Probability and Its Applications

Mu-Fa Chen

Eigenvalues, Inequalities, and Ergodic Theory

特征值,不等式和遍历理论

Springer

世界图书出版公司

www.wpcbj.com.cn

Mu-Fa Chen

Eigenvalues, Inequalities, and Ergodic Theory

Springer

图书在版编目（CIP）数据

特征值，不等式和遍历理论 = Eigenvalues，Inequalities，and Ergodic Theory：英文/陈木法著．—影印本．—北京：世界图书出版公司北京公司，2012.9
ISBN 978-7-5100-5267-5

Ⅰ.①特… Ⅱ.①陈… Ⅲ.①特征值—不等式—英文②遍历性理论—英文
Ⅳ.①O151.21②O177.99

中国版本图书馆 CIP 数据核字（2012）第 217142 号

书　　名：Eigenvalues，Inequalities，and Ergodic Theory
作　　者：Mu-Fa Chen
中 译 名：特征值，不等式和遍历理论
责任编辑：高蓉　刘慧

出 版 者：世界图书出版公司北京公司
印 刷 者：三河市国英印务有限公司
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话：010-64021602，010-64015659
电子信箱：kjb@wpcbj.com.cn

开　　本：24 开
印　　张：10.5
版　　次：2013 年 1 月
版权登记：图字：01-2012-5688

书　　号：978-7-5100-5267-5　　　定　　价：45.00 元

Probability and Its Applications

Anderson: Continuous-Time Markov Chains.
Azencott/Dacunha-Castelle: Series of Irregular Observations.
Bass: Diffusions and Elliptic Operators.
Bass: Probabilistic Techniques in Analysis.
Chen: Eigenvalues, Inequalities, and Ergodic Theory
Choi: ARMA Model Identification.
Daley/Vere-Jones: An Introduction to the Theory of Point Processes. Volume I: Elementary Theory and Methods, Second Edition.
de la Peña/Giné: Decoupling: From Dependence to Independence.
Del Moral: Feynman Kac Formulae: Genealogical and Interacting Particle Systems with Applications.
Durrett: Probability Models for DNA Sequence Evolution.
Galambos/Simonelli: Bonferroni-type Inequalities with Applications.
Gani (Editor): The Craft of Probabilistic Modelling.
Grandell: Aspects of Risk Theory.
Gut: Stopped Random Walks.
Guyon: Random Fields on a Network.
Kallenberg: Foundations of Modern Probability, Second Edition.
Last/Brandt: Marked Point Processes on the Real Line.
Leadbetter/Lindgren/Rootzén: Extremes and Related Properties of Random Sequences and Processes.
Nualart: The Malliavin Calculus and Related Topics.
Rachev/Rüschendorf: Mass Transportation Problems. Volume I: Theory.
Rachev/Rüschendofr: Mass Transportation Problems. Volume II: Applications.
Resnick: Extreme Values, Regular Variation and Point Processes.
Shedler: Regeneration and Networks of Queues.
Silvestrov: Limit Theorems for Randomly Stopped Stochastic Processes.
Thorisson: Coupling, Stationarity, and Regeneration.
Todorovic: An Introduction to Stochastic Processes and Their Applications.

Preface

First, let us explain the precise meaning of the compressed title. The word "eigenvalues" means the first nontrivial Neumann or Dirichlet eigenvalues, or the principal eigenvalues. The word "inequalities" means the Poincaré inequalities, the logarithmic Sobolev inequalities, the Nash inequalities, and so on. Actually, the first eigenvalues can be described by some Poincaré inequalities, and so the second topic has a wider range than the first one. Next, for a Markov process, corresponding to its operator, each inequality describes a type of ergodicity. Thus, study of the inequalities and their relations provides a way to develop the ergodic theory for Markov processes. Due to these facts, from a probabilistic point of view, the book can also be regarded as a study of "ergodic convergence rates of Markov processes," which could serve as an alternative title of the book. However, this book is aimed at a larger class of readers, not only probabilists.

The importance of these topics should be obvious. On the one hand, the first eigenvalue is the leading term in the spectrum, which plays an important role in almost every branch of mathematics. On the other hand, the ergodic convergence rates constitute a recent research area in the theory of Markov processes. This study has a very wide range of applications. In particular, it provides a tool to describe the phase transitions and the effectiveness of random algorithms, which are now a very fashionable research area.

This book surveys, in a popular way, the main progress made in the field by our group. It consists of ten chapters plus two appendixes. The first chapter is an overview of the second to the eighth ones. Mainly, we study several different inequalities or different types of convergence by using three mathematical tools: a probabilistic tool, the coupling methods (Chapters 2 and 3); a generalized Cheeger's method originating in Riemannian geometry (Chapter 4); and an approach coming from potential theory and harmonic analysis (Chapters 6 and 7). The explicit criteria for different types of convergence and the explicit estimates of the convergence rates (or the optimal constants in the inequalities) in dimension one are given in Chapters 5 and 6; some generalizations are given in Chapter 7. The proofs of a diagram of nine types of ergodicity (Theorem 1.9) are presented in Chapter 8. Very often, we deal with one-dimensional elliptic operators or tridiagonal matrices (which can be infinite) in detail, but we also handle general differential and integral operators. To avoid heavy technical details, some proofs are split among several locations in the text. This also provides different views of the same problem at different levels. The topics of the last two chapters (9 and 10) are different but closely related. Chapter 9 surveys the study of a class of interacting particle systems (from which a large part of the problems studied in this book are motivated), and illustrates some applications. In the last chapter, one can see an interesting application of the first eigenvalue, its eigenfunctions, and an ergodic theorem to stochastic models of economics. Some related open problems are included in each chapter. Moreover, an effort is made to make each chapter, except the first one, more or less self-contained. Thus, once one has read about the program in Chapter 1, one may freely go on to the other chapters. The main exception is Chapter 3, which depends heavily on Chapter 2. As usual, a quick way to get an impression about what is done in the book is to look at the summaries given at the beginning of each chapter.

One should not be disappointed if one cannot find an answer in the book for one's own model. The complete solutions to our problems have only recently been obtained in dimension one. Nevertheless, it is hoped that the three methods studied in the book will be helpful. Each method has its own advantages and disadvantages. In principle, the coupling method can produce sharper estimates than the other two methods, but additional work is required to figure out a suitable coupling and, more seriously, a good distance. The Cheeger and capacitary methods work in a very general setup and are powerful qualitatively, but they leave the estimation of isoperimetric constants to the reader. The last task is usually quite hard in higher-dimensional situations.

This book serves as an introduction to a developing field. We emphasize the ideas through simple examples rather than technical proofs, and most of them are only sketched. It is hoped that the book will be readable by nonspecialists. In the past ten years or more, the author has tried rather hard to make acceptable lectures; the present book is based on these lecture notes: Chen (1994b; 1997a; 1998a; 1999c; 2001a; 2002b; 2002c; 2003b; 2004a; 2004b) [see Chen (2001c)]. Having presented eleven lectures in Japan in 2002, the author understood that it would be worthwhile to publish a short book, and then the job was started.

Since each topic discussed in the book has a long history and contains a great number of publications, it is impossible to collect a complete list of references. We emphasize the recent progress and related references. It is hoped that the bibliography is still rich enough that the reader can discover a large number of contributors in the field and more related references.

Beijing, The People's Republic of China — Mu-Fa Chen, October 2004

Acknowledgments

As mentioned before, this book is based on lecture notes presented over the past ten years or so. Thus, the book should be dedicated, with the author's deep acknowledgment, to the mathematicians and their universities/institutes whose kind invitations, financial support, and warm hospitality made those lectures possible. Without their encouragement and effort, the book would never exist. With the kind permission of his readers, the author is happy to list some of the names below (since 1993), with an apology to those that are missing:

- Z.M. Ma and J.A. Yan, Institute of Applied Mathematics, Chinese Academy of Sciences. D.Y. Chen, G.Q. Zhang, J.D. Chen, and M.P. Qian, Beijing (Peking) University. T.S. Chiang, C.R. Hwang, Y.S. Chow, and S.J. Sheu, Institute of Mathematics, Academy Sinica, Taipei. C.H. Chen, Y.S. Chow, A.C. Hsiung, W.T. Huang, W.Q. Liang, and C.Z. Wei, Institute of Statistical Science, Academy Sinica, Taipei. H. Chen, National Taiwan University. T.F. Lin, Soochow University. Y.J. Lee and W.J. Huang, National University of Kaohsiung. C.L. Wang, National Dong Hwa University.
- D.A. Dawson and S. Feng [McMaster University], Carleton University. G. O'Brien, N. Madras, and J.M. Sun, York University. D. McDonald, University of Ottawa. M. Barlow, E.A. Perkins, and S.J. Luo, University of British Columbia.
- E. Scacciatelli, G. Nappo, and A. Pellegrinotti [University of Roma III], University of Roma I. L. Accardi, University of Roma II. C. Boldrighini, University of Camerino [University of Roma I]. V. Capasso and Y.G. Lu, University of Bari.
- B. Grigelionis, Akademijios, Lithuania.
- L. Stettner and J. Zabczyk, Polish Academy of Sciences.
- W.Th.F. den Hollander, Utrecht University [Universiteit Leiden].
- Louis H.Y. Chen, K.P. Choi, and J.H. Lou, Singapore University.
- R. Durrett, L. Gross, and Z.Q. Chen [University of Washington Seattle], Cornell University. D.L. Burkholder, University of Illinois. C. Heyde, K. Sigman, and Y.Z. Shao, Columbia University.

- D. Elworthy, Warwick University. S. Kurylev, C. Linton, S. Veselov, and H.Z. Zhao, Loughborough University. T.S. Zhang, University of Manchester. G. Grimmett, Cambridge University. Z. Brzezniak and P. Busch, University of Hull. T. Lyons, University of Oxford. A. Truman, N. Jacod, and J.L. Wu, University of Wales Swansea.
- F. Götze and M. Röckner, University of Bielefeld. S. Albeverio and K.-T. Sturm, University of Bonn. J.-D. Deuschel and A. Bovier, Technical University of Berlin.
- K.J. Hochberg, Bar-Ilan University. B. Granovsky, Technion-Israel Institute of Technology.
- B. Yart, Grenoble University [University, Paris V]. S. Fang and B. Schmit, University of Bourgogne. J. Bertoin and Z. Shi, University of Paris VI. L.M. Wu, Blaise Pascal University and Wuhan University.
- R.A. Minlos, E. Pechersky, and E. Zizhina, the Information Transmission Problems, Russian Academy of Sciences.
- A.H. Xia, University of New South Wales [Melbourne University]. C. Heyde, J. Gani, and W. Dai, Australian National University. E. Seneta, University of Sydney. F.C. Klebaner, University of Melbourne. Y.X. Lin, Wollongong University.
- I. Shigekawa, Y. Takahashi, T. Kumagai, N. Yosida, S. Watanabe, and Q.P. Liu, Kyoto University. M. Fukushima, S. Kotani, S. Aida, and N. Ikeda, Osaka University. H. Osada, S. Liang, and K. Sato, Nagoya University. T. Funaki and S. Kusuoka, Tokyo University.
- E. Bolthausen, University of Zurich, P. Embrechts and A.-S. Sznitman, ETH.
- London Mathematical Society for the visit to the United Kingdom during November 4–22, 2003.

Next, the author acknowledges the organizers of the following conferences (since 1993) for their invitations and financial support.

- The Sixth International Vilnuis Conference on Probability and Mathematical Statistics (June 1993, Vilnuis).
- The International Conference on Dirichlet Forms and Stochastic Processes (October 1993, Beijing).
- The 23rd and 25th Conferences on Stochastic Processes and Their Applications (June 1995, Singapore and July 1998, Oregon).
- The Symposium on Probability Towards the Year 2000 (October 1995, New York).
- Stochastic Differential Geometry and Infinite-Dimensional Analysis (April 1996, Hangzhou).
- Workshop on Interacting Particle Systems and Their Applications (June 1996, Haifa).
- IMS Workshop on Applied Probability (June 1999, Hong Kong).

- The Second Sino-French Colloquium in Probability and Applications (April 2001, Wuhan).
- The Conference on Stochastic Analysis on Large Scale Interacting Systems (July 2002, Japan).
- Stochastic Analysis and Statistical Mechanics, Yukawa Institute (July 2002, Kyoto).
- International Congress of Mathematicians (August 2002, Beijing).
- The First Sino-German Conference on Stochastic Analysis—A Satellite Conference of ICM 2002 (September 2002, Beijing).
- Stochastic Analysis in Infinite Dimensional Spaces (November 2002, Kyoto)
- Japanese National Conference on Stochastic Processes and Related Fields (December 2002, Tokyo).

Thanks are given to the editors, managing editors, and production editors, of the Springer Series in Statistics, Probability and Its Applications, especially J. Gani and S. Harding for their effort in publishing the book, and to the copyeditor D. Kramer for the effort in improving the English language.

Thanks are also given to World Scientific Publishing Company for permission to use some material from the author's previous book (1992a, 2004: 2nd edition).

The continued support of the National Natural Science Foundation of China, the Research Fund for Doctoral Program of Higher Education, as well as the Qiu Shi Science and Technology Foundation, and the 973 Project are also acknowledged.

Finally, the author is grateful to the colleagues in our group: F.Y. Wang, Y.H. Zhang, Y.H. Mao, and Y.Z. Wang for their fruitful cooperation. The suggestions and corrections to the earlier drafts of the book by a number of friends, especially J.W. Chen and H.J. Zhang, and a term of students are also appreciated. Moreover, the author would like to acknowledge S.J. Yan, Z.T. Hou, Z.K. Wang, and D.W. Stroock for their teaching and advice.

Mu-Fa Chen
Department of Mathematics, Beijing Normal University, Beijing 100875,
The People's Republic of China

Series Editors

J. Gani
Stochastic Analysis Group CMA
Australian National University
Canberra ACT 0200
Australia

C.C. Heyde
Stochastic Analysis Group, CMA
Australian National University
Canberra ACT 0200
Australia

T.G. Kurtz
Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, WI 53706
USA

P. Jagers
Mathematical Statistics
Chalmers University of Technology
S-41296 Göteborg
Sweden

Mathematics Subject Classification (2000): 60J25, 60K35, 37A25, 37A30, 47A45, 58C40, 34B24, 34L15, 35P15, 91B02

British Library Cataloguing in Publication Data
Chen, Mufa
Eigenvalues, inequalities and ergodic theory.
(Probability and its applications)
1. Eigenvalues 2. Inequalities (Mathematics) 3. Ergodic theory
I. Title
512.9′436
ISBN 1852338687

Library of Congress Cataloging-in-Publication Data
Chen, Mu-fa.
Eigenvalues, inequalities, and ergodic theory / Mu-Fa Chen.
p. cm. — (Probability and its applications)
Includes bibliographical references and indexes.
ISBN 1-85233-868-7 (alk. paper)
1. Eigenvalues. 2. Inequalities (Mathematics) 3. Ergodic theory. I. Title. II. Probability and its applications (Springer-Verlag).
QA193.C44 2004
512.9′436—dc22 2004049193

ISBN 1-85233-868-7 Springer-Verlag London Berlin Heidelberg
Springer Science+Business Media
springeronline.com

Reprint from English language edition:
Eigenvalues, Inequalities, and Ergodic Theory
by Mu-Fa Chen

Contents

Chapter 1

An Overview of the Book

This chapter is an overview of the book, especially of the first eight chapters. It consists of four sections. In the first section, we explain what eigenvalues we are interested in and show the difficulties in studying the first (nontrivial) eigenvalue through elementary examples. The second section presents some new (dual) variational formulas and explicit bounds for the first eigenvalue of the Laplacian on Riemannian manifolds or Jacobi matrices (Markov chains), and explains the main idea of the proof, which is a probabilistic approach: the coupling methods. In the third section, we introduce some recent lower bounds of several basic inequalities, based on a generalization of Cheeger's approach which comes from Riemannian geometry. In the last section, a diagram of nine different types of ergodicity and a table of explicit criteria for them are presented. The criteria are motivated by the weighted Hardy inequality, which comes from harmonic analysis.

1.1 Introduction

Let me now explain what eigenvalue we are talking about.

Definition. The first (nontrivial) eigenvalue

Consider a tridiagonal matrix (or in probabilistic language, a birth–death process with state space $E = \{0, 1, 2, \ldots\}$ and Q-matrix)

$$Q = (q_{ij}) = \begin{pmatrix} -b_0 & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1 + b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2 + b_2) & b_2 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix},$$

where a_k, $b_k > 0$. Since the sum of each row equals 0, we have $Q\mathbf{1} = \mathbf{0} = 0 \cdot \mathbf{1}$, where $\mathbf{1}$ is the vector having elements 1 everywhere and $\mathbf{0}$ is the zero vector.

This means that the Q-matrix has an eigenvalue 0 with eigenvector $\mathbf{1}$. Next, consider the finite case $E_n = \{0, 1, \dots, n\}$. Then, the eigenvalues of $-Q$ are discrete: $0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n$. We are interested in the first (nontrivial) eigenvalue $\lambda_1 = \lambda_1 - \lambda_0 =: \text{gap}\,(Q)$ (also called the *spectral gap* of Q). In the infinite case, $\lambda_1 := \inf\{\{\text{Spectrum of } (-Q)\} \setminus \{0\}\}$ can be 0. Certainly, one can consider a self-adjoint elliptic operator in $\mathbb{R}^d$ or the Laplacian Δ on manifolds or an infinite-dimensional operator as in the study of interacting particle systems.

Since the spectral theory is of central importance in many branches of mathematics and the first nontrivial eigenvalue is the leading term of the spectrum, it should not be surprising that the study of λ_1 has a very wide range of applications.

Difficulties

To get a concrete feeling about the difficulties of the topic, let us look at the following examples with finite state spaces.

When $E = \{0, 1\}$, it is trivial that $\lambda_1 = a_1 + b_0$. Everyone is happy to see this result, since if either a_1 or b_0 increases, so does λ_1. If we go one more step, $E = \{0, 1, 2\}$, then we have four parameters, b_0, b_1 and a_1, a_2. In this case, $\lambda_1 = 2^{-1}\big[a_1 + a_2 + b_0 + b_1 - \sqrt{(a_1 - a_2 + b_0 - b_1)^2 + 4a_1 b_1}\,\big]$. It is disappointing to see this result, since parameters effect on λ_1 is not clear at all. When $E = \{0, 1, 2, 3\}$, we have six parameters: $b_0, b_1, b_2, a_1, a_2, a_3$. The solution is expressed by the three quantities B, C, and D:

$$\lambda_1 = \frac{D}{3} - \frac{C}{3 \cdot 2^{1/3}} + \frac{2^{1/3}\,(3\,B - D^2)}{3\,C},$$

where the quantities D, B, and C are not too complicated:

$$\begin{aligned}
D &= a_1 + a_2 + a_3 + b_0 + b_1 + b_2,\\
B &= a_3\, b_0 + a_2\, (a_3 + b_0) + a_3\, b_1 + b_0\, b_1 + b_0\, b_2 + b_1\, b_2 + a_1\, (a_2 + a_3 + b_2),\\
C &= \Big(A + \sqrt{4(3\,B - D^2)^3 + A^2}\,\Big)^{1/3}.
\end{aligned}$$

However, in the last expression, another quantity, A, is involved. What, then, is A?

$$\begin{aligned}
A = &-2\,a_1^3 - 2\,a_2^3 - 2\,a_3^3 + 3\,a_3^2 b_0 + 3a_3 b_0^2 - 2b_0^3 + 3a_3^2 b_1 - 12\,a_3\, b_0 b_1 + 3b_0^2 b_1\\
&+ 3\,a_3\, b_1^2 + 3\,b_0\, b_1^2 - 2\,b_1^3 - 6\,a_3^2 b_2 + 6\,a_3 b_0 b_2 + 3\,b_0^2 b_2 + 6\,a_3 b_1 b_2 - 12\,b_0 b_1 b_2\\
&+ 3b_1^2\, b_2 - 6a_3 b_2^2 + 3b_0\, b_2^2 + 3b_1 b_2^2 - 2b_2^3 + 3a_1^2\,(a_2 + a_3 - 2\,b_0 - 2\,b_1 + b_2)\\
&+ 3\,a_2^2\,[a_3 + b_0 - 2\,(b_1 + b_2)]\\
&+ 3a_2\,\big[a_3^2 + b_0^2 - 2\,b_1^2 - b_1\, b_2 - 2b_2^2 - a_3(4b_0 - 2b_1 + b_2) + 2b_0(b_1 + b_2)\big]\\
&+ 3\,a_1\,\big[a_2^2 + a_3^2 - 2\,b_0^2 - b_0\, b_1 - 2\,b_1^2 - a_2(4\,a_3 - 2\,b_0 + b_1 - 2\,b_2)\\
&+ 2\,b_0\, b_2 + 2\,b_1\, b_2 + b_2^2 + 2\,a_3(b_0 + b_1 + b_2)\big],
\end{aligned}$$

computed using Mathematica. One should be shocked, at least I was, to see this result, since the roles of the parameters are completely hidden! Of course, everyone understands that it is impossible to compute λ_1 explicitly when the size of the matrix is greater than five!

Now, how about the estimation of λ_1? To see this, let us consider the perturbation of the eigenvalues and eigenfunctions. We consider the infinite state space $E = \{0, 1, 2, \ldots\}$. Denote by g and Degree(g), respectively, the eigenfunction of λ_1 and the degree of g when g is polynomial. Three examples of the perturbation of λ_1 and Degree(g) are listed in Table 1.1.

Table 1.1 Three examples of the perturbation of λ_1 and Degree(g)

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	**Degree** (g)
$i + c\,(c > 0)$	$2i$	1	1
$i + 1$	$2i + 3$	2	2
$i + 1$	$2i + (4 + \sqrt{2})$	3	3

The first line is the well-known linear model, for which $\lambda_1 = 1$, independent of the constant $c > 0$, and g is linear. Next, keeping the same birth rate, $b_i = i + 1$, the perturbation of the death rate a_i from $2i$ to $2i + 3$ (respectively, $2i + 4 + \sqrt{2}$) leads to the change of λ_1 from one to two (respectively, three). More surprisingly, the eigenfunction g is changed from linear to quadratic (respectively, cubic). For the intermediate values of a_i between $2i$, $2i + 3$, and $2i + 4 + \sqrt{2}$, λ_1 is unknown, since g is nonpolynomial. As seen from these examples, the first eigenvalue is very sensitive. Hence, in general, it is very hard to estimate λ_1.

Hopefully, we have presented enough examples to show the extreme difficulties of the topic. Very fortunately, at last, we are able to present a complete solution to this problem in the present context. Please be patient; the result will be given only later.

For a long period, we did not know how to proceed. So we visited several branches of mathematics. Finally, we found that the topic was well studied in Riemannian geometry.

1.2 New variational formula for the first eigenvalue

A story of estimating λ_1 in geometry

Here is a short story about the study of λ_1 in geometry.

Consider the Laplacian Δ on a connected compact Riemannian manifold (M, g), where g is the Riemannian metric. The spectrum of Δ is discrete: $\cdots \leqslant -\lambda_2 \leqslant -\lambda_1 < -\lambda_0 = 0$ (may be repeated). Estimating these eigenvalues λ_k (especially λ_1) is an important chapter in modern geometry. As far as we know, five books, excluding books on general spectral theory, have been devoted to this topic: I. Chavel (1984), P.H. Bérard (1986), R. Schoen and S.T. Yau (1988), P. Li (1993), and C.Y. Ma (1993). About 2000 references are collected in the second quoted book. Thus, it is impossible for us to introduce an overview of what has been done in geometry. Instead, we would like to show the reader ten of the most beautiful lower bounds. For a manifold M, denote its dimension, diameter, and the lower bound of Ricci curvature by d, D, and K ($\text{Ricci}_M \geqslant Kg$), respectively. The simplest example is the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$, for which $D = \pi$ and $K = d-1$. We are interested in estimating λ_1 in terms of these three geometric quantities. It is relatively easy to obtain an upper bound by applying a test function $f \in C^1(M)$ to the classical variational formula

$$\lambda_1 = \inf\left\{\int_M \|\nabla f\|^2 \mathrm{d}x:\ f \in C^1(M),\ \int_M f\mathrm{d}x = 0,\ \int_M f^2 \mathrm{d}x = 1\right\}, \tag{1.0}$$

where "dx" is the Riemannian volume element. To obtain the lower bound, however, is much harder. In Table 1.2, we list ten of the strongest lower bounds that have been derived in the past, using various sophisticated methods.

Table 1.2 Ten lower bounds of λ_1

Author(s)	Lower bound	
A. Lichnerowicz (1958)	$\frac{d}{d-1}K, \quad K \geqslant 0$	(1.1)
P.H. Bérard, G. Besson, & S. Gallot (1985)	$d\left\{\frac{\int_0^{\pi/2}\cos^{d-1}t\mathrm{d}t}{\int_0^{D/2}\cos^{d-1}t\mathrm{d}t}\right\}^{2/d}, \quad K = d-1 > 0$	(1.2)
P. Li & S.T. Yau (1980)	$\frac{\pi^2}{2D^2}, \quad K \geqslant 0$	(1.3)
J.Q. Zhong & H.C. Yang (1984)	$\frac{\pi^2}{D^2}, \quad K \geqslant 0$	(1.4)
D.G. Yang (1999)	$\frac{\pi^2}{D^2} + \frac{K}{4}, \quad K \geqslant 0$	(1.5)
P. Li & S.T. Yau (1980)	$\frac{1}{D^2(d-1)\exp\left[1+\sqrt{1+16\alpha^2}\right]}, \quad K \leqslant 0$	(1.6)
K.R. Cai (1991)	$\frac{\pi^2}{D^2} + K, \quad K \leqslant 0$	(1.7)
D. Zhao (1999)	$\frac{\pi^2}{D^2} + 0.52K, \quad K \leqslant 0.$	(1.8)
H.C. Yang (1990) & F. Jia (1991)	$\frac{\pi^2}{D^2}e^{-\alpha}, \quad \text{if } d \geqslant 5, \quad K \leqslant 0$	(1.9)
H.C. Yang (1990) & F. Jia (1991)	$\frac{\pi^2}{2D^2}e^{-\alpha'}, \quad \text{if } 2 \leqslant d \leqslant 4, \quad K \leqslant 0$	(1.10)