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# Advanced Mechanics of Piezoelectricity

## 压电材料高等力学（英文版）

Qing-Hua Qin

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Advances in Materials and Mechanics

# Advanced Mechanics of Piezoelectricity

## 压电材料高等力学（英文版）

YADIAN CAILIAO GAODENG LIXUE

Qing-Hua Qin



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*Author*
Prof. Qing-Hua Qin
Research School of Engineering
Australian National University
Canberra, Australia
E-mail: qinghua.qin@anu.edu.au

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# Advances in Materials and Mechanics 6 (AMM 6)

# 材料与力学进展 6

# *Advances in Materials and Mechanics (AMM)*

# Preface

This book contains a comprehensive treatment of piezoelectric materials using linear electroelastic theory, the symplectic model, and various special solution methods. The volume summarizes the current state of practice and presents the most recent research outcomes in piezoelectricity. Our hope in preparing this book is to present a stimulating guide and then to attract interested readers and researchers to a new field that continues to provide fascinating and technologically important challenges. You will benefit from the authors' thorough coverage of general principles for each topic, followed by detailed mathematical derivations and worked examples as well as tables and figures in appropriate positions.

The study of piezoelectricity was initiated by Jacques Curie and Pierre Curie in 1880. They found that certain crystalline materials generate an electric charge proportional to a mechanical stress. Since then new theories and applications of the field have been constantly advanced. These advances have resulted in a great many publications including journal papers and monographs. Although many concepts and theories have been included in earlier monographs, numerous new developments in piezoelectricity over the last two decades have made it increasingly necessary to collect significant information and to present a unified treatment of these useful but scattered results. These results should be made available to professional engineers, research scientists, workers and postgraduate students in applied mechanics and material engineering.

The objective of this book is to fill this gap, so that readers can obtain a sound knowledge of the solution methods for piezoelectric materials. This volume details the development of solution methods for piezoelectric composites and is written for researchers, postgraduate students, and professional engineers in the areas of solid mechanics, physical science and engineering, applied mathematics, mechanical engineering, and materials science. Little mathematical knowledge besides the usual calculus is required, although conventional matrixes, vectors, and tensor presentations are used throughout the book.

Chapter 1 provides a brief description of piezocomposites and the linear theory of piezoelectric materials in order to establish notation and fundamental concepts for reference in later chapters. Chapter 2 presents various solution methods for piezoelectric composites which can be taken as a common source for subsequent chapters. It includes the potential function method, Lekhnitskii formalism, techniques of Fourier transformation, Trefftz finite element method, integral equation approach, shear-lag model, and symplectic method. Chapter 3 deals with problems of fibrous piezoelectric composites, beginning with a discussion of piezoelectric fiber push-out and pull-out, and ending with a brief description of the solution for a piezoelectric composite with an elliptic fiber. Chapter 4 is concerned with applications of Trefftz method to piezoelectric materials. Trefftz finite element method, Trefftz boundary element method, and Trefftz boundary-collocation method are presented. Chapter 5 describes some solutions of piezoelectric problems using a symplectic approach. Chapter 6 presents Saint-Venant decay analysis of piezoelectric materials by way of symplectic formulation and the state space method. Chapter 7 reviews solutions for piezoelectric materials containing penny-shaped cracks. Chapter 8 describes solution methods for functionally graded piezoelectric materials.

I am indebted to a number of individuals in academic circles and organizations who have contributed in different, but important, ways to the preparation of this book. In particular, I wish to extend appreciation to my postgraduate students for their assistance in preparing this book. Special thanks go to Ms. Jianbo Liu of Higher Education Press for her commitment to the publication of this book. Finally, we wish to acknowledge the individuals and organizations cited in the book for permission to use their materials.

I would be grateful if readers would be so kind as to send reports of any typographical and other errors, as well as their more general comments.

Qing-Hua Qin
Canberra, Australia
May 2012

# Notation

## English symbols

| | |
|---|---|
| $a_{ij}$, $b_{ij}$ | reduced material constants defined in Eq. (1.26) |
| $B_i$ | magnetic flux |
| $\mathbf{c}_i$ | unknown coefficients in Eq. (4.9) and elastic stiffness constants in Chapter 3 |
| $c_{ijkl}$, $c_{ij}$ | elastic stiffness constants |
| $d_{ij}$ | piezoelectric charge constants |
| $D_i$ | electric displacements |
| $e_{ijk}$, $e_{ij}$ | piezoelectric constants |
| $\tilde{e}_{ij}$ | piezomagnetic coefficient |
| $E_i$ | electric field |
| $f_i$ | mechanical body forces |
| $f_{ij}$ | elastic compliances |
| $g_{ij}$ | piezoelectric voltage constants |
| $H_i$ | magnetic field intensity |
| $m_{ij}$ | reduced material constants defined in Eq. (6.5) |
| $q_s$ | surface charge |
| $Q$ | electric charge density |
| $t_i$ | surface tractions |
| $u$, $v$, $w$ | displacement in $x$, $y$, $z$ directions, respectively |
| $u_i$ | displacements |

## Greek symbols

| | |
|---|---|
| $\alpha_{ij}$ | magnetoelectric coupling coefficient |
| $\Delta$ | $=c_{55}\kappa_{11}+e_{15}^2$ |
| $\Delta_n$ | $=c_{55}^{(n)}\kappa_{11}^{(n)}+\left(e_{15}^{(n)}\right)^2$ for $n=0,1,2,\cdots$, defined in Eq. (5.73) |
| $\Delta_*$ | $=\beta_{11}\nu_{11}-\lambda_{11}^2$ defined in Eq. (5.119) |
| $\varepsilon_{ij}$ | elastic strains |
| $\theta$ | temperature change |
| $\kappa_{ij}$ | dielectric constants |
| $\mu_{ij}$ | magnetic permeability |
| $\sigma_{ij}$ | stresses |
| $\upsilon$ | Poisson's ratio |
| $\phi$ | electric potential |
| $\psi$ | magnetic potential |

**Other symbols**

| | |
|---|---|
| $\partial/\partial x$ | partial derivative of a variable with respect to $x$ |
| [ ] | denotes a rectangular or a square matrix |
| { } | denotes a column vector |
| $[\,]^{-1}$ | denotes the inverse of a matrix |
| $[\,]^{\mathrm{T}}$ | denotes the transpose of a matrix |
| $(\bar{\ })$ | a bar over a variable represents the variable being prescribed or complex conjugate |
| $\nabla$ | $=\partial^2/\partial x^2+\partial^2/\partial y^2$ |

# Contents

**Chapter 1 Introduction to Piezoelectricity** ··· 1

1.1 Background ··· 1
1.2 Linear theory of piezoelectricity ··· 4
1.2.1 Basic equations in rectangular coordinate system ··· 4
1.2.2 Boundary conditions ··· 7
1.3 Functionally graded piezoelectric materials ··· 8
1.3.1 Types of gradation ··· 9
1.3.2 Basic equations for two-dimensional FGPMs ··· 9
1.4 Fibrous piezoelectric composites ··· 11
References ··· 17

**Chapter 2 Solution Methods** ··· 21

2.1 Potential function method ··· 21
2.2 Solution with Lekhnitskii formalism ··· 23
2.3 Techniques of Fourier transformation ··· 28
2.4 Trefftz finite element method ··· 31
2.4.1 Basic equations ··· 31
2.4.2 Assumed fields ··· 31
2.4.3 Element stiffness equation ··· 33
2.5 Integral equations ··· 34
2.5.1 Fredholm integral equations ··· 34
2.5.2 Volterra integral equations ··· 36
2.5.3 Abel's integral equation ··· 37
2.6 Shear-lag model ··· 39
2.7 Hamiltonian method and symplectic mechanics ··· 41
2.8 State space formulation ··· 47
References ··· 51

**Chapter 3 Fibrous Piezoelectric Composites** ··· 53

3.1 Introduction ··· 53
3.2 Basic formulations for fiber push-out and pull-out tests ··· 55
3.3 Piezoelectric fiber pull-out ··· 59
3.3.1 Relationships between matrix stresses and interfacial shear stress ··· 59
3.3.2 Solution for bonded region ··· 61
3.3.3 Solution for debonded region ··· 62

3.3.4 Numerical results ······ 63
3.4 Piezoelectric fiber push-out ······ 63
3.4.1 Stress transfer in the bonded region ······ 64
3.4.2 Frictional sliding ······ 66
3.4.3 PFC push-out driven by electrical and mechanical loading ······ 69
3.4.4 Numerical assessment ······ 70
3.5 Interfacial debonding criterion ······ 76
3.6 Micromechanics of fibrous piezoelectric composites ······ 81
3.6.1 Overall elastoelectric properties of FPCs ······ 81
3.6.2 Extension to include magnetic and thermal effects ······ 89
3.7 Solution of composite with elliptic fiber ······ 94
3.7.1 Conformal mapping ······ 94
3.7.2 Solutions for thermal loading applied outside an elliptic fiber ······ 95
3.7.3 Solutions for holes and rigid fibers ······ 104
References ······ 105

**Chapter 4 Trefftz Method for Piezoelectricity ······ 109**

4.1 Introduction ······ 109
4.2 Trefftz FEM for generalized plane problems ······ 109
4.2.1 Basic field equations and boundary conditions ······ 109
4.2.2 Assumed fields ······ 111
4.2.3 Modified variational principle ······ 113
4.2.4 Generation of the element stiffness equation ······ 115
4.2.5 Numerical results ······ 117
4.3 Trefftz FEM for anti-plane problems ······ 118
4.3.1 Basic equations for deriving Trefftz FEM ······ 118
4.3.2 Trefftz functions ······ 119
4.3.3 Assumed fields ······ 119
4.3.4 Special element containing a singular corner ······ 121
4.3.5 Generation of element matrix ······ 123
4.3.6 Numerical examples ······ 125
4.4 Trefftz boundary element method for anti-plane problems ······ 127
4.4.1 Indirect formulation ······ 127
4.4.2 The point-collocation formulations of Trefftz boundary element method ······ 129
4.4.3 Direct formulation ······ 129
4.4.4 Numerical examples ······ 132
4.5 Trefftz boundary-collocation method for plane piezoelectricity ······ 137
4.5.1 General Trefftz solution sets ······ 137
4.5.2 Special Trefftz solution set for a problem with elliptic holes ······ 138

4.5.3 Special Trefftz solution set for impermeable crack problems ····· 140
4.5.4 Special Trefftz solution set for permeable crack problems ········ 142
4.5.5 Boundary collocation formulation ······ 144
References ······ 145

**Chapter 5 Symplectic Solutions for Piezoelectric Materials** ······ 149

5.1 Introduction ······ 149
5.2 A symplectic solution for piezoelectric wedges ······ 150
5.2.1 Hamiltonian system by differential equation approach ······ 150
5.2.2 Hamiltonian system by variational principle approach ······ 153
5.2.3 Basic eigenvalues and singularity of stress and electric fields ···· 154
5.2.4 Piezoelectric bimaterial wedge ······ 159
5.2.5 Multi-piezoelectric material wedge ······ 162
5.3 Extension to include magnetic effect ······ 166
5.3.1 Basic equations and their Hamiltonian system ······ 166
5.3.2 Eigenvalues and eigenfunctions ······ 167
5.3.3 Particular solutions ······ 170
5.4 Symplectic solution for a magnetoelectroelastic strip ······ 171
5.4.1 Basic equations ······ 171
5.4.2 Hamiltonian principle ······ 172
5.4.3 The zero-eigenvalue solutions ······ 175
5.4.4 Nonzero-eigenvalue solutions ······ 179
5.5 Three-dimensional symplectic formulation for piezoelectricity ······ 182
5.5.1 Basic formulations ······ 182
5.5.2 Hamiltonian dual equations ······ 183
5.5.3 The zero-eigenvalue solutions ······ 184
5.5.4 Sub-symplectic system ······ 187
5.5.5 Nonzero-eigenvalue solutions ······ 190
5.6 Symplectic solution for FGPMs ······ 192
5.6.1 Basic formulations ······ 192
5.6.2 Eigenvalue properties of the Hamiltonian matrix **H** ······ 194
5.6.3 Eigensolutions corresponding to $\mu$=0 and–$\alpha$ ······ 194
5.6.4 Extension to the case of magnetoelectroelastic materials ······ 197
References ······ 201

**Chapter 6 Saint-Venant Decay Problems in Piezoelectricity** ······ 205

6.1 Introduction ······ 205
6.2 Saint-Venant end effects of piezoelectric strips ······ 206
6.2.1 Hamiltonian system for a piezoelectric strip ······ 206
6.2.2 Decay rate analysis ······ 211

6.2.3 Numerical illustration ······216
6.3 Saint-Venant decay in anti-plane dissimilar laminates ······218
6.3.1 Basic equations for anti-plane piezoelectric problem ······218
6.3.2 Mixed-variable state space formulation ······219
6.3.3 Decay rate of FGPM strip ······220
6.3.4 Two-layered FGPM laminates and dissimilar piezoelectric laminates ······226
6.4 Saint-Venant decay in multilayered piezoelectric laminates ······231
6.4.1 State space formulation ······231
6.4.2 Eigensolution and decay rate equation ······234
6.5 Decay rate of piezoelectric-piezomagnetic sandwich structures ······237
6.5.1 Basic equations and notations in multilayered structures ······238
6.5.2 Space state differential equations for analyzing decay rate ······239
6.5.3 Solutions to the space state differential equations ······241
References ······246

**Chapter 7 Penny-Shaped Cracks** ······249

7.1 Introduction ······249
7.2 An infinite piezoelectric material with a penny-shaped crack ······250
7.3 A penny-shaped crack in a piezoelectric strip ······255
7.4 A fiber with a penny-shaped crack embedded in a matrix ······258
7.5 Fundamental solution for penny-shaped crack problem ······263
7.5.1 Potential approach ······263
7.5.2 Solution for crack problem ······265
7.5.3 Fundamental solution for penny-shaped crack problem ······266
7.6 A penny-shaped crack in a piezoelectric cylinder ······268
7.6.1 Problem statement and basic equation ······269
7.6.2 Derivation of integral equations and their solution ······272
7.6.3 Numerical results and discussion ······277
7.7 A fiber with a penny-shaped crack and an elastic coating ······279
7.7.1 Formulation of the problem ······279
7.7.2 Fredholm integral equation of the problem ······285
7.7.3 Numerical results and discussion ······286
References ······287

**Chapter 8 Solution Methods for Functionally Graded Piezoelectric Materials** ······291

8.1 Introduction ······291
8.2 Singularity analysis of angularly graded piezoelectric wedge ······292
8.2.1 Basic formulations and the state space equation ······292

8.2.2 Two AGPM wedges ··· 298
8.2.3 AGPM-EM-AGPM wedge system ··· 301
8.2.4 Numerical results and discussion ··· 303
8.3 Solution to FGPM beams ··· 308
8.3.1 Basic formulation ··· 308
8.3.2 Solution procedure ··· 308
8.4 Parallel cracks in an FGPM strip ··· 312
8.4.1 Basic formulation ··· 313
8.4.2 Singular integral equations and field intensity factors ··· 315
8.5 Mode Ⅲ cracks in two bonded FGPMs ··· 318
8.5.1 Basic formulation of the problem ··· 318
8.5.2 Impermeable crack problem ··· 320
8.5.3 Permeable crack problem ··· 324
References ··· 325

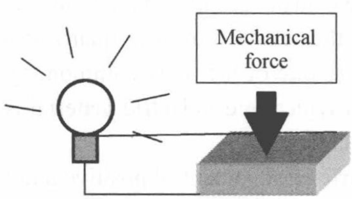
**Index** ··· 329

# Chapter 1 Introduction to Piezoelectricity

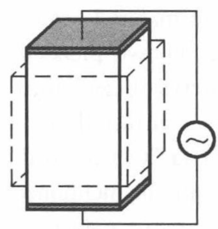
This chapter provides a basic introduction to piezoelectricity. It begins with a discussion of background and applications of piezoelectric materials. We then present the linear theory of piezoelectricity, functionally graded piezoelectric materials(FGPM), and fundamental knowledge of fibrous piezoelectric composites(FPC).

## 1.1 Background

Piezoelectric material is such that when it is subjected to a mechanical load, it generates an electric charge (see Fig. 1.1(a)). This effect is usually called the "piezoelectric effect". Conversely, when piezoelectric material is stressed electrically by a voltage, its dimensions change (see Fig. 1.1(b)). This phenomenon is known as the "inverse piezoelectric effect". The direct piezoelectric effect was first discovered by the brothers Pierre Curie and Jacques Curie more than a century ago [1]. They found out that when a mechanical stress was applied to crystals such as tourmaline, topaz, quartz, Rochelle salt and cane sugar, electrical charges appeared, and this voltage was proportional to the stress.




(a)



(b)

**Fig. 1.1** Electroelastic coupling in piezoelectricity.(a) Piezoelectric effect: voltage induced by force. (b) Inverse piezoelectric effect: strain induced by voltage.

The Curies did not, however, predict that crystals exhibiting the direct piezoelectric effect (electricity from applied stress) would also exhibit the inverse piezoelectric effect (strain in response to applied electric field). One year later that property was theoretically predicted on the basis of thermodynamic consideration by Lippmann [2], who proposed that converse effects must exist for piezoelectricity, pyroelectricity (see Fig. 1.2), etc. Subsequently, the inverse piezoelectric effect was confirmed experimentally by Curies [3], who proceeded to obtain quantitative proof of the complete reversibility of electromechanical deformations in piezoelectric crystals. These events above can be viewed as the beginning of the history of piezoelectricity. Based on them, Woldemar Voigt [4] developed the first complete and rigorous formulation of piezoelectricity in 1890. Since then several books on the phenomenon and theory of piezoelectricity have been published. Among them are the books by Cady [5], Tiersten [6], Parton and Kudryavtsev [7], Ikeda [8], Rogacheva [9], Qin [10,11], and Qin and Yang [12]. The first [5] treated the physical properties of piezoelectric crystals as well as their practical applications, the second [6] dealt with the linear equations of vibrations in piezoelectric materials, and the third and fourth [7,8] gave a more detailed description of the physical properties of piezoelectricity. Rogacheva [9] presented general theories of piezoelectric shells. Qin [10,11] discussed Green's functions and fracture mechanics of piezoelectric materials. Micromechanics of piezoelectricity were discussed in [12].

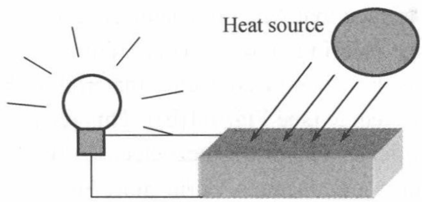



**Fig. 1.2** Illustration of pyroelectricity.

In general, the piezoelectric effect occurs only in nonconductive materials. Piezoelectric materials can be divided into two main groups: crystals and ceramics. The best known piezoelectric material in the crystal group is quartz ($SiO_2$), the trigonal crystallized silica which is known as one of the most common crystals on the earth's surface. In the ceramics group, a typical piezoelectric material is barium titanate ($BaTiO_3$), an oxide of barium and titanium.

It should be mentioned that an asymmetric arrangement of positive and negative ions imparts permanent electric dipole behavior to crystals. In order to "activate" the piezo properties of ceramics, a poling treatment is required. In that treatment the piezo ceramic material is first heated and an intense electric field (> 2 000 V/mm) is applied to it in the poling direction, forcing the ions to realign along this "poling" axis. When the ceramic cools and the field is removed, the ions "remember" this poling and the material now has a remanent polarization (which can be degraded by exceeding the mechanical, thermal and electrical limits of the material). Subsequently, when a voltage is applied to the poled piezoelectric material, the ions in the unit cells are shifted and, additionally, the domains change their degree of alignment. The result is a corresponding change of the dimensions (expansion, contraction) of the lead zirconate titanate (PZT) material. In the poling treatment, the Curie temperature is the critical temperature at which the crystal structure changes from a nonsymmetrical (piezoelectric) to a symmetrical (non-piezoelectric) form. Particularly, when the temperature is above the Curie temperature, each perovskite crystal (perovskite is a calcium titanium oxide mineral species composed of calcium titanate, with the chemical formula $CaTiO_3$) in the fired ceramic element exhibits a simple cubic symmetry with no dipole moment (Fig. 1.3(a)). At temperatures below the Curie point, however, each crystal has tetragonal or rhombohedral symmetry and a dipole moment (Fig. 1.3(b)).

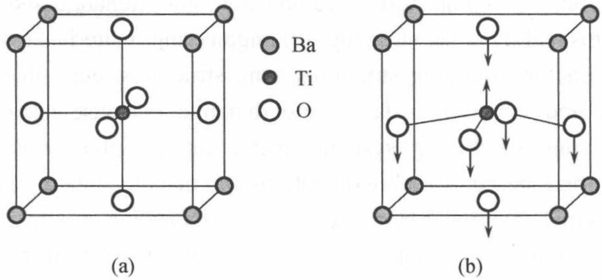



**Fig. 1.3** Crystal structures with the Curie temperature.(a) Temperature above Curie temperature: symmetric. (b) Temperature below Curie temperature: non-symmetric.

Although piezoelectricity was discovered in 1880 it remained a mere curiosity until the 1940s. The property of certain crystals to exhibit electrical charges under mechanical loading was of no practical use until very high input impedance amplifiers enabled engineers to amplify their signals. In 1951, several Japanese companies and universities formed a "competitively cooperative" association, established as the Barium Titanate Application Research Committee. This association set an organizational precedent not only for successfully surmounting technical challenges and manufacturing hurdles, but also for defining new market areas. Persistent efforts in materials research created new piezoceramic families which were competitive with Vernitron's PZT. With these materials available, Japanese manufacturers quickly developed several types of piezoelectric signal filters, which addressed needs arising from television, radio, and communications equipment markets; and piezoelectric igniters for natural gas/butane appliances. As time progressed, the markets for these products continued to grow, and other similarly lucrative ones were found. Most notable were audio buzzers (smoke alarms), air ultrasonic transducers (television remote controls and intrusion alarms) and devices employing surface acoustic wave effects to achieve high frequency signal filtering.

The commercial success of the Japanese efforts attracted the attention of industry in many other countries and spurred new efforts to develop successful piezoelectric products. There has been a large increase in relevant publications in China, India, Russia and the USA. Since the piezoelectric effect provides the ability to use these materials as both sensors and actuators, it has found relevant applications requiring accurate measurement and recording of dynamic changes in mechanical variables such as pressure, force and acceleration. The list of applications continues