Jiongmin Yong
Xun Yu Zhou

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Stochastic
Modelling
and
Applied
Probability

Stochastic Controls

Hamiltonian Systems and HJB Equations

随机控制

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by Jiongmin Yong and Xun Yu Zhou

献给我们的父母：

雍文耀　陈湘霞

周庆基　沈采真

To Our Parents

Wenyao Yong and Xiangxia Chen

Qingji Zhou and Caizhen Shen

Preface

As is well known, Pontryagin's maximum principle and Bellman's dynamic programming are the two principal and most commonly used approaches in solving stochastic optimal control problems.* An interesting phenomenon one can observe from the literature is that these two approaches have been developed separately and independently. Since both methods are used to investigate the same problems, a natural question one will ask is the following:

(Q) *What is the relationship between the maximum principle and dynamic programming in stochastic optimal controls?*

There did exist some researches (prior to the 1980s) on the relationship between these two. Nevertheless, the results usually were stated in heuristic terms and proved under rather restrictive assumptions, which were not satisfied in most cases.

In the statement of a Pontryagin-type maximum principle there is an adjoint equation, which is an ordinary differential equation (ODE) in the (finite-dimensional) deterministic case and a stochastic differential equation (SDE) in the stochastic case. The system consisting of the adjoint equation, the original state equation, and the maximum condition is referred to as an *(extended) Hamiltonian system*. On the other hand, in Bellman's dynamic programming, there is a partial differential equation (PDE), of first order in the (finite-dimensional) deterministic case and of second order in the stochastic case. This is known as a *Hamilton Jacobi-Bellman (HJB) equation*. This leads to the following question, which is essentially a rephrase of Question (Q):

(Q′) *What is the relationship between Hamiltonian systems and HJB equations?*

Or, even more generally,

(Q″) *What is the relationship between ODEs/SDEs and PDEs?*

Once the question is asked this way, one will immediately realize that similar questions have already been or are being addressed and studied in other fields. Let us briefly recall them below.

Analytic Mechanics. Using Hamilton's principle and Legendre's transformation, one can describe dynamics of a system of particles by a

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* Here, by a stochastic optimal control problem we mean a completely observed control problem with a state equation of the Itô type and with a cost functional of the Bolza type; see Chapter 2 for details. When the diffusion coefficient is identically zero, and the controls are restricted to deterministic functions, the problem is reduced to a deterministic optimal control problem.

family of ODEs called *Hamilton's canonical system* or the *Hamiltonian system*. On the other hand, by introducing *Hamilton's principal function*, one can describe the particle system by a PDE called the *Hamilton–Jacobi (HJ) equation*. These two ways are in fact equivalent in the sense that the solutions of the canonical system can be represented by that of the HJ equation, and vice versa. One easily sees a strong analogy between optimal control and analytic mechanics. This is not surprising, however, since the classical calculus of variations, which is the foundation of analytic mechanics, is indeed the origin of optimal control theory.

Partial Differential Equations. There is a classical *method of characteristics* in solving PDEs. More specifically, for a first-order PDE, there is an associated family of ODEs for curves, called *characteristic strips*, by which the solutions to the PDE can be constructed. In the context of (deterministic) optimal controls, the Hamiltonian system involved in the maximum principle serves as the characteristics for the HJB equation involved in the dynamic programming.

Stochastic Analysis. The stochastic version of the method of characteristics is the *Feynman–Kac formula*, which represents the solutions to a *linear* second-order parabolic or elliptic PDE by those to some SDEs. On the other hand, a reversed representation has been recently developed via the so-called *four-step scheme*, which represents the solutions to a coupled forward–backward SDE by those to a PDE. A deterministic version of this is closely related to the so-called *invariant embedding*, which was studied by Bellman-Kalaba-Wing.

Economics. The key to understanding the economic interpretation of optimal control theory is the *shadow price* of a resource under consideration. The very definition of the shadow price originates from one of the relationships between the maximum principle and dynamic programming, namely, the shadow price (adjoint variable) is the rate of change of the performance measure (value function) with respect to the change of the resource (state variable).

Finance. The celebrated Black–Scholes formula indeed gives nothing but a way of representing the option price (which is the solution to a backward SDE) by the solution to the Black–Scholes equation (which is a parabolic PDE).

Interestingly enough, all the relationships described above can be captured by the following simple, generic mathematical formula:

$$y(t) = \theta(t, x(t)),$$

where $(x(t), y(t))$ satisfies some ODE/SDE and θ satisfies some PDE. For example, in the relationship between the maximum principle and dynamic programming, $(x(t), y(t))$ is the solution to the Hamiltonian system and $[-\theta]$ is the gradient in the spatial variable of the value function (which is the solution to the HJB equation). In the Black–Scholes model, $y(t)$ is the option price, $x(t)$ is the underlying stock price, and θ is the solution to the Black–Scholes PDE.

Before studying Question (Q), one has first to resolve the following two problems:

(P1) *What is a general stochastic maximum principle if the diffusion depends on the control and the control domain is not necessarily convex?*

This problem has been investigated since the 1960s. However, almost all the results prior to 1980 assume that the diffusion term does not depend on the control variable and/or the diffusion depends on the control but the control domain is convex. Under these assumptions, the statements of the maximum principle and their proofs are very much parallel to those of the deterministic case. One does not see much essential difference between stochastic and deterministic systems from those results. The stochastic maximum principle for systems with control-dependent diffusion coefficients and possibly nonconvex control domains had long been an outstanding open problem until 1988.

(P2) *How is one to deal with the inherent nonsmoothness when studying the relationship between the maximum principle and dynamic programming?*

The relationship unavoidably involves the derivatives of the value functions, which as is well known could be nonsmooth in even very simple cases.

During 1987–1989, a group led by Xunjing Li at the Institute of Mathematics, Fudan University, including Ying Hu, Jin Ma, Shige Peng, and the two authors of the present book, was studying those problems and related issues in their weekly seminars. They insisted on tackling the control-dependent diffusion cases, and this insistence was based on the following belief: Only when the controls/decisions could or would influence the scale of uncertainty (as is indeed the case in many practical systems, especially in the area of finance) do the stochastic problems differ from the deterministic ones. In the stimulating environment of the seminars, Problems (P1) and (P2) were solved almost at the same time in late 1988, based on the introduction of the so-called *second-order adjoint equation*. Specifically, Peng, then a postdoctoral fellow at Fudan, solved Problem (P1) by considering the quadratic terms in the Taylor expansion of the spike variation, via which he established a new form of maximum principle for stochastic optimal controls. On the other hand, Zhou (who was then a Ph.D. student at Fudan) found a powerful way for solving Problem (P2). By utilizing viscosity solution theory, he managed to disclose the relationship between the first-order (respectively, second-order) adjoint equations and the first-order (respectively, second-order) derivatives of the value functions. After 1989, members of the Fudan group went to different places in the world, but the research they carried out at Fudan formed the foundation of their further research. In particular, studies on nonlinear backward and forward–backward

SDEs by Pardoux–Peng, and Ma–Yong are natural extensions of those on the (linear) adjoint equations in the stochastic maximum principle. This fashionable theory soon became a notable topic among probabilists and control theorists, and found interesting applications in stochastic analysis, PDE theory, and mathematical finance. The remarkable work on the nonlinear Feynman–Kac formula (representing solutions to *nonlinear* PDEs by those to backward SDEs) by Peng and the four-step scheme (representing solutions to forward–backward SDEs by those to PDEs) by Ma–Protter–Yong once again remind us about their analogy in stochastic controls, namely, the relationship between stochastic Hamiltonian systems and HJB equations. On the other hand, (stochastic) verification theorems by Zhou and Zhou–Yong–Li by means of viscosity solutions are extensions of the relationship between the maximum principle and dynamic programming from open-loop controls to feedback controls. These verification theorems lead to optimal feedback synthesis without involving derivatives of the value functions. Finally, the recent work by Chen–Li–Zhou, Chen–Yong, and Chen–Zhou on stochastic linear quadratic (LQ) controls with *indefinite* control weighting matrices in costs demonstrates how fundamentally different it is when the control enters into the diffusion term. The LQ case also provides an important example where the maximum principle and dynamic programming are *equivalent* via the *stochastic Riccati equation*.

The purpose of this book is to give a systematic and self-contained presentation of the work done by the Fudan group and related work done by others, with the core being the study on Question (Q) or (Q′). In other words, the theme of the book is to unify the maximum principle and dynamic programming, and to demonstrate that viscosity solution theory provides a nice framework to unify them. While the main context is in stochastic optimal controls, we try whenever possible to disclose some intrinsic relationship among ODEs, SDEs, and PDEs that may go beyond control theory. When writing the book, we paid every attention to the coherence and consistency of the materials presented, so that all the chapters are closely related to each other to support the central theme. In some sense, the idea of the whole book may be boiled down to the single formula $y(t) = \theta(t, x(t))$, which was mentioned earlier. That said, we do not mean to trivialize things; rather we want to emphasize the common ground of seemingly different theories in different areas. In this perspective, the Black–Scholes formula, for instance, would not surprise a person who is familiar with mechanics or the Feynman–Kac formula.

Let us now sketch the main contents of each chapter of the book.

Chapter 1. Since the book is intended to be self-contained, some preliminary materials on stochastic calculus are presented. Specifically, this chapter collects notions and results in stochastic calculus scattered around in the literature that are related to stochastic controls. It also unifies terminology and notation (which may differ in different papers/books) that are to be used in later chapters. These materials are mainly for beginners (say, graduate students). They also serve as a quick reference for knowledgeable readers.

Chapter 2. The stochastic optimal control problem is formulated and some examples of real applications are given. However, the chapter starts with the deterministic case. This practice of beginning with deterministic problems is carried out in Chapters 3–6 as well. The reasons for doing so are not only that the deterministic case itself may contain important and interesting results, but also that readers can see the essential difference between the deterministic and stochastic systems. In the formulation of stochastic control problems we introduce strong and weak formulations and emphasize the difference between the two, which is not usually spelled out explicitly in the literature. Stochastic control models other than the "standard" one studied in this book are also briefly discussed. This chapter finally provides a very extensive literature review ranging from the very origin of optimal control problems to all the models and applied examples presented in this chapter.

Chapter 3. A stochastic Hamiltonian system is introduced that consists of two backward SDEs (adjoint equations) and one forward SDE (the original state equation) along with a maximum condition. The general stochastic maximum principle is then stated and proved. Cases with terminal state constraints and sufficiency of the maximum principle are discussed.

Chapter 4. First, a stochastic version of Bellman's principle of optimality is *proved* by virtue of the weak formulation, based on which HJB equations are derived. The viscosity solution is introduced as the tool to handle the inherent nonsmoothness of the value functions. Some properties of the value functions and viscosity solutions of the HJB equations are then studied. It is emphasized that the time variable here plays a special role due to the nonanticipativeness of the underlying system. Finally, a simplified proof (compared with the existing ones) of the uniqueness of the viscosity solutions is presented. Notice that the verification technique involved in the dynamic programming is deferred to Chapter 5.

Chapter 5. Classical Hamilton–Jacobi theory in mechanics is reviewed first to demonstrate the origin of the study of the relationship between the maximum principle and dynamic programming. The relationship for deterministic systems is investigated and is compared with the method of characteristics in PDE theory, the Feynman–Kac formula in probability theory, and the shadow price in economics. The relationship for stochastic systems is then studied. It starts with the case where the value function is smooth to give some insights, followed by a detailed analysis for the nonsmooth case. Finally, stochastic verification theorems workable for the nonsmooth situation are given, and the construction of optimal feedback controls is discussed.

Chapter 6. This chapter investigates a special case of optimal control problems, namely, the linear quadratic optimal control problems (LQ problems). They constitute an extremely important class of optimal control problems, and the solutions of LQ problems exhibit elegant properties due to their simple and nice structures. They also nicely exemplify the general theory developed in Chapters 3–5. In the chapter an LQ problem is first treated as an optimization problem in an infinite-dimensional space, and abstract results are obtained to give insights. Then linear optimal state feedback is established via the so-called stochastic Riccati equation. It is pointed out that both the maximum principle and dynamic programming can lead to the stochastic Riccati equation, by which one can see more clearly the relationship between the maximum principle and dynamic programming (actually, these two approaches are *equivalent* in the LQ case). We emphasize that the control weighting matrices in the cost are allowed to be *indefinite* in our formulation. Therefore, it is essentially different from the deterministic case. Stochastic Riccati equations are extensively studied for various cases. Finally, as an example, a mean–variance portfolio selection is solved by the LQ method developed.

Chapter 7. This chapter presents the latest development on backward and forward–backward SDEs, with an emphasis on the relationship between nonlinear SDEs and nonlinear PDEs. Although the topics in this chapter go beyond the scope of stochastic controls, they originate from stochastic controls as mentioned earlier. The chapter begins with the original argument of Bismut for studying linear backward SDEs by using the martingale representation theorem. Then the existence and uniqueness of solutions to nonlinear backward SDEs are investigated for two types of time durations, finite deterministic horizon and random horizon, by virtue of two different methods. Feynman–Kac-type formulae with respect to both forward and backward SDEs are presented. Next, a kind of inverse of the Feynman–Kac-type formulae, the so-called four-step scheme, which represents solutions to forward–backward SDEs by those to PDEs, is discussed. Solvability and nonsolvability of forward–backward SDEs are also analyzed. Finally, the Black–Scholes formula in option pricing is derived by the four-step scheme.

The idea of writing such a book was around in late 1994 when JY was visiting XYZ in Hong Kong. While discussing the stochastic verification theorems, they realized that the series of works done by the Fudan group were rich enough for a book, and there *should* be a book as a systematic account of these results. The plan became firm with encouragement from Wendell Fleming (Brown), Ioannis Karatzas (Columbia), and Xunjing Li (Fudan). The authors are greatly indebted to Robert Elliott (Alberta), Wendell Fleming (Brown), Ulrich Haussmann (British Columbia), Ioannis Karatzas (Columbia), Thomas Kurtz (Wisconsin-Madison), Mete Soner (Princeton), and Michael Taksar (SUNY-Stony Brook), who substantially reviewed some or all chapters, which led to a much improved version. Michael Kohlmann (Konstanz) and Andrew Lim (CUHK) read carefully large portions of the manuscript and offered numerous helpful suggestions. During various stages in the prolonged, four-year course of the project, many experts and friends have shown their concern and encouragement, conveyed their comments, or sent their research works for the book. Among them the following deserve special mention: Alain Bensoussan (CNES), Leonard Berkovitz (Purdue), Giuseppe Da Prato (Scuola Normale Superiore), Darrell Duffie (Stanford), Tyrone Duncan (Kansas), Fausto Gozzi (Scuola Normal Superiore), Suzanne Lenhart (Tennessee), Zhuangyi Liu (Minnesota-Duluth), John Moore (ANU), Makiko Nisio (Kobe), Bozenna Pasik-Duncan (Kansas), Thomas Seidman (Maryland-Baltimore County), Hiroshi Tanaka (Keio), Wing Wong (CUHK), Jia-an Yan (Academia Sinica), George Yin (Wayne State), and Qing Zhang (Georgia). Especially, both authors would like to express appreciation to their long-time teachers and/or colleagues in the Fudan group: Shuping Chen (Zhejiang), Ying Hu (Rennes), Xunjing Li (Fudan), Jin Ma (Purdue), Shige Peng (Shandong), and Shanjian Tang (Fudan), whose elegant research work provided a rich source for this book.

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JY, Shanghai
XYZ, Hong Kong

November 1998

Notation

The following notation is frequently used in the book.

$\mathbb{R}^n$ — n-dimensional real Euclidean space.

$\mathbb{R}^{n\times m}$ — the set of all $(n \times m)$ real matrices.

$\mathcal{S}^n$ — the set of all $(n \times n)$ symmetric matrices.

$\mathcal{S}^n_+$ — the set of all $(n \times n)$ nonnegative definite matrices.

$\widehat{\mathcal{S}}^n_+$ — the set of all $(n \times n)$ positive definite matrices.

$\operatorname{tr}(A)$ — the trace of the square matrix A.

$x^\top$ — the transpose of the vector (or matrix) x.

$\langle\cdot,\cdot\rangle$ — inner product in some Hilbert space.

$\mathbf{Q}$ — the set of all rational numbers.

$\mathbb{N}$ — the set of natural numbers.

$|N|$ — Lebesgue measure of the set N.

$\stackrel{\Delta}{=}$ — Defined to be (see below).

I_A — the indicator function of the set A: $I_A(x) \stackrel{\Delta}{=} \begin{cases} 1, & x\in A, \\ 0, & x\notin A. \end{cases}$

$\varphi^+ \stackrel{\Delta}{=} \max\{\varphi,0\}, \qquad \varphi^- \stackrel{\Delta}{=} -\min\{\varphi,0\}.$

$a\vee b \stackrel{\Delta}{=} \max\{a,b\}, \qquad a\wedge b \stackrel{\Delta}{=} \min\{a,b\}.$

2^Ω — the set of all subsets of Ω.

ϕ — the empty set.

$C([0,T];\mathbb{R}^n)$ — the set of all continuous functions $\varphi:[0,T]\to\mathbb{R}^n$.

$C([0,\infty);\mathbb{R}^n)$ — the set of all continuous functions $\varphi:[0,\infty)\to\mathbb{R}^n$.

$C_b(U)$ — the set of all uniformly bounded, continuous functions on U.

$L^p(0,T;\mathbb{R}^n)$ — the set of Lebesgue measurable functions $\varphi:[0,T]\to\mathbb{R}^n$ such that $\int_0^T|\varphi(t)|^p dt<\infty$ ($p\in[1,\infty)$).

$L^\infty(0,T;\mathbb{R}^n)$ — the set of essentially bounded measurable functions $\varphi:[0,T]\to\mathbb{R}^n$.

$(\Omega,\mathcal{F},\mathbf{P})$ — probability space.

$\{\mathcal{F}_t\}_{t\geq 0}$ — filtration.

$(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbf{P})$ — filtered probability space.

$\mathcal{B}(U)$ — the Borel σ-field generated by all the open sets in U.

$\sigma(\xi)\stackrel{\Delta}{=}\xi^{-1}(\mathcal{F})$ — the σ-field generated by the random variable ξ.

$\sigma(\mathcal{A})$ — the smallest σ-field containing the class $\mathcal{A}$,

$\bigvee_\alpha \mathcal{F}_\alpha \stackrel{\Delta}{=} \sigma(\bigcup_\alpha \mathcal{F}_\alpha), \qquad \bigwedge_\alpha \mathcal{F}_\alpha \stackrel{\Delta}{=} \bigcap_\alpha \mathcal{F}_\alpha.$

$\mathbf{P}_\xi = \mathbf{P}\circ\xi^{-1}$ — the probability measure induced by the random variable ξ.

EX — the expectation of the random variable X.

$\text{Cov}(X,Y) \triangleq E[(X-EX)(Y-EY)^\top]$, $\quad$ $\text{Var}\,X \triangleq \text{Cov}(X,X)$.

$E(X|\mathcal{G})$ — conditional expectation of X given $\mathcal{G}$.

$L^p_{\mathcal{G}}(\Omega;\mathbb{R}^n)$ — the set of $\mathbb{R}^n$-valued $\mathcal{G}$-measurable random variables X such that $E|X|^p < \infty$ ($p \in [1,\infty)$).

$L^\infty_{\mathcal{G}}(\Omega;\mathbb{R}^n)$ — the set of bounded $\mathbb{R}^n$-valued $\mathcal{G}$-measurable random variables.

$L^p_{\mathcal{F}}(0,T;\mathbb{R}^n)$ — the set of all $\{\mathcal{F}\}_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes $X(\cdot)$ such that $E\int_0^T |X(t)|^p dt < \infty$.

$L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^n)$ — the set of $\{\mathcal{F}\}_{t\geq 0}$-adapted $\mathbb{R}^n$-valued essentially bounded processes.

$L^p_{\mathcal{F}}(\Omega;C([0,T];\mathbb{R}^n))$ — the set of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^n$-valued continuous processes $X(\cdot)$ such that $E\sup_{t\in[0,T]}|X(t)|^p < \infty$ ($p \in [1,\infty)$).

$\mathcal{M}^2[0,T]$ — the set of square-integrable martingales.

$\mathcal{M}^2_c[0,T]$ — the set of square-integrable continuous martingales.

$\mathcal{M}^{2,loc}[0,T]$ — the set of square-integrable local martingales.

$\mathcal{M}^{2,loc}_c[0,T]$ — the set of square-integrable continuous local martingales.

$\mathbf{W}^n[0,T] \triangleq C([0,T];\mathbb{R}^n)$, $\quad$ $\mathbf{W}^n \triangleq C([0,\infty);\mathbb{R}^n)$.

$\mathbf{C}_s$ — the set of Borel cylinders in $\mathbf{W}^n[0,s]$.

$\mathbf{C}$ — the set of Borel cylinders in $\mathbf{W}^n$.

$\mathbf{W}^n_t[0,T] \triangleq \{\zeta(\cdot\wedge t) \mid \zeta(\cdot) \in \mathbf{W}^n[0,T]\}$, $\quad t \in [0,T]$.

$\mathcal{B}_t(\mathbf{W}^n[0,T]) \triangleq \mathcal{B}(\mathbf{W}^n_t[0,T])$, $\quad t \in [0,T]$.

$\mathcal{B}_{t+}(\mathbf{W}^n[0,T]) \triangleq \bigcap_{s>t}\mathcal{B}_s(\mathbf{W}^n[0,T])$, $\quad t \in [0,T)$.

$\mathbf{W}^n_t \triangleq \{\zeta(\cdot\wedge t) \mid \zeta(\cdot) \in \mathbf{W}^n\}$, $\quad t \geq 0$.

$\mathcal{B}_t(\mathbf{W}^n) \triangleq \mathcal{B}(\mathbf{W}^n_t)$, $\quad t \geq 0$.

$\mathcal{B}_{t+}(\mathbf{W}^n) \triangleq \bigcap_{s>t}\mathcal{B}_s(\mathbf{W}^n)$, $\quad t \geq 0$.

$\mathcal{A}^n_T(U)$ — the set of all $\{\mathcal{B}_{t+}(\mathbf{W}^n[0,T])\}_{t\geq 0}$-progressively measurable processes $\eta : [0,T]\times\mathbf{W}^n[0,T] \to U$.

$\mathcal{A}^n(U)$ — the set of all $\{\mathcal{B}_{t+}(\mathbf{W}^n\}_{t\geq 0}$-progressively measurable processes $\eta : [0,\infty)\times\mathbf{W}^n \to U$.

$\mathcal{V}[0,T]$ — the set of all measurable functions $u : [0,T] \to U$.

$\mathcal{U}[0,T]$ — the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $u : [0,T]\times\Omega \to U$.

$\mathcal{V}_{ad}[0,T]$ — the set of deterministic admissible controls.

$\mathcal{U}^s_{ad}[0,T]$ — the set of (stochastic) strong admissible controls.

$\mathcal{U}^w_{ad}[0,T]$ — the set of (stochastic) weak admissible controls.

Assumption Index

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