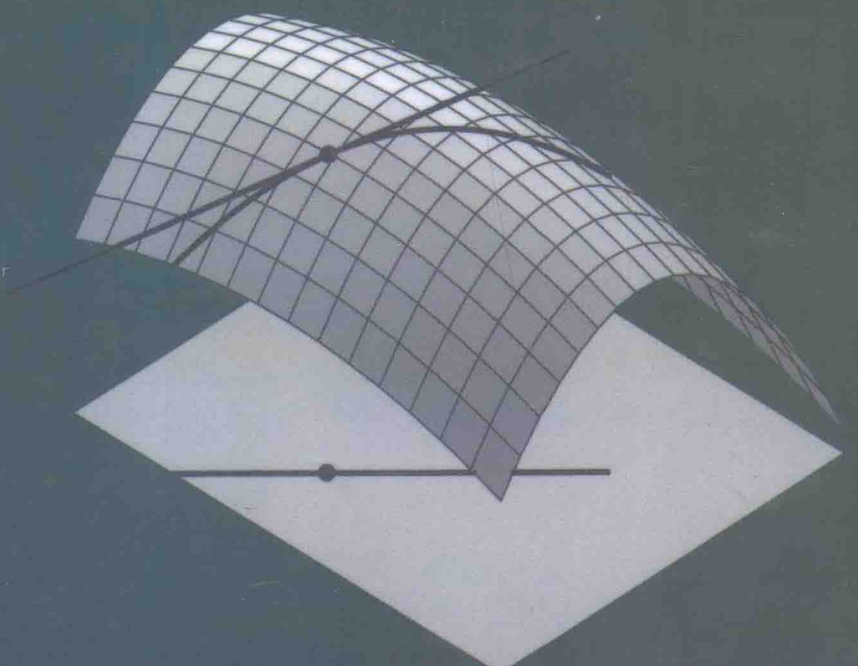
Herbert Amann
Joachim Escher

Analysis II

分析 第2卷



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Analysis II
by Herbert Amann and Joachim Escher

Foreword

As with the first, the second volume contains substantially more material than can be covered in a one-semester course. Such courses may omit many beautiful and well-grounded applications which connect broadly to many areas of mathematics. We of course hope that students will pursue this material independently; teachers may find it useful for undergraduate seminars.

For an overview of the material presented, consult the table of contents and the chapter introductions. As before, we stress that doing the numerous exercises is indispensable for understanding the subject matter, and they also round out and amplify the main text.

In writing this volume, we are indebted to the help of many. We especially thank our friends and colleages Pavol Quittner and Gieri Simonett. They have not only meticulously reviewed the entire manuscript and assisted in weeding out errors but also, through their valuable suggestions for improvement, contributed essentially to the final version. We also extend great thanks to our staff for their careful perusal of the entire manuscript and for tracking errata and inaccuracies.

Our most heartfelt thank extends again to our "typesetting perfectionist", without whose tireless effort this book would not look nearly so nice.[1] We also thank Andreas for helping resolve hardware and software problems.

Finally, we extend thanks to Thomas Hintermann and to Birkhäuser for the good working relationship and their understanding of our desired deadlines.

Zürich and Kassel, March 1999 H. Amann and J. Escher

[1]The text was set in LaTeX, and the figures were created with CorelDRAW! and Maple.

Foreword to the second edition

In this version, we have corrected errors, resolved imprecisions, and simplified several proofs. These areas for improvement were brought to our attention by readers. To them and to our colleagues H. Crauel, A. Ilchmann and G. Prokert, we extend heartfelt thanks.

Zürich and Hannover, December 2003 H. Amann and J. Escher

Foreword to the English translation

It is a pleasure to express our gratitude to Silvio Levy and Matt Cargo for their careful and insightful translation of the original German text into English. Their effective and pleasant cooperation during the process of translation is highly appreciated.

Zürich and Hannover, March 2008 H. Amann and J. Escher

Contents

Chapter VII Multivariable differential calculus

Chapter VIII Line integrals

Chapter VI

Integral calculus in one variable

Integration was invented for finding the area of shapes. This, of course, is an ancient problem, and the basic strategy for solving it is equally old: divide the shape into rectangles and add up their areas.

A mathematically satisfactory realization of this clear, intuitive idea is amazingly subtle. We note in particular that is a vast number of ways a given shape can be approximated by a union of rectangles. It is not at all self-evident they all lead to the same result. For this reason, we will not develop the rigorous theory of measures until Volume III.

In this chapter, we will consider only the simpler case of determining the area between the graph of a sufficiently regular function of one variable and its axis. By laying the groundwork for approximating a function by a juxtaposed series of rectangles, we will see that this boils down to approaching the function by a series of staircase functions, that is, functions that are piecewise constant. We will show that this idea for approximations is extremely flexible and is independent of its original geometric motivation, and we will arrive at a concept of integration that applies to a large class of vector-valued functions of a real variable.

To determine precisely the class of functions to which we can assign an integral, we must examine which functions can be approximated by staircase functions. By studying the convergence under the supremum norm, that is, by asking if a given function can be approximated uniformly on the entire interval by staircase functions, we are led to the class of jump continuous functions. Section 1 is devoted to studying this class.

There, we will see that an integral is a linear map on the vector space of staircase functions. There is then the problem of extending integration to the space of jump continuous functions; the extension should preserve the elementary properties of this map, particularly linearity. This exercise turns out to be a special case of the general problem of uniquely extending continuous maps. Because the extension problem is of great importance and enters many areas of mathematics, we will discuss it at length in Section 2. From the fundamental extension theorem for uniformly continuous maps, we will derive the theorem of continuous extensions of continuous linear maps. This will give us an opportunity to introduce the important concepts of bounded linear operators and the operator norm, which play a fundamental role in modern analysis.

After this groundwork, we will introduce in Section 3 the integral of jump continuous functions. This, the Cauchy–Riemann integral, extends the elementary integral of staircase functions. In the sections following, we will derive its fundamental properties. Of great importance (and you can tell by the name) is the fundamental theorem of calculus, which, to oversimplify, says that integration reverses differentiation. Through this theorem, we will be able to explicitly calculate a great many integrals and develop a flexible technique for integration. This will happen in Section 5.

In the remaining sections — except for the eighth — we will explore applications of the so-far developed differential and integral calculus. Since these are not essential for the overall structure of analysis, they can be skipped or merely sampled on first reading. However, they do contain many of the beautiful results of classical mathematics, which are needed not only for one's general mathematical literacy but also for numerous applications, both inside and outside of mathematics.

Section 6 will explore the connection between integrals and sums. We derive the Euler–Maclaurin sum formula and point out some of its consequences. Special mention goes to the proof of the formulas of de Moivre and Sterling, which describe the asymptotic behavior of the factorial function, and also to the derivation of several fundamental properties of the famous Riemann ζ function. The latter is important in connection to the asymptotic behavior of the distribution of prime numbers, which, of course, we can go into only very briefly.

In Section 7, we will revive the problem — mentioned at the end of Chapter V — of representing periodic functions by trigonometric series. With help from the integral calculus, we can specify a complete solution of this problem for a large class of functions. We place the corresponding theory of Fourier series in the general framework of the theory of orthogonality and inner product spaces. Thereby we achieve not only clarity and simplicity but also lay the foundation for a number of concrete applications, many of which you can expect see elsewhere. Naturally, we will also calculate some classical Fourier series explicitly, leading to some surprising results. Among these is the formula of Euler, which gives an explicit expression for the ζ function at even arguments; another is an interesting expression for the sine as an infinite product.

Up to this point, we have will have concentrated on the integration of jump continuous functions on compact intervals. In Section 8, we will further extend the domain of integration to cover functions that are defined (and integrated) on infinite intervals or are not bounded. We content ourselves here with simple but important results which will be needed for other applications in this volume because, in Volume III, we will develop an even broader and more flexible type of integral, the Lebesgue integral.

Section 9 is devoted to the theory of the gamma function. This is one of the most important nonelementary functions, and it comes up in many areas of mathematics. Thus we have tried to collect all the essential results, and we hope you will find them of value later. This section will show in a particularly nice way the strength of the methods developed so far.

1 Jump continuous functions

In many concrete situations, particularly in the integral calculus, the constraint of continuity turns out to be too restrictive. Discontinuous functions emerge naturally in many applications, although the discontinuity is generally not very pathological. In this section, we will learn about a simple class of maps which contains the continuous functions and is especially useful in the integral calculus in one independent variable. However, we will see later that the space of jump continuous functions is still too restrictive for a flexible theory of integration, and, in the context of multidimensional integration, we will have to extend the theory into an even broader class containing the continuous functions.

In the following, suppose

- $E := (E, \|\cdot\|)$ is a Banach space;
  $I := [\alpha, \beta]$ is a compact perfect interval.

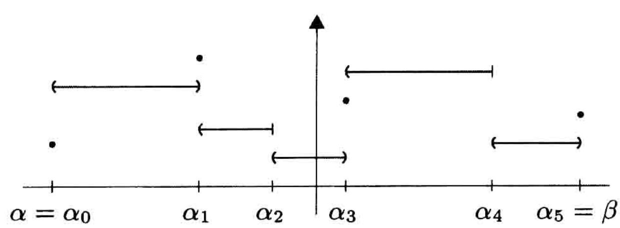
Staircase and jump continuous functions

We call $\mathfrak{Z} := (\alpha_0, \ldots, \alpha_n)$ a **partition** of I, if $n \in \mathbb{N}^\times$ and

$$\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_n = \beta\ .$$

If $\{\alpha_0, \ldots, \alpha_n\}$ is a subset of the partition $\overline{\mathfrak{Z}} := (\beta_0, \ldots, \beta_k)$, $\overline{\mathfrak{Z}}$ is called a **refinement** of $\mathfrak{Z}$, and we write $\mathfrak{Z} \le \overline{\mathfrak{Z}}$.

The function $f : I \to E$ is called a **staircase function** on I if I has a partition $\mathfrak{Z} := (\alpha_0, \ldots, \alpha_n)$ such that f is constant on every (open) interval (α_{j-1}, α_j). Then we say $\mathfrak{Z}$ is a partition for f, or we say f is a staircase function on the partition $\mathfrak{Z}$.



A staircase function

If $f : I \to E$ is such that the limits $f(\alpha + 0)$, $f(\beta - 0)$, and

$$f(x \pm 0) := \lim_{\substack{y \to x \pm 0 \\ y \ne x}} f(y)$$

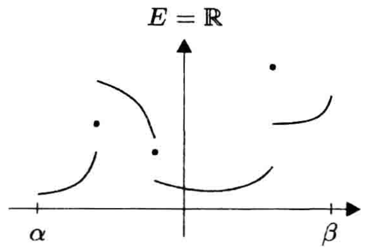
exist for all $x \in \mathring{I}$, we call f **jump continuous**.[1] A jump continuous function is **piecewise continuous** if it has only finitely many discontinuities ("jumps"). Finally,

[1]Note that, in general, $f(x + 0)$ and $f(x - 0)$ may differ from $f(x)$.

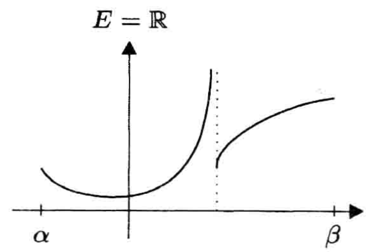
we denote by

$$\mathcal{T}(I,E), \qquad \mathcal{S}(I,E), \qquad \mathcal{SC}(I,E)$$

the sets of all functions $f : I \to E$ that are staircase, jump continuous, and piecewise continuous, respectively.[2]



A piecewise continuous function



Not a jump continuous function

1.1 Remarks **(a)** Given partitions $\mathfrak{Z} := (\alpha_0, \ldots, \alpha_n)$ and $\overline{\mathfrak{Z}} := (\beta_0, \ldots, \beta_m)$ of I, the union $\{\alpha_0, \ldots, \alpha_n\} \cup \{\beta_0, \ldots, \beta_m\}$ will naturally define another partition $\mathfrak{Z} \vee \overline{\mathfrak{Z}}$ of I. Obviously, $\mathfrak{Z} \le \mathfrak{Z} \vee \overline{\mathfrak{Z}}$ and $\overline{\mathfrak{Z}} \le \mathfrak{Z} \vee \overline{\mathfrak{Z}}$. In fact, $\le$ is an ordering on the set of partitions of I, and $\mathfrak{Z} \vee \overline{\mathfrak{Z}}$ is the largest from $\{\mathfrak{Z}, \overline{\mathfrak{Z}}\}$.

(b) If f is a staircase function on a partition $\mathfrak{Z}$, every refinement of $\mathfrak{Z}$ is also a partition for f.

(c) If $f : I \to E$ is jump continuous, neither $f(x+0)$ nor $f(x-0)$ need equal $f(x)$ for $x \in I$.

(d) $\mathcal{S}(I,E)$ is a vector subspace of $B(I,E)$.

Proof The linearity of one-sided limits implies immediately that $\mathcal{S}(I,E)$ is a vector space. If $f \in \mathcal{S}(I,E) \backslash B(I,E)$, we find a sequence (x_n) in I with

$$\|f(x_n)\| \ge n \quad \text{for } n \in \mathbb{N} . \tag{1.1}$$

Because I is compact, there is a subsequence (x_{n_k}) of (x_n) and $x \in I$ such that $x_{n_k} \to x$ as $k \to \infty$. By choosing a suitable subsequence of (x_{n_k}), we find a sequence (y_n), that converges monotonically to x.[3] If f is jump continuous, there is a $v \in E$ with $\lim f(y_n) = v$ and thus $\lim \|f(y_n)\| = \|v\|$ (compare with Example III.1.3(j)). Because every convergent sequence is bounded, we have contradicted (1.1). Therefore $\mathcal{S}(I,E) \subset B(I,E)$. ■

(e) We have sequences of vector subspaces

$$\mathcal{T}(I,E) \subset \mathcal{SC}(I,E) \subset \mathcal{S}(I,E) \quad \text{and} \quad C(I,E) \subset \mathcal{SC}(I,E) .$$

(f) Every monotone function $f : I \to \mathbb{R}$ is jump continuous.

[2]We usually abbreviate $\mathcal{T}(I) := \mathcal{T}(I, \mathbb{K})$ etc, if the context makes clear which of the fields $\mathbb{R}$ or $\mathbb{C}$ we are dealing with.

[3]Compare with Exercise II.6.3.