ALM 24
Advanced Lectures in Mathematics

Handbook of Moduli

(Volume I)

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Editors: Gavril Farkas · Ian Morrison

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The Handbook of Moduli was designed by Ian Morrison in LaTeX using a variant of the standard style files of the Higher Education Press. The text of the Handbook is set in ITC Giovanni and the mathematics in AMS Euler.

Giovanni was designed by Robert Slimbach in 1989 for ITC and was one of the early faces that earned him the Prix Charles Peignot, the Fields Medal of type design awarded "to a designer under the age of 35 who has made an outstanding contribution to type design". It combines the basic proportions of traditional oldstyle designs with the more even color and higher x-height of modern digital fonts to produce an inconspicuous but legible typeface.

Euler was designed in 1981 by Hermann Zapf, a major figure in 20th type design and a pioneer in digital typography, working in close cooperation with Donald Knuth, as an upright, cursive symbol font that would give the effect of mathematics handwritten on a blackboard. In 2008, Zapf reshaped many of the glyphs, with the assistance of Hans Hagen, Taco Hoekwater, and Volker RW Schaa, in order to harmonize the designs and bring them into line with contemporary standards of digital typography.

The Handbook of Moduli is dedicated to the memory of Eckart Viehweg, whose untimely death precluded a planned contribution, and to David Mumford, who first proposed the project, for all that they both did to nurture its subject; and to Angela Ortega and Jane Reynolds for everything that they do to sustain its editors.

Preface

Gavril Farkas and Ian Morrison

The title of these volumes might lead unwary readers to expect an encyclopedic survey for experts in the study of moduli problems in algebraic geometry. What they will discover is rather different. Our aims here are, first, to clarify the audience that we hope the Handbook *will* serve and the approach it *does* takes to its subject and, second, to thank all those who have assisted us in helping it realize these aims.

To begin with, a bit of history. The idea for a Handbook of Moduli originated in a discussion between David Mumford and Lizhen Ji at Michigan in 2006. Lizhen and David produced a draft table of contents that was circulated at the Symposium marking David's retirement from Brown in 2007. The Handbook was originally to have been edited by Ching-Li Chai and Amnon Neeman, but the demands of their work with Takahiro Shiota as editors of the second volume of Mumford's Collected Papers took priority and, at their urging, we agreed to take over editorship in the spring of 2009.

We quickly reached the conclusion that what was needed for many topics was not a discussion of the latest results aimed at specialists, but a survey aimed at a broad community of producers (and even some consumers from cognate areas) of algebraic geometry, most of whom had little prior familiarity with the area. Our goal became a Handbook that would introduce the techniques, examples and results essential to each topic, and say enough about recent developments to prepare the reader to tackle the primary literature in the area. We particularly sought to elicit contributions that illustrated "secret handshakes", yogas and heuristics that experts use privately to guide intuition or simplify calculation but that are replaced by more formal arguments, or simply do not appear, in articles aimed at other specialists.

For many topics, the Handbook succeeds much better than we dared to hope. The credit is due entirely to the hard work of the Handbook's many authors in producing articles that conformed to the goals we had set. Again and again, we were delighted to find that authors, instead of taking the easy course of cutting and pasting from earlier surveys and primary references, had made the substantially greater effort to write the original treatments needed to bridge gaps in the literature and make important problems accessible to a wide audience for the first time.

We expect that they will reap a just reward and that their articles will be widely read and referenced. Here we want to offer them not only our sincerest thanks, but also those of the Handbook's readers, for their exceptional generosity. Many

Handbook articles were also improved by extensive and thoughtful referees' reports. We are grateful for all work that the referees did to improve the Handbook and take this opportunity to thank them collectively on behalf of the contributors.

We must, however, disclaim that the Handbook's coverage is often incomplete, in extreme cases, non-existent. The blame for these gaps is mostly ours. When we solicited contributions to the Handbook, each invitation was accompanied by a suggested topic, and we selected contributors who we thought would be able to cover their topics in the spirit discussed above. The results reflect both our knowledge and taste—of topics and of experts in them—and also, in some cases, our ignorance.

In some areas, we found it easy to produce candidate contributor–topic pairs, and to recruit the contributors we had identified. The Handbook's discussion of, for example, moduli spaces of curves is, therefore, particularly complete—some will say, not without a certain justice, excessive.

In other areas, we had more difficulty both in identifying and in enlisting candidates. A few of the more obvious gaps arose when authors who had accepted our invitation backed out after it was too late to find replacements. A more deeply felt loss—one that impacts the whole subject of moduli—was the untimely death of Eckart Viehweg, who had been one of the first to agree to contribute.

We also omitted a few topics as a courtesy to the authors of monographs devoted to them that we knew to be in preparation, others because papers treating them in the spirit we were seeking had recently appeared, and yet others because we felt that they were developing so rapidly that any contribution dealing with them would have a limited shelf-life. In hindsight, not all of these decisions were well taken.

As a result, the Handbook's treatment of moduli has some major lacunae (mirror symmetry, wall crossing formulae) and there are other topics (moduli of sheaves and bundles) which are discussed but not in the depth that their importance merits. We apologize to readers who may have hoped to find more about these subjects in the Handbook, and (with Lizhen's encouragement) we challenge experts who feel that their areas deserve a fuller exposition to offer him proposals for additional Handbook volumes devoted to them.

The Handbook also benefitted from the efforts of many other colleagues. Amnon Neeman showed considerable doggedness in recruiting us to succeed him and Ching-Li as editors. Scott Wolpert provided valuable advice on the cat-herding elements of the editor's job. Dave Bayer helped enormously in setting up the final production process both to automate complex and error prone operations and to prevent inconsistencies between the LaTeX installations on our home systems and those at Higher Education Press.

Brian Bianchini, International Press' General Manager, made sure that we had the resources we needed throughout the Handbook's growth from the single volume originally projected to the present three. The Advanced Mathematics series editor, Lizhen Ji, was always ready to answer our questions, help with practical difficulties, and adjust his schedule for the series to adapt to changes in ours. Liping Wang and her production staff at the Higher Education Press were unfailingly accommodating and helpful to us in resolving LaTeX issues—even reTeXing several submissions to bring them into conformity with the Handbook style—and made every effort to ensure that the appearance of the Handbook volumes was up to the standard of their contents.

To all of them, and to many others who provided more informal help, we here offer our sincerest thanks.

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Logarithmic geometry and moduli

Dan Abramovich, Qile Chen, Danny Gillam, Yuhao Huang, Martin Olsson, Matthew Satriano, and Shenghao Sun

Abstract. We discuss the role played by logarithmic structures in the theory of moduli.

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1. Introduction

Logarithmic structures in algebraic geometry

It can be said that Logarithmic Geometry is concerned with a method of finding and using "hidden smoothness" in singular varieties. The original insight comes from consideration of de Rham cohomology, where logarithmic differentials can reveal such hidden smoothness. Since singular varieties naturally occur "at the boundary" of many moduli problems, logarithmic geometry was soon applied in the theory of moduli.

Foundations for this theory were first given by Kazuya Kato in [27], following ideas of Fontaine and Illusie. The main body of work on logarithmic geometry

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has been concerned with deep applications in the cohomological study of p-adic and arithmetic schemes. This gave the theory an aura of "yet another extremely complicated theory". The treatments of the theory are however quite accessible. We hope to convince the reader here that the theory is simple enough and useful enough to be considered by anybody interested in moduli of singular varieties, indeed enough to be included in a Handbook of Moduli.

Normal crossings and logarithmic smoothness

So what is the original insight? Let X be a nonsingular irreducible complex variety, S a smooth curve with a point s and $f : X \to S$ a dominant morphism smooth away from s, in such a way that the fiber $f^{-1}s = X_s = Y_1 \cup \ldots \cup Y_m$ is a reduced simple normal crossings divisor. Then of course $\Omega_{X/S} = \Omega_X/f^*\Omega_S$ fails to be locally free at the singular points of f. But consider instead the sheaves $\Omega_X(\log(X_s))$ of differential forms with at most logarithmic poles along the Y_i, and similarly $\Omega_S(\log(s))$. Then there is an injective sheaf homomorphism $f^*\Omega_S(\log(s)) \to \Omega_X(\log(X_s))$, and *the quotient sheaf* $\Omega_X(\log(X_s))/\Omega_S(\log(s))$ *is locally free.*

So in terms of logarithmic forms, *the morphism* f *is as good as a smooth morphism.*

There is much more to be said: first, this $\Omega_X(\log(X_s))/\Omega_S(\log(s))$ can be extended to a logarithmic de Rham complex, and its hypercohomology, while not recovering the cohomology of the singular fibers, does give rise to the limiting Hodge structure. So it is evidently worth considering.

Second, the picture is quite a bit more general, and can be applied to all toric and toroidal maps between toric varieties or toroidal embeddings (with a little caveat about the characteristic of the residue fields). So there is some flexibility in choosing $X \to S$.

The search for a structure

Since we are considering moduli, then as soon as we consider $X \to S$ as above we must also consider the normal crossings fiber $X_s \to \{s\}$. But what structure should we put on this variety? The notion of differentials with logarithmic poles along X_s is not in itself intrinsic to X_s. Also the normal crossings variety X_s is not in itself toric or toroidal, so a new structure is needed to incorporate it into the picture.

One is tempted to consider varieties which are assembled from nice varieties by some sort of gluing, as normal crossings varieties are. But already normal crossings varieties do not give a satisfactory answer in general, because their deformation spaces have "bad" components. Here is a classical example: consider a smooth projective variety Z such that $\mathrm{Pic}^0(Z)$ is nontrivial. Let L be a line bundle on Z and set $Y = \mathbb{P}(\mathcal{O} \oplus L)$, with zero section $Z \subset Y$. Let X be the blowing up of $Z \times 0 \subset Y \times \mathbb{A}^1$. We have a flat morphism $f : X \to \mathbb{A}^1$ with fiber $X_0 = f^{-1}(0) \simeq Y \cup Y$, where the two copies of Y are glued with the zero section of one attached to the ∞ section of the other.

So clearly X_0 is a normal crossings variety with a nice smoothing to a copy of Y. But there are other deformations: the variety $Y \cup Y$ also deforms to $Y \cup Y'$ where $Y' = \mathbb{P}(\mathcal{O} \oplus L')$ and L' a deformation of the line bundle L. And it is not hard to see that $Y \cup Y'$ does not have a smoothing. Ideally one really does not want to see this deformation $Y \cup Y'$ in the picture - and ideally X_0 should have a natural structure whose deformation space excludes $Y \cup Y'$ automatically.

Such a structure was proposed by Friedman in [10], where the notion of *d-semistable varieties* was introduced. This structure is somewhat subtle, and while it solves the issue in this case, it is not quite as flexible as one could wish. As we will see in Section 5, logarithmic structures subsume d-semistability and do provide an appropriate flexibility.

Organization of this chapter

In this chapter we briefly describe logarithmic structures and indicate where they can be useful in the study of moduli spaces. Section 2 gives the basic definitions of logarithmic structures, and Section 3 discusses logarithmic differentials and log smooth deformations, which are important in considering moduli spaces.

Section 4 gives the first example where logarithmic geometry fits well with moduli spaces: the moduli space of stable curves is the moduli space of log smooth curves. The issue of d-semistability does not arise since a nodal curve is automatically d-semistable. So the theory for curves is simple. Turning to higher dimensions, Section 5 shows how d-semistability can be described using logarithmic structures.

If one is to enlarge algebraic geometry to include logarithmic structures, the task of generalizing the techniques of algebraic geometry to logarithmic structure can certainly seem daunting. In Section 6 we show how to encode logarithmic structure in terms of certain algebraic stacks. This allows us to reduce various constructions to the case of algebraic stacks. (One can argue that the theory of stacks is not simple either, but at least in the theory of moduli they have come to be accepted, with some exceptions [34].)

In Section 7 we make use of logarithmic stacks to describe the complexes which govern deformations and obstructions for logarithmic structures even in the non-smooth case. This comes in handy later. For instance, even when studying moduli of log-smooth schemes, the moduli spaces tend to be singular, and their cotangent complexes are necessary ingredients in constructing virtual fundamental classes.

Section 8 describes a beautiful construction, similar to polar coordinates, in which families of complex log smooth varieties give rise canonically to families of topological manifolds. Differential geometers have used polar coordinates on nodal curves to "make space" for monodromy to act by Dehn twists. Rounding (using Ogus's terminology) is a magnificent way to generalize this.

The immediate implications of logarithmic structures for De Rham cohomology and Hodge structures are described in Section 9.

We conclude by describing three applications, where logarithmic structures serve as the proverbial "magic powder" (term suggested by Kato and Ogus) to clarify or remove unwanted behavior from moduli spaces.

Section 10 describes a number of cases where the main irreducible component of a moduli space can be separated from other "unwanted" components by sprinkling the objects with a bit of logarithmic structure.

In Section 11 we introduce twisted curves, a central object of orbifold stable maps, and show how logarithmic structures give a palatable way to construct the moduli stack of twisted curves.

Section 12 gives background for the work of B. Kim, in which Jun Li's moduli space of relative stable maps, with its obstruction theory and virtual fundamental class, is beautifully simplified using logarithmic structures.

Notation

Following the lead of Ogus [45], we try whenever possible to denote a logarithmic scheme by a regular letter (such as X) and the underlying scheme by $\underline{X}$. When this is impossible we write X for the underlying scheme and $(X, \mathcal{M}_X)$ for a logarithmic scheme over it.

Acknowledgements

This chapter originated from lectures given by Olsson at the School and Workshop on Aspects of Moduli, June 15-28, 2008 at the De Giorgi Center at the Scuola Normale Superiore in Pisa, Italy. The material was revisited and expanded in our seminar during the Algebraic Geometry program at MSRI, 2009. We thank the De Giorgi Center, MSRI, their staff and program organizers for providing these opportunities. Thanks are due to Arthur Ogus and Phillip Griffiths, who lectured on two topics at the MSRI seminar. While no new material is intended here, we acknowledge that research by Abramovich, Gillam and Olsson is supported by the NSF.

2. Definitions and basic properties

In this section we introduce the basic definitions of logarithmic geometry in the sense of [27]. Good introductions are given in [27] and [45]. Further technique is developed in [12].

Logarithmic structures

The basic definitions are as follows:

Definition 2.1. A *monoid* is a commutative semi-group with a unit. A morphism of monoids is required to preserve the unit element. We use $\mathcal{M}on$ to denote the category of Monoids.

Definition 2.2. Let $\underline{X}$ be a scheme. A *pre-logarithmic structure* on $\underline{X}$ is a sheaf of monoids $\mathcal{M}_X$ on the étale site $\underline{X}_{\acute{e}t}$ combined with a morphism of sheaves of monoids: $\alpha : \mathcal{M}_X \longrightarrow \mathcal{O}_{\underline{X}}$, called the structure morphism, where we view $\mathcal{O}_{\underline{X}}$ as a monoid under multiplication. A pre-log structure is called a *log structure* if $\alpha^{-1}(\mathcal{O}^*_{\underline{X}}) \cong \mathcal{O}^*_{\underline{X}}$ via α. The pair $(\underline{X}, \mathcal{M}_X)$ is called a *log scheme*, and will be denoted by X.

Note that, given a log structure $\mathcal{M}_X$ on $\underline{X}$, we can view $\mathcal{O}^*_{\underline{X}}$ as a subsheaf $\mathcal{M}_X$.

Definition 2.3. Given a log scheme X, the quotient sheaf $\overline{\mathcal{M}}_X = \mathcal{M}_X/\mathcal{O}^*_{\underline{X}}$ is called the *characteristic of the log structure* $\mathcal{M}_X$.

Definition 2.4. Let $\mathcal{M}$ and $\mathcal{N}$ be pre-log structures on $\underline{X}$. A *morphism* between them is a morphism $\mathcal{M} \to \mathcal{N}$ of sheaves of monoids which is compatible with the structure morphisms.

How should one think of such a beast? There are two extreme cases:

(1) If an element $m \in \mathcal{M}$ has $\alpha(m) = x \neq 0$, one often thinks of m as some sort of partial data of a "branch of the logarithm of x". Evidently no data is added if x is invertible, but some is added otherwise. In particular, we will see later that m permits us to take the logarithmic differential dx/x of x.
(2) If $\alpha(m) = 0$ it is often the case that it m comes by restricting the log structure of an ambient space, and serves as the "ghost" of a logarithmic cotangent vector coming from that space. So the log structure "remembers" deformations that are lost when looking at the underlying scheme.

The log structure associated to a pre-log structure

We have a natural inclusion

$$i : (\text{log structures on } \underline{X}) \hookrightarrow (\text{pre-log structures on } \underline{X})$$

by viewing a log structure as a pre-log structure. We now construct a left adjoint.

Let $\alpha : \mathcal{M} \to \mathcal{O}_{\underline{X}}$ be a pre-log structure on $\underline{X}$. We define the *associated log structure* $\mathcal{M}^a$ to be the push-out of

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}^*_{\underline{X}}) & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}^*_{\underline{X}} & & \end{array}$$

in the category of sheaves of monoids on $\underline{X}_{\acute{e}t}$, endowed with

$$\mathcal{M}^a \to \mathcal{O}_{\underline{X}} \quad (a, b) \mapsto \alpha(a)b \ (a \in \mathcal{M}, b \in \mathcal{O}^*_{\underline{X}}).$$

In this way, we obtain a functor a : (pre-log structures on $\underline{X}$) $\to$ (log structures on $\underline{X}$). From the universal property of push-out, any morphism of pre-log structure from a pre-log structure $\mathcal{M}$ to a log structure on $\underline{X}$ factor through $\mathcal{M}^a$ uniquely.

Lemma 2.5. [45, 1.1.5] *The functor* a *is left adjoint to* i.

Example 2.6. The category of log structures on $\underline{X}$ has an initial object, called the trivial log structure, given by the inclusion $\mathcal{O}^*_{\underline{X}} \hookrightarrow \mathcal{O}_{\underline{X}}$. It also has a final object, given by the identity map $\mathcal{O}_{\underline{X}} \to \mathcal{O}_{\underline{X}}$. Trivial log structures are quite useful as they make the category of schemes into a full subcategory of the category of log schemes (see Definition 2.9). The final object is rarely used since it is not fine, see Definition 2.16.

Example 2.7. Let $\underline{X}$ be a regular scheme, and $D \subset \underline{X}$ a divisor. We can define a log structure $\mathcal{M}$ on $\underline{X}$ associated to the divisor D as

$$\mathcal{M}(U) = \{ g \in \mathcal{O}_{\underline{X}}(U) : g|_{U \setminus D} \in \mathcal{O}^*_{\underline{X}}(U \setminus D) \} \subset \mathcal{O}_{\underline{X}}(U).$$

The case where D is a normal crossings divisor is special - we will see later that it is *log smooth*.

Note that the concept of normal crossing is local in the étale topology. This is one reason we use the étale topology instead of the Zariski topology.

Example 2.8. Let P be a monoid, R a ring, and denote by R[P] the monoid algebra. Let $\underline{X} = \operatorname{Spec} R[P]$. Then $\underline{X}$ has a canonical log structure associated to the canonical map $P \to R[P]$. We denote by Spec $(P \to R[P])$ the log scheme with underlying $\underline{X}$, and the canonical log structure.

The inverse image and the category of log schemes

Let $f : \underline{X} \to \underline{Y}$ be a morphism of schemes. Given a log structure $\mathcal{M}_Y$ on $\underline{Y}$, we can define a log structure on $\underline{X}$, called the inverse image of $\mathcal{M}_Y$, to be the log structure associated to the pre-log structure $f^{-1}(\mathcal{M}_Y) \to f^{-1}(\mathcal{O}_{\underline{Y}}) \to \mathcal{O}_{\underline{X}}$. This is usually denoted by $f^*(\mathcal{M}_Y)$. Using the inverse image of log structures, we can give the following definition.

Definition 2.9. A *morphism of log schemes* $X \to Y$ consists of a morphism of underlying schemes $f : \underline{X} \to \underline{Y}$, and a morphism $f^\flat : f^*\mathcal{M}_Y \to \mathcal{M}_X$ of log structures on $\underline{X}$.

We denote by LSch the category of log schemes.

Example 2.10. In Example 2.8, the log structure on Spec $(P \to R[P])$ can be viewed as the inverse image of the log structure on Spec $(P \to \mathbb{Z}[P])$ via the canonical map Spec $(R[P]) \to$ Spec $(\mathbb{Z}[P])$.

Example 2.11. Let k be a field, $\underline{Y}$=Spec $k[x_1, \cdots, x_n]$, D=$V(x_1 \cdots x_r)$. Note that D is a normal crossing divisor in $\underline{Y}$. By Example 2.7, we have a log structure $\mathcal{M}_Y$ on $\underline{Y}$ associated to the divisor D. In fact, $\mathcal{M}_Y$ can be viewed as a submonoid of $\mathcal{O}_{\underline{Y}}$ generated by $\mathcal{O}^*_{\underline{Y}}$ and $\{x_1, \cdots, x_r\}$.

Consider the inclusion $j : p = \operatorname{Spec} k \hookrightarrow \underline{Y}$ sending the point to the origin of $\underline{Y}$. Then $j^*\mathcal{M}_Y = k^* \oplus \mathbb{N}^r$, and the structure map $j^*\mathcal{M} \longrightarrow \mathcal{O}_{\underline{X}}$ is given by $(a, n_1, \cdots, n_r) \mapsto a \cdot 0^{n_1 + \cdots + n_r}$, where we define $0^0 = 1$ and $0^n = 0$ if $n \neq 0$. Such point with the log structure above is call a *logarithmic point*; when $r = 1$ we call it the *standard logarithmic point*.