

William J. Stewart

# Probability, Markov Chains, Queues, and Simulation

The Mathematical Basis of Performance Modeling

概率论、马尔科夫链、排队和模拟



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# **PROBABILITY, MARKOV CHAINS, QUEUES, AND SIMULATION**

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**This book is dedicated to all those  
whom I love, especially**

*My dear wife, Kathie,  
and my wonderful children  
Nicola, Stephanie, Kathryn, and William*

*My father, William J. Stewart and  
the memory of my mother, Mary (Marshall) Stewart*

# Preface and Acknowledgments

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This book has been written to provide a complete, yet elementary and pedagogic, treatment of the mathematical basis of systems performance modeling. Performance modeling is of fundamental importance to many branches of the mathematical sciences and engineering as well as to the social and economic sciences. Advances in methodology and technology have now provided the wherewithal to build and solve sophisticated models. The purpose of this book is to provide the student and teacher with a modern approach for building and solving probability based models with confidence.

The book is divided into four major parts, namely, “Probability,” “Markov Chains,” “Queueing Models,” and “Simulation.” The eight chapters of Part I provide the student with a comprehensive and thorough knowledge of probability theory. Part I is self-contained and complete and should be accessible to anyone with a basic knowledge of calculus. Newcomers to probability theory as well as those whose knowledge of probability is rusty should be equally at ease in their progress through Part I. The first chapter provides the fundamental concepts of set-based probability and the probability axioms. Conditional probability and independence are stressed as are the laws of total probability and Bayes’ rule. Chapter 2 introduces combinatorics—the art of counting—which is so important for the correct evaluation of probabilities. Chapter 3 introduces the concepts of random variables and distribution functions including functions of a random variable and conditioned random variables. This chapter prepares the ground work for Chapters 4 and 5: Chapter 4 introduces joint and conditional distributions and Chapter 5 treats expectations and higher moments. Discrete distribution functions are the subject of Chapter 6 while their continuous counterparts, continuous distribution functions, are the subject of Chapter 7. Particular attention is paid to phase-type distributions due to the important role they play in modeling scenarios and the chapter also includes a section on fitting phase-type distributions to given means and variances. The final chapter in Part I is devoted to bounds and limit theorems, including the laws of large numbers and the central limit theorem.

Part II contains two rather long chapters on the subject of Markov chains, the first on theoretical aspects of Markov chains, and the second on their numerical solution. In Chapter 9, the basic concepts of discrete and continuous-time Markov chains and their underlying equations and properties are discussed. Special attention is paid to irreducible Markov chains and to the potential, fundamental, and reachability matrices in reducible Markov chains. This chapter also contains sections on random walk problems and their applications, the property of reversibility in Markov chains, and renewal processes. Chapter 10 deals with numerical solutions, from Gaussian elimination and basic iterative-type methods for stationary solutions to ordinary differential equation solvers for transient solutions. Block methods and iterative aggregation-disaggregation methods for nearly completely decomposable Markov chains are considered. A section is devoted to matrix geometric and matrix analytic methods for structured Markov chains. Algorithms and computational considerations are stressed throughout this chapter.

Queueing models are presented in the five chapters that constitute Part III. Elementary queueing theory is presented in Chapter 11. Here an introduction to the basic terminology and definitions is followed by an analysis of the simplest of all queueing models, the  $M/M/1$  queue. This is then generalized to birth-death processes, which are queueing systems in which the underlying Markov chain matrix is tridiagonal. Chapter 12 deals with queues in which the arrival process need no longer

be Poisson and the service time need not be exponentially distributed. Instead, interarrival times and service times can be represented by phase-type distributions and the underlying Markov chain is now block tridiagonal. The following chapter, Chapter 13, explores the  $z$ -transform approach for solving similar types of queues. The  $M/G/1$  and  $G/M/1$  queues are the subject of Chapter 14. The approach used is that of the embedded Markov chain. The Pollaczek-Khintchine mean value and transform equations are derived and a detailed discussion of residual time and busy period follows. A thorough discussion of nonpreemptive and preempt-resume scheduling policies as well as shortest-processing-time-first scheduling is presented. An analysis is also provided for the case in which only a limited number of customers can be accommodated in both the  $M/G/1$  and  $G/M/1$  queues. The final chapter of Part III, Chapter 15, treats queueing networks. Open networks are introduced via Burke's theorem and Jackson's extensions to this theorem. Closed queueing networks are treated using both the convolution algorithm and the mean value approach. The "flow-equivalent server" approach is also treated and its potential as an approximate solution procedure for more complex networks is explored. The chapter terminates with a discussion of product form in queueing networks and the BCMP theorem for open, closed, and mixed networks.

The final part of the text, Part IV, deals with simulation. Chapter 16 explores how uniformly distributed random numbers can be applied to obtain solutions to probabilistic models and other time-independent problems—the "Monte Carlo" aspect of simulation. Chapter 17 describes the modern approaches for generating uniformly distributed random numbers and how to test them to ensure that they are indeed uniformly distributed and independent of each other. The topic of generating random numbers that are not uniformly distributed, but satisfy some other distribution such as Erlang or normal, is dealt with in Chapter 18. A large number of possibilities exist and not all are appropriate for every distribution. The next chapter, Chapter 19, provides guidelines for writing simulation programs and a number of examples are described in detail. Chapter 20 is the final chapter in the book. It concerns simulation measurement and accuracy and is based on sampling theory. Special attention is paid to the generation of confidence intervals and to variance reduction techniques, an important means of keeping the computational costs of simulation to a manageable level.

The text also includes two appendixes; the first is just a simple list of the letters of the Greek alphabet and their spellings; the second is a succinct, yet complete, overview of the linear algebra used throughout the book.

## Genesis and Intent

This book saw its origins in two first-year graduate level courses that I teach, and have taught for quite some time now, at North Carolina State University. The first is entitled "An Introduction to Performance Evaluation;" it is offered by the Computer Science Department and the Department of Electrical and Computer Engineering. This course is required for our networking degrees. The second is a course entitled "Queues and Stochastic Service Systems" and is offered by the Operations Research Program and the Industrial and Systems Engineering Department. It follows then that this book has been designed for students from a variety of academic disciplines in which stochastic processes constitute a fundamental concept, disciplines that include not only computer science and engineering, industrial engineering, and operations research, but also mathematics, statistics, economics, and business, the social sciences—in fact all disciplines in which stochastic performance modeling plays a primary role. A calculus-based probability course is a prerequisite for both these courses so it is expected that students taking these classes are already familiar with probability theory. However, many of the students who sign up for these courses are returning students, and it is often the case that it has been several years and in some cases a decade or more, since they last studied probability. A quick review of probability is hardly sufficient to bring them



up to the required level. Part I of the book has been designed with them in mind. It provides the prerequisite probability background needed to fully understand and appreciate the material in the remainder of the text. The presentation, with its numerous examples and exercises, is such that it facilitates an independent review so the returning student in a relatively short period of time, preferably prior to the beginning of class, will once again have mastered probability theory. Part I can then be used as a reference source as and when needed.

The entire text has been written at a level that is suitable for upper-level undergraduate students or first-year graduate students and is completely self-contained. The entirety of the text can be covered in a two-semester sequence, such as the stochastic processes sequence offered by the Industrial Engineering (IE) Department and the Operations Research (OR) Program at North Carolina State University. A two-semester sequence is appropriate for classes in which students have limited (or no) exposure to probability theory. In such cases it is recommended that the first semester be devoted to the Chapters 1–8 on probability theory, the first five sections of Chapter 9, which introduce the fundamental concepts of discrete-time Markov chains, and the first three sections of Chapter 11, which concern elementary queueing theory. With this background clearly understood, the student should have no difficulty in covering the remaining topics of the text in the second semester.

The complete content of Parts II–IV might prove to be a little too much for some one-semester classes. In this case, an instructor might wish to omit the later sections of Chapter 10 on the numerical solution of Markov chains, perhaps covering only the basic direct and iterative methods. In this case the material of Chapter 12 should also be omitted since it depends on a knowledge of the matrix geometric method of Chapter 10. Because of the importance of computing numerical solutions, it would be a mistake to omit Chapter 10 in its entirety. Some of the material in Chapter 18 could also be eliminated: for example, an instructor might include only the first three sections of this chapter. In my own case, when teaching the OR/IE course, I concentrate on covering all of the Markov chain and queueing theory chapters. These students often take simulation as an individual course later on. When teaching the computer science and engineering course, I omit some of the material on the numerical solution of Markov chains so as to leave enough time to cover simulation.

Numerous examples with detailed explanations are provided throughout the text. These examples are designed to help the student more clearly understand the theoretical and computational aspects of the material and to be in a position to apply the acquired knowledge to his/her own areas of interest. A solution manual is available for teachers who adopt this text for their courses. This manual contains detailed explanations of the solution of all the exercises.

Where appropriate, the text contains program modules written in Matlab or in the Java programming language. These programs are not meant to be robust production code, but are presented so that the student may experiment with the mathematical concepts that are discussed. To free the student from the hassle of copying these code segments from the book, a listing of all of the code used can be freely downloaded from the web page:

<http://press.princeton.edu/titles/8844.html>

## Acknowledgments

As mentioned just a moment ago, this book arose out of two courses that I teach at North Carolina State University. It is, therefore, ineluctable that the students who took these courses contributed immeasurably to its content and form. I would like to express my gratitude to them for their patience and input. I would like to cite, in particular, Nishit Gandhi, Scott Gerard, Rong Huang, Kathryn Peding, Amirhosein Norouzi, Robert Shih, Hui Wang, Song Yang, and Shengfan Zhang for their helpful comments. My own doctoral students, Shep Barge, Tugrul Dayar, Amy Langville, Ning Liu, and Bin Peng, were subjected to different versions of the text and I owe them a particular

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I would like to thank Vickie Kearn and the editorial and production staff at Princeton University Press for their help and guidance in producing this book. It would be irresponsible of me not to mention the influence that my teachers, colleagues, and friends have had on me. I owe them a considerable debt of gratitude for helping me understand the vital role that mathematics plays, not only in performance modeling, but in all aspects of life.

Finally, and most of all, I would like to thank my wife Kathie and our four children, Nicola, Stephanie, Kathryn, and William, for all the love they have shown me over the years.

# Contents

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<i>Preface and Acknowledgments</i>	xv
<b>I PROBABILITY</b>	<b>1</b>
<b>1 Probability</b>	<b>3</b>
1.1 Trials, Sample Spaces, and Events	3
1.2 Probability Axioms and Probability Space	9
1.3 Conditional Probability	12
1.4 Independent Events	15
1.5 Law of Total Probability	18
1.6 Bayes' Rule	20
1.7 Exercises	21
<b>2 Combinatorics—The Art of Counting</b>	<b>25</b>
2.1 Permutations	25
2.2 Permutations with Replacements	26
2.3 Permutations without Replacement	27
2.4 Combinations without Replacement	29
2.5 Combinations with Replacements	31
2.6 Bernoulli (Independent) Trials	33
2.7 Exercises	36
<b>3 Random Variables and Distribution Functions</b>	<b>40</b>
3.1 Discrete and Continuous Random Variables	40
3.2 The Probability Mass Function for a Discrete Random Variable	43
3.3 The Cumulative Distribution Function	46
3.4 The Probability Density Function for a Continuous Random Variable	51
3.5 Functions of a Random Variable	53
3.6 Conditioned Random Variables	58
3.7 Exercises	60
<b>4 Joint and Conditional Distributions</b>	<b>64</b>
4.1 Joint Distributions	64
4.2 Joint Cumulative Distribution Functions	64
4.3 Joint Probability Mass Functions	68
4.4 Joint Probability Density Functions	71
4.5 Conditional Distributions	77
4.6 Convolutions and the Sum of Two Random Variables	80
4.7 Exercises	82
<b>5 Expectations and More</b>	<b>87</b>
5.1 Definitions	87
5.2 Expectation of Functions and Joint Random Variables	92
5.3 Probability Generating Functions for Discrete Random Variables	100

5.4	Moment Generating Functions	103
5.5	Maxima and Minima of Independent Random Variables	108
5.6	Exercises	110
<b>6</b>	<b>Discrete Distribution Functions</b>	<b>115</b>
6.1	The Discrete Uniform Distribution	115
6.2	The Bernoulli Distribution	116
6.3	The Binomial Distribution	117
6.4	Geometric and Negative Binomial Distributions	120
6.5	The Poisson Distribution	124
6.6	The Hypergeometric Distribution	127
6.7	The Multinomial Distribution	128
6.8	Exercises	130
<b>7</b>	<b>Continuous Distribution Functions</b>	<b>134</b>
7.1	The Uniform Distribution	134
7.2	The Exponential Distribution	136
7.3	The Normal or Gaussian Distribution	141
7.4	The Gamma Distribution	145
7.5	Reliability Modeling and the Weibull Distribution	149
7.6	Phase-Type Distributions	155
7.6.1	The Erlang-2 Distribution	155
7.6.2	The Erlang- $r$ Distribution	158
7.6.3	The Hypoexponential Distribution	162
7.6.4	The Hyperexponential Distribution	164
7.6.5	The Coxian Distribution	166
7.6.6	General Phase-Type Distributions	168
7.6.7	Fitting Phase-Type Distributions to Means and Variances	171
7.7	Exercises	176
<b>8</b>	<b>Bounds and Limit Theorems</b>	<b>180</b>
8.1	The Markov Inequality	180
8.2	The Chebychev Inequality	181
8.3	The Chernoff Bound	182
8.4	The Laws of Large Numbers	182
8.5	The Central Limit Theorem	184
8.6	Exercises	187
<b>II</b>	<b>MARKOV CHAINS</b>	<b>191</b>
<b>9</b>	<b>Discrete- and Continuous-Time Markov Chains</b>	<b>193</b>
9.1	Stochastic Processes and Markov Chains	193
9.2	Discrete-Time Markov Chains: Definitions	195
9.3	The Chapman-Kolmogorov Equations	202
9.4	Classification of States	206
9.5	Irreducibility	214
9.6	The Potential, Fundamental, and Reachability Matrices	218
9.6.1	Potential and Fundamental Matrices and Mean Time to Absorption	219
9.6.2	The Reachability Matrix and Absorption Probabilities	223

9.7	Random Walk Problems	228
9.8	Probability Distributions	235
9.9	Reversibility	248
9.10	Continuous-Time Markov Chains	253
9.10.1	Transition Probabilities and Transition Rates	254
9.10.2	The Chapman-Kolmogorov Equations	257
9.10.3	The Embedded Markov Chain and State Properties	259
9.10.4	Probability Distributions	262
9.10.5	Reversibility	265
9.11	Semi-Markov Processes	265
9.12	Renewal Processes	267
9.13	Exercises	275
<b>10</b>	<b>Numerical Solution of Markov Chains</b>	<b>285</b>
10.1	Introduction	285
10.1.1	Setting the Stage	285
10.1.2	Stochastic Matrices	287
10.1.3	The Effect of Discretization	289
10.2	Direct Methods for Stationary Distributions	290
10.2.1	Iterative versus Direct Solution Methods	290
10.2.2	Gaussian Elimination and <i>LU</i> Factorizations	291
10.3	Basic Iterative Methods for Stationary Distributions	301
10.3.1	The Power Method	301
10.3.2	The Iterative Methods of Jacobi and Gauss-Seidel	305
10.3.3	The Method of Successive Overrelaxation	311
10.3.4	Data Structures for Large Sparse Matrices	313
10.3.5	Initial Approximations, Normalization, and Convergence	316
10.4	Block Iterative Methods	319
10.5	Decomposition and Aggregation Methods	324
10.6	The Matrix Geometric/Analytic Methods for Structured Markov Chains	332
10.6.1	The Quasi-Birth-Death Case	333
10.6.2	Block Lower Hessenberg Markov Chains	340
10.6.3	Block Upper Hessenberg Markov Chains	345
10.7	Transient Distributions	354
10.7.1	Matrix Scaling and Powering Methods for Small State Spaces	357
10.7.2	The Uniformization Method for Large State Spaces	361
10.7.3	Ordinary Differential Equation Solvers	365
10.8	Exercises	375
<b>III</b>	<b>QUEUEING MODELS</b>	<b>383</b>
<b>11</b>	<b>Elementary Queueing Theory</b>	<b>385</b>
11.1	Introduction and Basic Definitions	385
11.1.1	Arrivals and Service	386
11.1.2	Scheduling Disciplines	395
11.1.3	Kendall's Notation	396
11.1.4	Graphical Representations of Queues	397
11.1.5	Performance Measures—Measures of Effectiveness	398
11.1.6	Little's Law	400

11.2	Birth-Death Processes: The $M/M/1$ Queue	402
11.2.1	Description and Steady-State Solution	402
11.2.2	Performance Measures	406
11.2.3	Transient Behavior	412
11.3	General Birth-Death Processes	413
11.3.1	Derivation of the State Equations	413
11.3.2	Steady-State Solution	415
11.4	Multiserver Systems	419
11.4.1	The $M/M/c$ Queue	419
11.4.2	The $M/M/\infty$ Queue	425
11.5	Finite-Capacity Systems—The $M/M/1/K$ Queue	425
11.6	Multiserver, Finite-Capacity Systems—The $M/M/c/K$ Queue	432
11.7	Finite-Source Systems—The $M/M/c//M$ Queue	434
11.8	State-Dependent Service	437
11.9	Exercises	438
<b>12</b>	<b>Queues with Phase-Type Laws: Neuts' Matrix-Geometric Method</b>	<b>444</b>
12.1	The Erlang- $r$ Service Model—The $M/E_r/1$ Queue	444
12.2	The Erlang- $r$ Arrival Model—The $E_r/M/1$ Queue	450
12.3	The $M/H_2/1$ and $H_2/M/1$ Queues	454
12.4	Automating the Analysis of Single-Server Phase-Type Queues	458
12.5	The $H_2/E_3/1$ Queue and General $Ph/Ph/1$ Queues	460
12.6	Stability Results for $Ph/Ph/1$ Queues	466
12.7	Performance Measures for $Ph/Ph/1$ Queues	468
12.8	Matlab code for $Ph/Ph/1$ Queues	469
12.9	Exercises	471
<b>13</b>	<b>The <math>z</math>-Transform Approach to Solving Markovian Queues</b>	<b>475</b>
13.1	The $z$ -Transform	475
13.2	The Inversion Process	478
13.3	Solving Markovian Queues using $z$ -Transforms	484
13.3.1	The $z$ -Transform Procedure	484
13.3.2	The $M/M/1$ Queue Solved using $z$ -Transforms	484
13.3.3	The $M/M/1$ Queue with Arrivals in Pairs	486
13.3.4	The $M/E_r/1$ Queue Solved using $z$ -Transforms	488
13.3.5	The $E_r/M/1$ Queue Solved using $z$ -Transforms	496
13.3.6	Bulk Queueing Systems	503
13.4	Exercises	506
<b>14</b>	<b>The <math>M/G/1</math> and <math>G/M/1</math> Queues</b>	<b>509</b>
14.1	Introduction to the $M/G/1$ Queue	509
14.2	Solution via an Embedded Markov Chain	510
14.3	Performance Measures for the $M/G/1$ Queue	515
14.3.1	The Pollaczek-Khintchine Mean Value Formula	515
14.3.2	The Pollaczek-Khintchine Transform Equations	518
14.4	The $M/G/1$ Residual Time: Remaining Service Time	523
14.5	The $M/G/1$ Busy Period	526
14.6	Priority Scheduling	531
14.6.1	$M/M/1$ : Priority Queue with Two Customer Classes	531
14.6.2	$M/G/1$ : Nonpreemptive Priority Scheduling	533

14.6.3	<i>M/G/1</i> : Preempt-Resume Priority Scheduling	536
14.6.4	A Conservation Law and SPTF Scheduling	538
14.7	The <i>M/G/1/K</i> Queue	542
14.8	The <i>G/M/1</i> Queue	546
14.9	The <i>G/M/1/K</i> Queue	551
14.10	Exercises	553
<b>15</b>	<b>Queueing Networks</b>	<b>559</b>
15.1	Introduction	559
15.1.1	Basic Definitions	559
15.1.2	The Departure Process—Burke's Theorem	560
15.1.3	Two <i>M/M/1</i> Queues in Tandem	562
15.2	Open Queueing Networks	563
15.2.1	Feedforward Networks	563
15.2.2	Jackson Networks	563
15.2.3	Performance Measures for Jackson Networks	567
15.3	Closed Queueing Networks	568
15.3.1	Definitions	568
15.3.2	Computation of the Normalization Constant: Buzen's Algorithm	570
15.3.3	Performance Measures	577
15.4	Mean Value Analysis for Closed Queueing Networks	582
15.5	The Flow-Equivalent Server Method	591
15.6	Multiclass Queueing Networks and the BCMP Theorem	594
15.6.1	Product-Form Queueing Networks	595
15.6.2	The BCMP Theorem for Open, Closed, and Mixed Queueing Networks	598
15.7	Java Code	602
15.8	Exercises	607
<b>IV</b>	<b>SIMULATION</b>	<b>611</b>
<b>16</b>	<b>Some Probabilistic and Deterministic Applications of Random Numbers</b>	<b>613</b>
16.1	Simulating Basic Probability Scenarios	613
16.2	Simulating Conditional Probabilities, Means, and Variances	618
16.3	The Computation of Definite Integrals	620
16.4	Exercises	623
<b>17</b>	<b>Uniformly Distributed "Random" Numbers</b>	<b>625</b>
17.1	Linear Recurrence Methods	626
17.2	Validating Sequences of Random Numbers	630
17.2.1	The Chi-Square "Goodness-of-Fit" Test	630
17.2.2	The Kolmogorov-Smirnov Test	633
17.2.3	"Run" Tests	634
17.2.4	The "Gap" Test	640
17.2.5	The "Poker" Test	641
17.2.6	Statistical Test Suites	644
17.3	Exercises	644

<b>18 Nonuniformly Distributed “Random” Numbers</b>	<b>647</b>
18.1 The Inverse Transformation Method	647
18.1.1 The Continuous Uniform Distribution	649
18.1.2 “Wedge-Shaped” Density Functions	649
18.1.3 “Triangular” Density Functions	650
18.1.4 The Exponential Distribution	652
18.1.5 The Bernoulli Distribution	653
18.1.6 An Arbitrary Discrete Distribution	653
18.2 Discrete Random Variates by Mimicry	654
18.2.1 The Binomial Distribution	654
18.2.2 The Geometric Distribution	655
18.2.3 The Poisson Distribution	656
18.3 The Accept-Reject Method	657
18.3.1 The Lognormal Distribution	660
18.4 The Composition Method	662
18.4.1 The Erlang- $r$ Distribution	662
18.4.2 The Hyperexponential Distribution	663
18.4.3 Partitioning of the Density Function	664
18.5 Normally Distributed Random Numbers	670
18.5.1 Normal Variates via the Central Limit Theorem	670
18.5.2 Normal Variates via Accept-Reject and Exponential Bounding Function	670
18.5.3 Normal Variates via Polar Coordinates	672
18.5.4 Normal Variates via Partitioning of the Density Function	673
18.6 The Ziggurat Method	673
18.7 Exercises	676
<b>19 Implementing Discrete-Event Simulations</b>	<b>680</b>
19.1 The Structure of a Simulation Model	680
19.2 Some Common Simulation Examples	682
19.2.1 Simulating the $M/M/1$ Queue and Some Extensions	682
19.2.2 Simulating Closed Networks of Queues	686
19.2.3 The Machine Repairman Problem	689
19.2.4 Simulating an Inventory Problem	692
19.3 Programming Projects	695
<b>20 Simulation Measurements and Accuracy</b>	<b>697</b>
20.1 Sampling	697
20.1.1 Point Estimators	698
20.1.2 Interval Estimators/Confidence Intervals	704
20.2 Simulation and the Independence Criteria	707
20.3 Variance Reduction Methods	711
20.3.1 Antithetic Variables	711
20.3.2 Control Variables	713
20.4 Exercises	716
<b>Appendix A: The Greek Alphabet</b>	<b>719</b>
<b>Appendix B: Elements of Linear Algebra</b>	<b>721</b>
B.1 Vectors and Matrices	721
B.2 Arithmetic on Matrices	721



B.3	Vector and Matrix Norms	723
B.4	Vector Spaces	724
B.5	Determinants	726
B.6	Systems of Linear Equations	728
	B.6.1 Gaussian Elimination and $LU$ Decompositions	730
B.7	Eigenvalues and Eigenvectors	734
B.8	Eigenproperties of Decomposable, Nearly Decomposable, and Cyclic Stochastic Matrices	738
	B.8.1 Normal Form	738
	B.8.2 Eigenvalues of Decomposable Stochastic Matrices	739
	B.8.3 Eigenvectors of Decomposable Stochastic Matrices	741
	B.8.4 Nearly Decomposable Stochastic Matrices	743
	B.8.5 Cyclic Stochastic Matrices	744
	<b>Bibliography</b>	745
	<b>Index</b>	749