Graduate Texts in Mathematics

Bruce E. Sagan

The Symmetric Group

Representations, Combinatorial Algorithms, and Symmetric Functions

Second Edition

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Second Edition

With 31 Figures

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Preface to the 2nd Edition

I have been very gratified by the response to the first edition, which has resulted in it being sold out. This put some pressure on me to come out with a second edition and now, finally, here it is.

The original text has stayed much the same, the major change being in the treatment of the hook formula which is now based on the beautiful Novelli-Pak-Stoyanovskii bijection [NPS 97]. I have also added a chapter on applications of the material from the first edition. This includes Stanley's theory of differential posets [Stn 88, Stn 90] and Fomin's related concept of growths [Fom 86, Fom 94, Fom 95], which extends some of the combinatorics of S_n-representations. Next come a couple of sections showing how groups acting on posets give rise to interesting representations that can be used to prove unimodality results [Stn 82]. Finally, we discuss Stanley's symmetric function analogue of the chromatic polynomial of a graph [Stn 95, Stn ta].

I would like to thank all the people, too numerous to mention, who pointed out typos in the first edition. My computer has been severely reprimanded for making them. Thanks also go to Christian Krattenthaler, Tom Roby, and Richard Stanley, all of whom read portions of the new material and gave me their comments. Finally, I would like to give my heartfelt thanks to my editor at Springer, Ina Lindemann, who has been very supportive and helpful through various difficult times.

Ann Arbor, Michigan, 2000

Preface to the 1st Edition

In recent years there has been a resurgence of interest in representations of symmetric groups (as well as other Coxeter groups). This topic can be approached from three directions: by applying results from the general theory of group representations, by employing combinatorial techniques, or by using symmetric functions. The fact that this area is the confluence of several strains of mathematics makes it an exciting one in which to study and work. By the same token, it is more difficult to master.

The purpose of this monograph is to bring together, for the first time under one cover, many of the important results in this field. To make the work accessible to the widest possible audience, a minimal amount of prior knowledge is assumed. The only prerequisites are a familiarity with elementary group theory and linear algebra. All other results about representations, combinatorics, and symmetric functions are developed as they are needed. Hence this book could be read by a graduate student or even a very bright undergraduate. For researchers I have also included topics from recent journal articles and even material that has not yet been published.

Chapter 1 is an introduction to group representations, with special emphasis on the methods of use when working with the symmetric groups. Because of space limitations, many important topics that are not germane to the rest of the development are not covered. These subjects can be found in any of the standard texts on representation theory.

In Chapter 2, the results from the previous chapter are applied to the symmetric group itself, and more highly specialized machinery is developed to handle this case. I have chosen to take the elegant approach afforded by the Specht modules rather than working with idempotents in the group algebra.

The third chapter focuses on combinatorics. It starts with the two famous formulae for the dimensions of the Specht modules: the Frame-Robinson-Thrall hook formula and the Frobenius-Young determinantal formula. The centerpiece is the Robinson-Schensted-Knuth algorithm, which allows us to describe some of the earlier theorems in purely combinatorial terms. A thorough discussion of Schützenberger's jeu de taquin and related matters is included.

Chapter 4 recasts much of the previous work in the language of symmetric functions. Schur functions are introduced, first combinatorially as the generating functions for semistandard tableaux and then in terms of symmetric group characters. The chapter concludes with the famous Littlewood-Richardson and Murnaghan-Nakayama rules.

My debt to several other books will be evident. Much of Chapter 1 is based on Ledermann's exquisite text on group characters [Led 77]. Chapter

2 borrows heavily from the monograph of James [Jam 78], whereas Chapter 4 is inspired by Macdonald's already classic book [Mac 79]. Finally, the third chapter is a synthesis of material from the research literature.

There are numerous applications of representations of groups, and in particular of the symmetric group, to other areas. For example, they arise in physics [Boe 70], probability and statistics [Dia 88], topological graph theory [Whi 84], and the theory of partially ordered sets [Stn 82]. However, to keep the length of this text reasonable, I have discussed only the connections with combinatorial algorithms.

This book grew out of a course that I taught while visiting the Université du Québec à Montréal during the fall of 1986. I would like to thank *l'équipe de combinatoire* for arranging my stay. I also presented this material in a class here at Michigan State University in the winter and spring of 1990. I thank my students in both courses for many helpful suggestions (and those at UQAM for tolerating my bad French). Francesco Brenti, Kathy Dempsey, Yoav Dvir, Kathy Jankoviak, and Scott Mathison have all pointed out ways in which the presentation could be improved. I also wish to express my appreciation of John Kimmel, Marlene Thom, and Linda Loba at Wadsworth and Brooks/Cole for their help during the preparation of the manuscript. Because I typeset this document myself, all errors can be blamed on my computer.

East Lansing, Michigan, 1991

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List of Symbols

Symbol	Meaning	Page
A	poset	58
A_k	kth rank of a graded poset	195
A_S	rank selected poset	204
a_μ	alternant corresponding to composition μ	164
$\alpha\backslash\alpha_1$	composition α with first part α_1 removed	180
β	set partition	215
$\mathbb{C}$	complex numbers	4
ch^n	characteristic map	167
C_n	cyclic group of order n	5
C_t	column-stabilizer group of Young tableau t	60
$\mathrm{Com}\, X$	commutant algebra of representation X	23
$\mathbb{C}[\mathbf{G}]$	group algebra of G over $\mathbb{C}$	8
$\mathbb{C}[\mathbf{S}]$	vector space generated by set S over $\mathbb{C}$	7
$\mathbb{C}[[x]]$	ring of formal power series in x over $\mathbb{C}$	142
c_x	column insertion operator for integer x	95
$c_{\mu\nu}^{\lambda}$	Littlewood-Richardson coefficient	175
$\chi(g)$	character of group element g	30
χ_K	character value on conjugacy class K	32
χ^{def}	defining character of S_n	31
χ^{reg}	regular character	31
$\chi\downarrow_H^G$	restriction of character χ from G to H	45
$\chi\uparrow_H^G$	induction of character χ from h to G	45
D	down operator of a poset	196
$d_k(\pi)$	length of π's longest k-decreasing subsequence	103
ΔQ	delta operator applied to tableau Q	121
$\deg X$	degree of representation X	4
$\mathrm{Des}\, P$	the descent set of a tableau P	206
e_λ	λth elementary symmetric function	154
e_n	nth elementary symmetric function	152
e_t	polytabloid associated with Young tableau t	61
$E(\Gamma)$	edges of graph Γ	213

Symbol	Meaning	Page
$\operatorname{End} V$	endomorphism algebra of module V	23
$\operatorname{ev} Q$	the evacuation tableau of Q	122
ϵ	identity element of a group	1
f_S	weight generating function of weighted set S	145
f^λ	number of standard λ-tableaux	73
ϕ^λ	character of permutation module M^λ	56
G	group	1
$g_{A,B}$	Garnir element of a pair of sets A, B	70
GL_d	$d \times d$ complex general linear group	4
GP	generalized permutations	169
GP$'$	generalized permutations, no repeated columns	172
Γ	graph	213
h_n	nth complete homogeneous symmetric function	152
h_λ	λth complete homogeneous symmetric function	154
$\operatorname{Hom}(V, W)$	all module homomorphisms from V to W	22
$h_{i,j}$	hooklength of cell (i, j)	124
$H_{i,j}$	hook of cell (i, j)	124
h_v	hooklength of cell v	124
H_v	hook of cell v	124
I	identity matrix	23
I_d	$d \times d$ identity matrix	24
$i_k(\pi)$	length of π's longest k-increasing subsequence	103
$i_\lambda(V)$	number of stable partitions of V of type λ	215
$\operatorname{im} \theta$	image of θ	21
$j(P)$	jeu de taquin tableau of P	116
$j_c(P)$	forward slide on tableau P into cell c	113
$j^c(P)$	backward slide on tableau P into cell c	113
$j_Q(P)$	sequence of forward slides on P into Q	118
$j^P(Q)$	sequence of backward slides on Q into P	118
$\ker \theta$	kernel of θ	21
K_g	conjugacy class of a group element g	3
k_λ	size of K_λ	3
K_λ	conjugacy class in S_n corresponding to λ	3
$K_{\lambda\mu}$	Kostka number	85
κ_t	signed column sum for Young tableau t	61

Symbol	Meaning	Page
$l(\lambda)$	length (number of parts) of partition λ	152
$ll(\xi)$	leg length of rim hook ξ	180
λ	integer partition	2
	integer composition	67
λ'	conjugate or transpose of partition λ	103
$(\lambda_1, \lambda_2, \ldots, \lambda_l)$	partition given as a sequence	2
	composition given as a sequence	15
λ/μ	skew shape	112
$\lambda \backslash \xi$	partition λ with rim hook ξ removed	180
$\lambda(\beta)$	type of the set partition β	215
Λ	algebra of symmetric functions	151
Λ_l	symmetric functions in l variables	163
Λ^n	symmetric functions homogeneous of degree n	152
Mat_d	full $d \times d$ matrix algebra	4
m_λ	λth monomial symmetric function	151
M^λ	permutation module corresponding to λ	56
P	partial tableau	92
$p(n)$	number of partitions of n	143
p_n	nth power sum symmetric function	152
p_λ	λth power sum symmetric function	154
$P(\pi)$	P-tableau of permutation π	93
P_Γ	chromatic polynomial of graph Γ	214
$\mathbb{P}$	positive integers	214
π	permutation	1
π_P	row word of the Young tableau P	101
π^r	reversal of permutation π	97
$\hat{\pi}$	top row of generalized permutation π	169
$\check{\pi}$	bottom row of generalized permutation π	169
$\pi \stackrel{\text{R-S}}{\longleftrightarrow} (P, Q)$	Robinson-Schensted map	92
$\pi \stackrel{\text{R-S-K}}{\longleftrightarrow} (P, Q)$	Robinson-Schensted-Knuth map	170
$\pi \stackrel{\text{R-S-K}'}{\longleftrightarrow} (P, Q)$	dual Robinson-Schensted-Knuth map	172
$Q(\pi)$	Q-tableau of permutation π	93
$R(G)$	vector space of class functions on G	32
R^n	class functions of S_n	167
R_t	row-stabilizer group of Young tableau t	60
r_x	row-insertion operator for the integer x	93
rk	the rank function in a poset	195

Symbol	Meaning	Page
$\mathcal{S}_A$	group of permutations of A	20
s_λ	Schur function associated with λ	155
$s_{\lambda/\mu}$	skew Schur function associated with λ/μ	175
$\mathcal{S}_\lambda$	Young subgroup associated with partition λ	54
S^λ	Specht module associated with partition λ	62
$\text{sgn}(\pi)$	sign of permutation π	4
$\text{sh}\, t$	shape of Young tableau t	55
$\mathcal{S}_n$	symmetric group on $\{1, 2, \ldots, n\}$	1
t	Young tableau	55
T	generalized Young tableau	78
$\{t\}$	row tabloid of t	55
$[t]$	column tabloid of t	72
$\mathcal{T}_{\lambda\mu}$	generalized tableaux, shape λ, content μ	78
$\mathcal{T}^0_{\lambda\mu}$	semistandard tableaux, shape λ, content μ	81
θ	homomorphism of modules	18
θ_T	homomorphism corresponding to tableau T	80
$\bar{\theta}_T$	restriction of θ_T to a Specht module	81
U	up operator of a poset	196
$v_Q(P)$	vacating tableau for $j_Q(P)$	118
$v^P(Q)$	vacating tableau for $j^P(Q)$	118
$V(\Gamma)$	vertices of graph Γ	213
$W^\perp$	orthogonal complement of W	15
wt	weight function	145
$\mathbf{x}$	the set of variables $\{x_1, x_2, x_3, \ldots\}$	151
$X(g)$	matrix of g in the representation X	4
$\mathbf{x}^T$	monomial weight of a tableau T	155
$\mathbf{x}^\mu$	monomial weight of a composition μ	155
$X\downarrow^G_H$	restriction of representation X from G to H	45
$X\uparrow^G_H$	induction of representation X from H to G	45
X_Γ	chromatic symmetric function of graph Γ	214
ξ	rim or skew hook	180
Y	Young's lattice	192
Z_A	center of algebra A	27
Z_g	centralizer of group element g	3
z_λ	size of Z_g where $g \in S_n$ has cycle type λ	3
$\mathbb{Z}$	integers	204

Symbol	Meaning	Page
$\hat{0}$	minimum of a poset	195
$\hat{1}$	maximum of a poset	204
1_G	trivial representation of G	5
$(1^{m_1}, 2^{m_2}, \ldots)$	partition given by multiplicities	2
$\cong$	equivalence of modules	19
	slide equivalence of tableaux	114
$\overset{*}{\cong}$	dual equivalence of tableaux	117
$\overset{K}{\cong}$	Knuth equivalence	100
$\overset{K^*}{\cong}$	dual Knuth equivalence	111
$\overset{P}{\cong}$	P-equivalence	99
$\overset{Q}{\cong}$	Q-equivalence	111
$\overset{1}{\cong}$	Knuth relation of the first kind	99
$\overset{1^*}{\cong}$	dual Knuth relation of the first kind	111
$\overset{2}{\cong}$	Knuth relation of the second kind	99
$\overset{2^*}{\cong}$	dual Knuth relation of the second kind	111
$\leq$	subgroup relation	9
	submodule relation	10
	lexicographic order on partitions	59
$\prec$	covering relation in a poset	58
$\trianglerighteq$	dominance order on partitions	58
	dominance order on tabloids	68
$\vdash$	is a partition of for integers	53
	is a partition of for sets	215
$\oplus$	direct sum of matrices	13
	direct sum of vector spaces	13
$\otimes$	tensor product of matrices	25
	tensor product of representations	43
	tensor product of vector spaces	26
$\langle \chi, \psi \rangle$	inner product of characters χ and ψ	34
$\chi \cdot \psi$	product of characters	168
$A \times B$	product of posets	198
$G \wr H$	wreath product of groups	212
$\cup$	union of tableaux	120
$\uplus$	disjoint union	45
$\lvert \cdot \rvert$	cardinality of a set	3
	sum of parts of a partition	54
$\wedge$	meet operation in a poset	192
$\vee$	join operation in a poset	192

Chapter 1

Group Representations

We begin our study of the symmetric group by considering its representations. First, however, we must present some general results about group representations that will be useful in our special case. Representation theory can be couched in terms of matrices or in the language of modules. We consider both approaches and then turn to the associated theory of characters. All our work will use the complex numbers as the ground field in order to make life as easy as possible.

We are presenting the material in this chapter so that this book will be relatively self-contained, although it all can be found in other standard texts. In particular, our exposition is modeled on the one in Ledermann [Led 77].

1.1 Fundamental Concepts

In this section we introduce some basic terminology and notation. We pay particular attention to the symmetric group.

Let G be a group written multiplicatively with identity ϵ. Throughout this work, G is finite unless stated otherwise. We assume that the reader is familiar with the elementary properties of groups (cosets, Lagrange's theorem, etc.) that can be found in any standard text such as Herstein [Her 64].

Our object of study is the *symmetric group*, S_n, consisting of all bijections from $\{1, 2, \ldots, n\}$ to itself using composition as the multiplication. The elements $\pi \in S_n$ are called *permutations*. We multiply permutations from right to left. (In fact, we compose all functions in this manner.) Thus $\pi\sigma$ is the bijection obtained by first applying σ, followed by π.

If π is a permutation, then there are three different notations we can use for this element. *Two-line notation* is the array

$$\pi = \begin{matrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{matrix} .$$

For example, if $\pi \in \mathcal{S}_5$ is given by

$$\pi(1) = 2, \qquad \pi(2) = 3, \qquad \pi(3) = 1, \qquad \pi(4) = 4, \qquad \pi(5) = 5,$$

then its two-line form is

$$\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{matrix}.$$

Because the top line is fixed, we can drop it to get *one-line notation.*

Lastly, we can display π using *cycle notation.* Given $i \in \{1, 2, \ldots, n\}$, the elements of the sequence $i, \pi(i), \pi^2(i), \pi^3(i), \ldots$ cannot all be distinct. Taking the first power p such that $\pi^p(i) = i$, we have the cycle

$$(i, \pi(i), \pi^2(i), \ldots, \pi^{p-1}(i)).$$

Equivalently, the cycle $(i, j, k, \ldots, l)$ means that π sends i to j, j to k, ... , and l back to i. Now pick an element not in the cycle containing i and iterate this process until all members of $\{1, 2, \ldots, n\}$ have been used. Our example from the last paragraph becomes

$$\pi = (1, 2, 3)(4)(5)$$

in cycle notation. Note that cyclically permuting the elements within a cycle or reordering the cycles themselves does not change the permutation. Thus

$$(1, 2, 3)(4)(5) = (2, 3, 1)(4)(5) = (4)(2, 3, 1)(5) = (4)(5)(3, 1, 2).$$

A *k-cycle*, or *cycle of length k*, is a cycle containing k elements. The preceding permutation consists of a 3-cycle and two 1-cycles. The *cycle type*, or simply the *type*, of π is an expression of the form

$$(1^{m_1}, 2^{m_2}, \ldots, n^{m_n}),$$

where m_k is the number of cycles of length k in π. The example permutation has cycle type

$$(1^2, 2^0, 3^1, 4^0, 5^0).$$

A 1-cycle of π is called a *fixedpoint.* The numbers 4 and 5 are fixedpoints in our example. Fixedpoints are usually dropped from the cycle notation if no confusion will result. An *involution* is a permutation such that $\pi^2 = \epsilon$. It is easy to see that π is an involution if and only if all of π's cycles have length 1 or 2.

Another way to give the cycle type is as a partition. A *partition of n* is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

where the λ_i are weakly decreasing and $\sum_{i=1}^{l} \lambda_i = n$. Thus k is repeated m_k times in the partition version of the cycle type of π. Our example corresponds to the partition

$$\lambda = (3, 1, 1).$$