i$ for $i = 1, 2, \cdots, n$. This is what is meant by the uniqueness assertion in F.

The properties A1–A4 state that $\Delta^{(n)}$ is a commutative group under the composition of addition. The properties S1–S4 are properties of the multiplication by elements of Δ and relations between this composition and the addition composition. Property F is the fundamental finiteness condition.

We shall now use these properties to define an *abstract vector space*. By this we mean a system consisting of 1) a commutative group $\mathfrak{R}$ (composition written as $+$), 2) a division ring Δ, 3) a function defined for all the pairs (ρ, x), ρ in Δ, x in $\mathfrak{R}$, having values ρx in $\mathfrak{R}$ such that S1–S4 hold. In analogy with the geometric case of n-tuples we call the elements of $\mathfrak{R}$ *vectors* and the elements of Δ *scalars*. In our discussion the emphasis will usually be placed on $\mathfrak{R}$. For this reason we shall also refer to $\mathfrak{R}$ somewhat inexactly as a "vector space over the division ring Δ." (Strictly speaking $\mathfrak{R}$ is only the group part of the vector space.) If F holds in addition to the other assumptions, then we say that $\mathfrak{R}$ is *finite dimensional*, or that $\mathfrak{R}$ *possesses a finite basis over* Δ.

The system consisting of $\Delta^{(n)}$, Δ, and the multiplication ρx defined above is an example of a finite dimensional vector space. We shall describe next a situation in the theory of rings which gives rise to vector spaces. Let $\mathfrak{R}$ be an arbitrary ring with an identity element 1 and suppose that $\mathfrak{R}$ contains a division subring Δ that contains 1. For the product ρx, ρ in Δ, and x in $\mathfrak{R}$ we take the ring product ρx. Then S1–S3 are consequences of the distributive and associative laws of multiplication, and S4 holds since the identity element of Δ is the identity of $\mathfrak{R}$. Hence the additive group $\mathfrak{R}$, the division ring Δ and the multiplication ρx constitute a vector space. This space may or may not be finite dimensional. For example, if $\mathfrak{R}$ is the field of complex numbers and Δ is the subfield of real numbers, then $\mathfrak{R}$ is finite dimensional; for any complex number can be written in one and only one way as $\xi + \eta\sqrt{-1}$ in terms of the "vectors" 1, $\sqrt{-1}$. Another example of this type is $\mathfrak{R} = \Delta[\lambda]$, the polynomial ring in the transcendental element (indeterminate) λ with coefficients in the division ring Δ. We shall see that this vector space is not finite dimensional (see Exercise 1, p. 13). Similarly we can regard the polynomial ring $\Delta[\lambda_1, \lambda_2, \cdots, \lambda_r]$ where the λ_i are algebraically independent (independent indeterminates) as a vector space over Δ.

Other examples of vector spaces can be obtained as subspaces of the spaces defined thus far. Let $\mathfrak{R}$ be any vector space over Δ and let $\mathfrak{S}$ be a subset of $\mathfrak{R}$ that is a subgroup and that is closed under multiplication by elements of Δ. By this we mean that if $y \varepsilon \mathfrak{S}$ and ρ is arbitrary in Δ then $\rho y \varepsilon \mathfrak{S}$. Then it is clear that the trio consisting of $\mathfrak{S}$, Δ and the multiplication ρy is a vector space; for, since S1–S4 hold in $\mathfrak{R}$, it is obvious that they hold also in the subset $\mathfrak{S}$. We call this a *subspace* of the given vector space, and also we shall call $\mathfrak{S}$ a subspace of $\mathfrak{R}$. As an example, let $\mathfrak{R} = \Delta[\lambda]$ and let $\mathfrak{S}$ be the subset of polynomials of degree $<n$. It is immediate that $\mathfrak{S}$ is a subspace. Moreover, it is

finite dimensional since any polynomial of degree $<n$ can be expressed in one and only one way as a linear combination of the polynomials $1, \lambda, \cdots, \lambda^{n-1}$.

EXERCISE

1. Show that the totality $\mathfrak{S}$ of homogeneous quadratic polynomials $\sum_{i \leqq j} \alpha_{ij}\lambda_i\lambda_j$, α_{ij} in Δ, is a finite dimensional subspace of $\Delta[\lambda_1, \lambda_2]$.

2. Right vector spaces. As we have pointed out at the beginning the system $\Delta^{(n)}$ of n-tuples can also be studied relative to addition and to right multiplication by scalars. This leads us to define the concept of a *right vector space.* By this we mean a system consisting of a commutative group $\mathfrak{R}'$, a division ring Δ and a function of pairs (ρ, x'), ρ in Δ, x' in $\mathfrak{R}'$, having values $x'\rho$ in $\mathfrak{R}'$ and satisfying:

S′1 $$(x' + y')\alpha = x'\alpha + y'\alpha.$$

S′2 $$x'(\alpha + \beta) = x'\alpha + x'\beta.$$

S′3 $$x'(\alpha\beta) = (x'\alpha)\beta.$$

S′4 $$x'1 = x' \text{ for all } x' \text{ in } \mathfrak{R}'.$$

Obviously the theory based on this definition will parallel that of left vector spaces. It should be noted, however, that a right space over Δ cannot be regarded as a left space over Δ if this division ring is not commutative. For if we write $\alpha x'$ for $x'\alpha$, then we have by S′3

$$(\alpha\beta)x' = x'(\alpha\beta) = (x'\alpha)\beta = \beta(\alpha x').$$

Hence S3: $(\beta\alpha)x' = \beta(\alpha x')$ holds only if

$$[(\alpha\beta) - (\beta\alpha)]x' = 0$$

for all x'. This together with S4 implies that $\alpha\beta = \beta\alpha$ for all α, β.

On the other hand, let Δ' be a division ring anti-isomorphic to Δ and let $\alpha \rightarrow \alpha'$ be any anti-isomorphism of Δ onto Δ'. Then if $\mathfrak{R}'$ is a right vector space over Δ, $\mathfrak{R}'$ may be considered a left