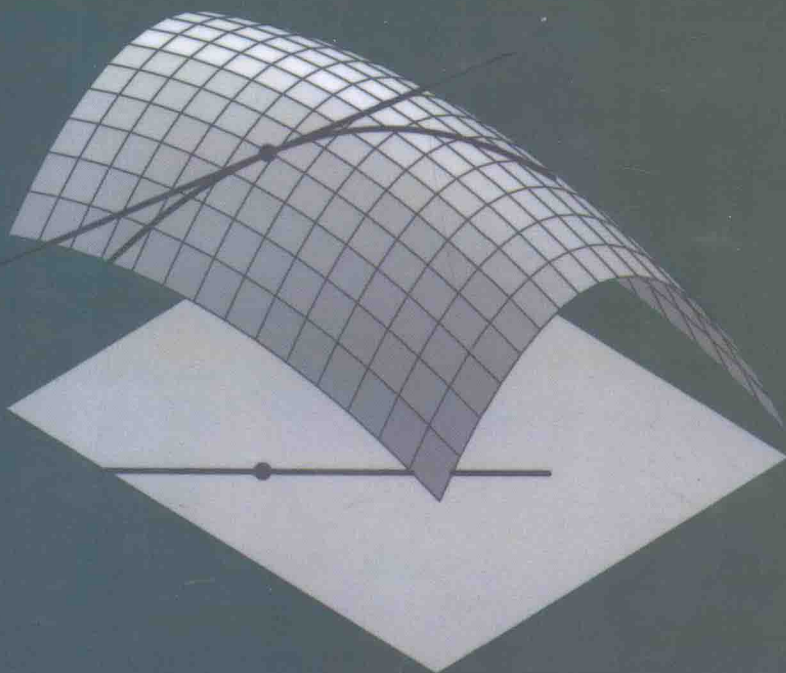


■ Herbert Amann
■ Joachim Escher

Analysis II

分析 第2卷



Birkhäuser

世界图书出版公司
www.wpcbj.com.cn

Herbert Amann
Joachim Escher

Analysis II

Translated from the German
by Silvio Levy and Matthew Cargo

Birkhäuser
Basel · Boston · Berlin

图书在版编目 (CIP) 数据

分析. 第2卷 = Analysis II: 英文 / (德) 阿莫恩 (Amann, H.) 著. —影印本.

—北京: 世界图书出版公司北京公司, 2012. 6

ISBN 978 - 7 - 5100 - 4799 - 2

I. ①分… II. ①阿… III. ①分析 (数学) —英文 IV. ①O17

中国版本图书馆 CIP 数据核字 (2012) 第 123568 号

书 名: Analysis II
作 者: Herbert Amann, Joachim Escher
中译名: 分析 第2卷
责任编辑: 高蓉 刘慧

出 版 者: 世界图书出版公司北京公司
印 刷 者: 三河市国英印务有限公司
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)
联系电话: 010 - 64021602, 010 - 64015659
电子信箱: kjb@wpcbj.com.cn

开 本: 16 开
印 张: 26
版 次: 2012 年 09 月
版权登记: 图字: 01 - 2012 - 4602

书 号: 978 - 7 - 5100 - 4799 - 2 定 价: 89.00 元

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Originally published in German under the same title by Birkhäuser Verlag, Switzerland
© 1999 by Birkhäuser Verlag

2000 Mathematics Subject Classification: 26-01, 26A42, 26Bxx, 30-01

Library of Congress Control Number: 2008926303

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <<http://dnb.ddb.de>>.

ISBN 3-7643-7472-3 Birkhäuser Verlag, Basel – Boston – Berlin

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Reprint from English language edition:

Analysis II

by Herbert Amann and Joachim Escher

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Foreword

As with the first, the second volume contains substantially more material than can be covered in a one-semester course. Such courses may omit many beautiful and well-grounded applications which connect broadly to many areas of mathematics. We of course hope that students will pursue this material independently; teachers may find it useful for undergraduate seminars.

For an overview of the material presented, consult the table of contents and the chapter introductions. As before, we stress that doing the numerous exercises is indispensable for understanding the subject matter, and they also round out and amplify the main text.

In writing this volume, we are indebted to the help of many. We especially thank our friends and colleagues Pavol Quittner and Gieri Simonett. They have not only meticulously reviewed the entire manuscript and assisted in weeding out errors but also, through their valuable suggestions for improvement, contributed essentially to the final version. We also extend great thanks to our staff for their careful perusal of the entire manuscript and for tracking errata and inaccuracies.

Our most heartfelt thank extends again to our “typesetting perfectionist”, without whose tireless effort this book would not look nearly so nice.¹ We also thank Andreas for helping resolve hardware and software problems.

Finally, we extend thanks to Thomas Hintermann and to Birkhäuser for the good working relationship and their understanding of our desired deadlines.

Zürich and Kassel, March 1999

H. Amann and J. Escher

¹The text was set in L^AT_EX, and the figures were created with CorelDRAW! and Maple.

Foreword to the second edition

In this version, we have corrected errors, resolved imprecisions, and simplified several proofs. These areas for improvement were brought to our attention by readers. To them and to our colleagues H. Crauel, A. Ilchmann and G. Prokert, we extend heartfelt thanks.

Zürich and Hannover, December 2003

H. Amann and J. Escher

Foreword to the English translation

It is a pleasure to express our gratitude to Silvio Levy and Matt Cargo for their careful and insightful translation of the original German text into English. Their effective and pleasant cooperation during the process of translation is highly appreciated.

Zürich and Hannover, March 2008

H. Amann and J. Escher

Contents

Foreword	v
Chapter VI Integral calculus in one variable	
1 Jump continuous functions	4
Staircase and jump continuous functions	4
A characterization of jump continuous functions	6
The Banach space of jump continuous functions	7
2 Continuous extensions	10
The extension of uniformly continuous functions	10
Bounded linear operators	12
The continuous extension of bounded linear operators	15
3 The Cauchy–Riemann Integral	17
The integral of staircase functions	17
The integral of jump continuous functions	19
Riemann sums	20
4 Properties of integrals	25
Integration of sequences of functions	25
The oriented integral	26
Positivity and monotony of integrals	27
Componentwise integration	30
The first fundamental theorem of calculus	30
The indefinite integral	32
The mean value theorem for integrals	33
5 The technique of integration	38
Variable substitution	38
Integration by parts	40
The integrals of rational functions	43

6	Sums and integrals	50
	The Bernoulli numbers	50
	Recursion formulas	52
	The Bernoulli polynomials	53
	The Euler–Maclaurin sum formula	54
	Power sums	56
	Asymptotic equivalence	57
	The Riemann ζ function	59
	The trapezoid rule	64
7	Fourier series	67
	The L_2 scalar product	67
	Approximating in the quadratic mean	69
	Orthonormal systems	71
	Integrating periodic functions	72
	Fourier coefficients	73
	Classical Fourier series	74
	Bessel’s inequality	77
	Complete orthonormal systems	79
	Piecewise continuously differentiable functions	82
	Uniform convergence	83
8	Improper integrals	90
	Admissible functions	90
	Improper integrals	90
	The integral comparison test for series	93
	Absolutely convergent integrals	94
	The majorant criterion	95
9	The gamma function	98
	Euler’s integral representation	98
	The gamma function on $\mathbb{C} \setminus (-\mathbb{N})$	99
	Gauss’s representation formula	100
	The reflection formula	104
	The logarithmic convexity of the gamma function	105
	Stirling’s formula	108
	The Euler beta integral	110

Chapter VII Multivariable differential calculus

1	Continuous linear maps	118
	The completeness of $\mathcal{L}(E, F)$	118
	Finite-dimensional Banach spaces	119
	Matrix representations	122
	The exponential map	125
	Linear differential equations	128
	Gronwall's lemma	129
	The variation of constants formula	131
	Determinants and eigenvalues	133
	Fundamental matrices	136
	Second order linear differential equations	140
2	Differentiability	149
	The definition	149
	The derivative	150
	Directional derivatives	152
	Partial derivatives	153
	The Jacobi matrix	155
	A differentiability criterion	156
	The Riesz representation theorem	158
	The gradient	159
	Complex differentiability	162
3	Multivariable differentiation rules	166
	Linearity	166
	The chain rule	166
	The product rule	169
	The mean value theorem	169
	The differentiability of limits of sequences of functions	171
	Necessary condition for local extrema	171
4	Multilinear maps	173
	Continuous multilinear maps	173
	The canonical isomorphism	175
	Symmetric multilinear maps	176
	The derivative of multilinear maps	177
5	Higher derivatives	180
	Definitions	180
	Higher order partial derivatives	183
	The chain rule	185
	Taylor's formula	185

	Functions of m variables	186
	Sufficient criterion for local extrema	188
6	Nemytskii operators and the calculus of variations	195
	Nemytskii operators	195
	The continuity of Nemytskii operators	195
	The differentiability of Nemytskii operators	197
	The differentiability of parameter-dependent integrals	200
	Variational problems	202
	The Euler–Lagrange equation	204
	Classical mechanics	207
7	Inverse maps	212
	The derivative of the inverse of linear maps	212
	The inverse function theorem	214
	Diffeomorphisms	217
	The solvability of nonlinear systems of equations	218
8	Implicit functions	221
	Differentiable maps on product spaces	221
	The implicit function theorem	223
	Regular values	226
	Ordinary differential equations	226
	Separation of variables	229
	Lipschitz continuity and uniqueness	233
	The Picard–Lindelöf theorem	235
9	Manifolds	242
	Submanifolds of $\mathbb{R}^n$	242
	Graphs	243
	The regular value theorem	243
	The immersion theorem	244
	Embeddings	247
	Local charts and parametrizations	252
	Change of charts	255
10	Tangents and normals	260
	The tangential in $\mathbb{R}^n$	260
	The tangential space	261
	Characterization of the tangential space	265
	Differentiable maps	266
	The differential and the gradient	269
	Normals	271
	Constrained extrema	272
	Applications of Lagrange multipliers	273

Chapter VIII Line integrals

1	Curves and their lengths	281
	The total variation	281
	Rectifiable paths	282
	Differentiable curves	284
	Rectifiable curves	286
2	Curves in $\mathbb{R}^n$	292
	Unit tangent vectors	292
	Parametrization by arc length	293
	Oriented bases	294
	The Frenet n -frame	295
	Curvature of plane curves	298
	Identifying lines and circles	300
	Instantaneous circles along curves	300
	The vector product	302
	The curvature and torsion of space curves	303
3	Pfaff forms	308
	Vector fields and Pfaff forms	308
	The canonical basis	310
	Exact forms and gradient fields	312
	The Poincaré lemma	314
	Dual operators	316
	Transformation rules	317
	Modules	321
4	Line integrals	326
	The definition	326
	Elementary properties	328
	The fundamental theorem of line integrals	330
	Simply connected sets	332
	The homotopy invariance of line integrals	333
5	Holomorphic functions	339
	Complex line integrals	339
	Holomorphism	342
	The Cauchy integral theorem	343
	The orientation of circles	344
	The Cauchy integral formula	345
	Analytic functions	346
	Liouville's theorem	348
	The Fresnel integral	349
	The maximum principle	350

Harmonic functions	351
Goursat's theorem	353
The Weierstrass convergence theorem	356
6 Meromorphic functions	360
The Laurent expansion	360
Removable singularities	364
Isolated singularities	365
Simple poles	368
The winding number	370
The continuity of the winding number	374
The generalized Cauchy integral theorem	376
The residue theorem	378
Fourier integrals	379
References	387
Index	389

Chapter VI

Integral calculus in one variable

Integration was invented for finding the area of shapes. This, of course, is an ancient problem, and the basic strategy for solving it is equally old: divide the shape into rectangles and add up their areas.

A mathematically satisfactory realization of this clear, intuitive idea is amazingly subtle. We note in particular that there is a vast number of ways a given shape can be approximated by a union of rectangles. It is not at all self-evident they all lead to the same result. For this reason, we will not develop the rigorous theory of measures until Volume III.

In this chapter, we will consider only the simpler case of determining the area between the graph of a sufficiently regular function of one variable and its axis. By laying the groundwork for approximating a function by a juxtaposed series of rectangles, we will see that this boils down to approaching the function by a series of staircase functions, that is, functions that are piecewise constant. We will show that this idea for approximations is extremely flexible and is independent of its original geometric motivation, and we will arrive at a concept of integration that applies to a large class of vector-valued functions of a real variable.

To determine precisely the class of functions to which we can assign an integral, we must examine which functions can be approximated by staircase functions. By studying the convergence under the supremum norm, that is, by asking if a given function can be approximated uniformly on the entire interval by staircase functions, we are led to the class of jump continuous functions. Section 1 is devoted to studying this class.

There, we will see that an integral is a linear map on the vector space of staircase functions. There is then the problem of extending integration to the space of jump continuous functions; the extension should preserve the elementary properties of this map, particularly linearity. This exercise turns out to be a special case of the general problem of uniquely extending continuous maps. Because the extension problem is of great importance and enters many areas of mathematics, we

will discuss it at length in Section 2. From the fundamental extension theorem for uniformly continuous maps, we will derive the theorem of continuous extensions of continuous linear maps. This will give us an opportunity to introduce the important concepts of bounded linear operators and the operator norm, which play a fundamental role in modern analysis.

After this groundwork, we will introduce in Section 3 the integral of jump continuous functions. This, the Cauchy–Riemann integral, extends the elementary integral of staircase functions. In the sections following, we will derive its fundamental properties. Of great importance (and you can tell by the name) is the fundamental theorem of calculus, which, to oversimplify, says that integration reverses differentiation. Through this theorem, we will be able to explicitly calculate a great many integrals and develop a flexible technique for integration. This will happen in Section 5.

In the remaining sections — except for the eighth — we will explore applications of the so-far developed differential and integral calculus. Since these are not essential for the overall structure of analysis, they can be skipped or merely sampled on first reading. However, they do contain many of the beautiful results of classical mathematics, which are needed not only for one's general mathematical literacy but also for numerous applications, both inside and outside of mathematics.

Section 6 will explore the connection between integrals and sums. We derive the Euler–Maclaurin sum formula and point out some of its consequences. Special mention goes to the proof of the formulas of de Moivre and Sterling, which describe the asymptotic behavior of the factorial function, and also to the derivation of several fundamental properties of the famous Riemann ζ function. The latter is important in connection to the asymptotic behavior of the distribution of prime numbers, which, of course, we can go into only very briefly.

In Section 7, we will revive the problem — mentioned at the end of Chapter V — of representing periodic functions by trigonometric series. With help from the integral calculus, we can specify a complete solution of this problem for a large class of functions. We place the corresponding theory of Fourier series in the general framework of the theory of orthogonality and inner product spaces. Thereby we achieve not only clarity and simplicity but also lay the foundation for a number of concrete applications, many of which you can expect see elsewhere. Naturally, we will also calculate some classical Fourier series explicitly, leading to some surprising results. Among these is the formula of Euler, which gives an explicit expression for the ζ function at even arguments; another is an interesting expression for the sine as an infinite product.

Up to this point, we have will have concentrated on the integration of jump continuous functions on compact intervals. In Section 8, we will further extend the domain of integration to cover functions that are defined (and integrated) on infinite intervals or are not bounded. We content ourselves here with simple but important results which will be needed for other applications in this volume

because, in Volume III, we will develop an even broader and more flexible type of integral, the Lebesgue integral.

Section 9 is devoted to the theory of the gamma function. This is one of the most important nonelementary functions, and it comes up in many areas of mathematics. Thus we have tried to collect all the essential results, and we hope you will find them of value later. This section will show in a particularly nice way the strength of the methods developed so far.

1 Jump continuous functions

In many concrete situations, particularly in the integral calculus, the constraint of continuity turns out to be too restrictive. Discontinuous functions emerge naturally in many applications, although the discontinuity is generally not very pathological. In this section, we will learn about a simple class of maps which contains the continuous functions and is especially useful in the integral calculus in one independent variable. However, we will see later that the space of jump continuous functions is still too restrictive for a flexible theory of integration, and, in the context of multidimensional integration, we will have to extend the theory into an even broader class containing the continuous functions.

In the following, suppose

- $E := (E, \|\cdot\|)$ is a Banach space;
- $I := [\alpha, \beta]$ is a compact perfect interval.

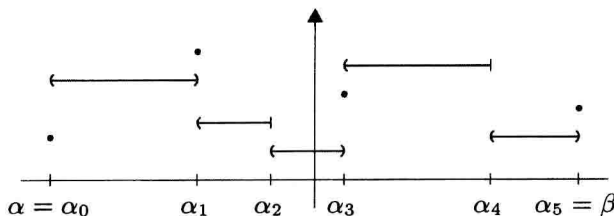
Staircase and jump continuous functions

We call $\mathfrak{J} := (\alpha_0, \dots, \alpha_n)$ a **partition** of I , if $n \in \mathbb{N}^\times$ and

$$\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_n = \beta.$$

If $\{\alpha_0, \dots, \alpha_n\}$ is a subset of the partition $\bar{\mathfrak{J}} := (\beta_0, \dots, \beta_k)$, $\bar{\mathfrak{J}}$ is called a **refinement** of $\mathfrak{J}$, and we write $\mathfrak{J} \leq \bar{\mathfrak{J}}$.

The function $f : I \rightarrow E$ is called a **staircase function** on I if I has a partition $\mathfrak{J} := (\alpha_0, \dots, \alpha_n)$ such that f is constant on every (open) interval (α_{j-1}, α_j) . Then we say $\mathfrak{J}$ is a partition for f , or we say f is a staircase function on the partition $\mathfrak{J}$.



A staircase function

If $f : I \rightarrow E$ is such that the limits $f(\alpha + 0)$, $f(\beta - 0)$, and

$$f(x \pm 0) := \lim_{\substack{y \rightarrow x \pm 0 \\ y \neq x}} f(y)$$

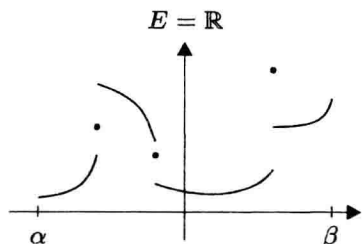
exist for all $x \in \overset{\circ}{I}$, we call f **jump continuous**.¹ A jump continuous function is **piecewise continuous** if it has only finitely many discontinuities ("jumps"). Finally,

¹Note that, in general, $f(x + 0)$ and $f(x - 0)$ may differ from $f(x)$.

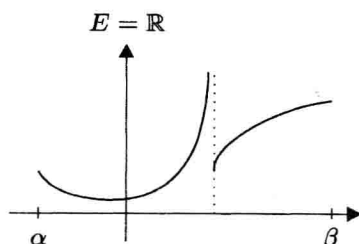
we denote by

$$\mathcal{T}(I, E), \quad \mathcal{S}(I, E), \quad \mathcal{SC}(I, E)$$

the sets of all functions $f: I \rightarrow E$ that are staircase, jump continuous, and piecewise continuous, respectively.²



A piecewise continuous function



Not a jump continuous function

1.1 Remarks (a) Given partitions $\mathfrak{J} := (\alpha_0, \dots, \alpha_n)$ and $\bar{\mathfrak{J}} := (\beta_0, \dots, \beta_m)$ of I , the union $\{\alpha_0, \dots, \alpha_n\} \cup \{\beta_0, \dots, \beta_m\}$ will naturally define another partition $\mathfrak{J} \vee \bar{\mathfrak{J}}$ of I . Obviously, $\mathfrak{J} \leq \mathfrak{J} \vee \bar{\mathfrak{J}}$ and $\bar{\mathfrak{J}} \leq \mathfrak{J} \vee \bar{\mathfrak{J}}$. In fact, $\leq$ is an ordering on the set of partitions of I , and $\mathfrak{J} \vee \bar{\mathfrak{J}}$ is the largest from $\{\mathfrak{J}, \bar{\mathfrak{J}}\}$.

(b) If f is a staircase function on a partition $\mathfrak{J}$, every refinement of $\mathfrak{J}$ is also a partition for f .

(c) If $f: I \rightarrow E$ is jump continuous, neither $f(x+0)$ nor $f(x-0)$ need equal $f(x)$ for $x \in I$.

(d) $\mathcal{S}(I, E)$ is a vector subspace of $B(I, E)$.

Proof The linearity of one-sided limits implies immediately that $\mathcal{S}(I, E)$ is a vector space. If $f \in \mathcal{S}(I, E) \setminus B(I, E)$, we find a sequence (x_n) in I with

$$\|f(x_n)\| \geq n \quad \text{for } n \in \mathbb{N}. \quad (1.1)$$

Because I is compact, there is a subsequence (x_{n_k}) of (x_n) and $x \in I$ such that $x_{n_k} \rightarrow x$ as $k \rightarrow \infty$. By choosing a suitable subsequence of (x_{n_k}) , we find a sequence (y_n) , that converges monotonically to x .³ If f is jump continuous, there is a $v \in E$ with $\lim f(y_n) = v$ and thus $\lim \|f(y_n)\| = \|v\|$ (compare with Example III.1.3(j)). Because every convergent sequence is bounded, we have contradicted (1.1). Therefore $\mathcal{S}(I, E) \subset B(I, E)$. ■

(e) We have sequences of vector subspaces

$$\mathcal{T}(I, E) \subset \mathcal{SC}(I, E) \subset \mathcal{S}(I, E) \quad \text{and} \quad C(I, E) \subset \mathcal{SC}(I, E).$$

(f) Every monotone function $f: I \rightarrow \mathbb{R}$ is jump continuous.

²We usually abbreviate $\mathcal{T}(I) := \mathcal{T}(I, \mathbb{K})$ etc, if the context makes clear which of the fields $\mathbb{R}$ or $\mathbb{C}$ we are dealing with.

³Compare with Exercise II.6.3.