ALM 25
Advanced Lectures in Mathematics

Handbook of Moduli

(Volume II)

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Editors: Gavril Farkas · Ian Morrison

高等教育出版社
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The Handbook of Moduli was designed by Ian Morrison in LaTeX using a variant of the standard style files of the Higher Education Press. The text of the Handbook is set in ITC Giovanni and the mathematics in AMS Euler.

Giovanni was designed by Robert Slimbach in 1989 for ITC and was one of the early faces that earned him the Prix Charles Peignot, the Fields Medal of type design awarded "to a designer under the age of 35 who has made an outstanding contribution to type design". It combines the basic proportions of traditional oldstyle designs with the more even color and higher x-height of modern digital fonts to produce an inconspicuous but legible typeface.

Euler was designed in 1981 by Hermann Zapf, a major figure in 20th type design and a pioneer in digital typography, working in close cooperation with Donald Knuth, as an upright, cursive symbol font that would give the effect of mathematics handwritten on a blackboard. In 2008, Zapf reshaped many of the glyphs, with the assistance of Hans Hagen, Taco Hoekwater, and Volker RW Schaa, in order to harmonize the designs and bring them into line with contemporary standards of digital typography.

The Handbook of Moduli is dedicated to the memory of Eckart Viehweg, whose untimely death precluded a planned contribution, and to David Mumford, who first proposed the project, for all that they both did to nurture its subject; and to Angela Ortega and Jane Reynolds for everything that they do to sustain its editors.

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Parameter spaces of curves

Joe Harris

Abstract. In this article I will try to survey the state of our knowledge (and the much greater area of our ignorance) of the geometry of spaces parametrizing curves in projective space.

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1. Introduction

Robin Hartshorne, in [6], describes the problem of classifying algebraic varieties as the guiding problem of algebraic geometry. I'd agree, for the most part; and, since you're currently reading a book entitled "Handbook of Moduli," presumably you would too.

But the question remains: what exactly are we classifying? To be specific, consider the problem of smooth, complete algebraic curves over $\mathbb{C}$. If you ask mathematicians today to describe the problem of classifying curves, they would naturally take "curve" to mean "abstract curve," in which case the answer to the problem, "classify all smooth complete curves" would consist of two parts. Algebraic

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curves are classified first by their sole discrete numerical invariant, the genus, which can assume any value $g \in \mathbb{N}$; and the set of curves of a given genus g has naturally the structure of an irreducible quasi-projective variety M_g. Beyond this, the problem of classifying algebraic curves consists of studying the geometry of the variety M_g, and of relating properties of curves to the loci in M_g of curves with that property.

If you had posed the same question to an algebraic geometer of the 19th century, however, it would of necessity have been interpreted differently. Abstract curves didn't exist then (or, depending on your philosophical point of view, they hadn't been discovered); the word "curve" would have been taken to mean a subset of projective space defined by polynomial equations, smooth and irreducible of dimension 1. As such, a curve had not one but three numerical invariants: its degree d; the dimension r of the projective space in which it lay (or, more properly, the dimension of its span); and of course its genus. The problem of classifying all algebraic curves would thus amount to two things:

(1) To say for which triples g, r and d there exists a smooth, irreducible and nondegenerate curve of degree d and genus g in $\mathbb{P}^r$; and
(2) To describe, for each such triple (g, r, d), the geometry of the space $\mathcal{H}^\circ_{g,r,d}$ parametrizing such curves: its irreducible components, their dimensions and so on.

In this volume, there are many articles that address aspects of the problem of classification in its modern sense. But the classical version is still very much of interest, and has many fascinating aspects that are not fully understood: we haven't answered the first of the questions above; and we know the answer to the second only in an extremely limited range of cases. The goal of this article is to give a survey of what we do know about this problem, and likewise to suggest some of the numerous open problems.

The remainder of this paper will consist of three parts. In Section 2, we'll discuss the notion of parameter spaces of curves, and compare the two most commonly used such spaces, primarily the Hilbert scheme and the Kontsevich space. This may in a sense not be necessary if we're only concerned with smooth, irreducible curves in projective space, since the Hilbert scheme and the Kontsevich space have a common open subset parametrizing such curves (and indeed the reader can skip this section and go directly to the following ones). But for many purposes it's useful to have a compactification of the space of curves, and here the Hilbert scheme and the Kontsevich space differ dramatically, as we'll see.

In Section 3, we'll describe the conjectured answer to the Existence Problem, the first of the two questions listed above. This actually tells us a lot about curves of high genus: when g is more than roughly half the maximal possible genus of an irreducible, nondegenerate curve of degree d in $\mathbb{P}^r$, in addition to simply saying which triples (g, r, d) occur, we learn about the geometry of such curves, and the dimension and irreducible components of their families. But for g below this bound, all we can say is that such curves exist; we can't say much about the spaces parametrizing them

Finally, in Section 4, we address this issue. We can in fact give a pretty explicit description of the spaces of curves of low genus, using what we know about the moduli space of abstract curves and Brill-Noether theory. Again, our knowledge—even conjectured—drops off as we approach the middle range of possible genera; we'll try to indicate what are some of the main unresolved questions in this area.

2. Parameter spaces

First of all, some terminology. We propose to call a space whose points correspond naturally to isomorphism classes of varieties or schemes X of a given type a *moduli space*; we'll call a space whose points correspond naturally to subschemes $X \subset Z$ of a fixed scheme Z (not up to isomorphism) a *parameter space*. There is not always a clear line dividing the two—for example, the Kontsevich space parameterizing stable maps has elements of both—but it does reflect an important duality in how we view geometric objects. One of the fundamental ideas underlying much recent progress in the theory of curves, for example, is the fact that whenever we have a one-parameter family $\{C_t \subset \mathbb{P}^r\}_{t \in \Delta}$ of curves in projective space, with C_t smooth for $t \neq t_0$, we have two distinct notions of the "limit" $\lim_{t \to t_0} C_t$ of the curves C_t: the *flat limit*, which is a subscheme of $\mathbb{P}^r$ whose geometry can be pretty much arbitrarily messy; and the *stable limit*, which is the limit of the abstract curves C_t and has at worst nodes as singularities. (Other articles in this volume discuss alternative notions of stability, and correspondingly alternative definitions of the limit of the abstract curves C_t; as for the flat limit, we really don't have much of an alternative to that.)

That said, what should we take as the parameter space for curves of degree d and genus g in $\mathbb{P}^r$? There are principally three answers to this question: the *Chow variety*, the *Hilbert scheme* and the *Kontsevich space*. These agree on the common open subset $\mathcal{H}^\circ_{g,r,d}$ parametrizing smooth curves (at least if we ignore the scheme structure on these spaces), but give very different compactifications of $\mathcal{H}^\circ_{g,r,d}$. Now, the questions we raised earlier—when do there exist such curves $C \subset \mathbb{P}^r$, what are the irreducible components of $\mathcal{H}^\circ_{g,r,d}$ and what are their dimensions—really don't depend on the choice of compactification, as long as we restrict our attention to the closure of $\mathcal{H}^\circ_{g,r,d}$ in each. But for many other questions it is important to have a complete parameter space, and so we start with a brief discussion of the properties of each. Actually, we'll pretty much ignore the Chow variety—in many ways, it has all the drawbacks of the Hilbert scheme and the Kontsevich space, and none of the virtues—and focus primarily on the other two.

The following discussion is adapted from a forthcoming book, *3264 and All That: Intersection Theory in Algebraic Geometry*, by the author and David Eisenbud.

2.1. Hilbert schemes

The Hilbert scheme $\mathcal{H} = \mathcal{H}_{g,r,d}(\mathbb{P}^r)$ is a parameter space for subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(m) = md - g + 1$; in the case of curves (one-dimensional subschemes) this means all subschemes with fixed degree and arithmetic genus. The Hilbert scheme has many good properties. For example, there is a useful cohomological description of its tangent spaces, and, beyond that, a deformation theory that in some cases can describe its local structure. And, of course, associated to a point on the Hilbert scheme is all the rich structure of a homogenous ideal in the ring $K[x_0, \ldots, x_r]$ and its resolution.

There is one circumstance where the Hilbert scheme is particularly nice: the Hilbert scheme parametrizing plane curves of degree d (Hilbert polynomial $p(m) = md - \binom{d-1}{2} + 1$) is simply the projective space $\mathbb{P}^{\binom{d+2}{2}-1}$ of homogeneous forms of degree d. Beyond that, the geometry of the Hilbert scheme ranges from the mysterious to the pathological; we'll illustrate with some examples.

Beyond hypersurfaces, the simplest case is the Hilbert scheme containing twisted cubic curves in $\mathbb{P}^3$ (Hilbert polynomial $H(m) = 3m + 1$.) It has one component of dimension 12 whose general point corresponds to a twisted cubic curve, but it has a second component, whose general point corresponds to the union of a plane cubic $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ and a point $p \in \mathbb{P}^3$. Moreover, this second component has dimension 15 (the choice of plane has 3 degrees of freedom; the cubic inside the plane 9 more; and the point gives an additional 3.) These two components meet along the 11-dimensional subscheme of singular plane cubics C with an embedded point at the singularity, not contained in the plane spanned by C.

2.1.1. Report card for the Hilbert scheme

The Hilbert scheme, as a compactification of the space of smooth curves, has drawbacks that sometimes make it difficult to use:

(1) **It has extraneous components, often of differing dimensions.** We see this phenomenon already in the case of twisted cubics, above. Of course we could take just the closure in the Hilbert scheme of the locus of smooth curves, but we would lose some of the nice properties, like the description of the tangent space. Thus while it is relatively easy to describe the singular locus of $\mathcal{H}$, we don't know how to describe singular locus of $\mathcal{H}^\circ$ along the locus where it intersects other components.

In fact, we don't know for curves of higher degree how many such extraneous components there are, or their dimensions: for $r \geqslant 3$ and large d the Hilbert scheme of zero-dimensional subschemes of degree d in $\mathbb{P}^r$ will have an unknown number of extraneous components of unknown dimensions, and this creates even more extraneous components in the Hilbert schemes of curves.

(2) **No one knows what's in the closure of the locus of smooth curves**. If we do choose to deal with the closure of the locus of smooth curves rather than the whole Hilbert scheme—as it seems we must—we face another problem: except in a few special cases, we can't tell if a given point in the Hilbert scheme is in this closure. That is, we don't know how to tell whether a given singular 1-dimensional scheme $C \subset \mathbb{P}^r$ is smoothable.

(3) **It has many singularities**. Vakil has shown that the singularities of the Hilbert scheme are, in a precise sense, arbitrarily bad: in[14] he proves that that the completion of every affine local K-algebra appears (up to adding variables) as the completion of a local ring on a Hilbert scheme of curves. But I want to focus here on a specific source of particularly bad singular points: points in the Hilbert scheme corresponding to curves that are flat limits of almost all other curves.

For example, start with an arbitrary smooth curve $C \subset \mathbb{P}^r$ of degree d and genus g, choose a general system of coordinates on $\mathbb{P}^r$ and apply the one-parameter group of automorphisms

$$A_t = \begin{pmatrix} t^{a_0} & 0 & 0 & \dots & 0 \\ 0 & t^{a_1} & 0 & \dots & 0 \\ 0 & 0 & t^{a_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t^{a_n} \end{pmatrix}$$

with $a_0 \ll a_1 \ll \cdots \ll a_n$. Let C_0 and C_∞ be the flat limits of the curves $C_t = A_t(C)$ as $t \to 0$ or ∞—that is, the schemes defined by the *initial ideal* with Hilbert polynomial $dm - g + 1$ with respect to the lexicographical and reverse lexicographical orderings. The corresponding points $[C_0]$ and $[C_\infty]$ in the Hilbert scheme will then be in the closure of the orbit of a general point under the action of PGL_{r+1}; so that the local geometry of the Hilbert scheme at the point $[C]$ necessarily encodes most of the global geometry of the Hilbert scheme itself, or at least its quotient by PGL_{r+1}.

2.2. The Kontsevich space

These drawbacks often make it difficult to study the global geometry of the Hilbert scheme. An alternative is the *Kontsevich space*; see [3] for a systematic treatment.

The Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ parametrizes what are called *stable maps* of degree d and genus g to $\mathbb{P}^r$. These are morphisms

$$f : C \to \mathbb{P}^r$$

with C a connected curve of arithmetic genus g having only nodes as singularities, such that the image $f_*[C]$ of the fundamental class of C is equal to d times the class of a line in $A_1(\mathbb{P}^r)$, and satisfying the one additional condition that the automorphism group of the map f—that it, automorphisms ϕ of C such that $f \circ \phi = f$—is finite. (This last condition is automatically satisfied if the map f is finite; it's relevant only for maps that are constant on an irreducible component of C, and amounts to saying that any smooth, rational component C_0 of C on which f is constant must intersect the rest of the curve C in at least three points.) Two such maps $f : C \to \mathbb{P}^r$ and $f' : C' \to \mathbb{P}^r$ are said to be the same if there exists an isomorphism $\alpha : C \to C'$ with $f' \circ \alpha = f$. There's an analogous notion of a family of stable maps, and the Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ is a coarse moduli space for the functor of families of stable maps. Note that we're taking the quotient by automorphisms of the domain, but not of the image, so that $\overline{M}_{g,0}(\mathbb{P}^r, d)$ shares with the Hilbert scheme $\mathcal{H}_{dm-g+1}(\mathbb{P}^r)$ a common subset parametrizing smooth curves $C \subset \mathbb{P}^r$ of degree d and genus g.

There are naturally variants of this: the space $\overline{M}_{g,n}(\mathbb{P}^r, d)$ parametrizes maps $f : C \to \mathbb{P}^r$ with C a nodal curve having n marked distinct smooth points $p_1, \ldots, p_n \in C$. (Here an automorphism of f is an automorphism of C fixing the points p_i and commuting with f; the condition of stability is thus that any smooth, rational component C_0 of C on which f is constant must have at least three distinguished points, counting both marked points and points of intersection with the rest of the curve C.) More generally, for any projective variety X and numerical equivalence class $\beta \in \mathrm{Num}_1(X)$, we have a space $\overline{M}_{g,n}(X, \beta)$ parametrizing maps $f : C \to X$ with fundamental class $f_*[C] = \beta$, again with C nodal and f having finite automorphism group.

The remarkable aspect of the Kontsevich space is simply that it is indeed proper: in other words, if $\mathcal{C} \subset \Delta \times \mathbb{P}^r$ is a flat family of subschemes of $\mathbb{P}^r$ parametrized by a smooth, one-dimensional base Δ, and the fiber C_t is a smooth curve for $t \neq 0$, then no matter what the singularities of C_0 there is a unique stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ which is the limit of the inclusions $\iota_t : C_t \hookrightarrow \mathbb{P}^r$. Note that this limiting stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ depends on the family, not just on the scheme C_0; the import of this in practice is that the Kontsevich space is often locally a blow-up of the Hilbert scheme along loci of curves with singularities worse than nodes. (This is not to say we have in general a regular map from the Kontsevich space to the Hilbert scheme; as we'll see in the examples below, the limiting stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ doesn't determine the flat limit C_0 either.) We'll see how this plays out in four relatively simple cases below.

2.2.1. Plane conics One indication of how useful the Kontsevich space can be is that, in the case of $\overline{M}_0(\mathbb{P}^2,2)$ (that is, plane conics), the Kontsevich space is actually equal to the space of complete conics:

To begin with, if $C \subset \mathbb{P}^2$ is a conic of rank 2 or 3—that is, anything but a double line—then the inclusion map $\iota : C \hookrightarrow \mathbb{P}^2$ is a stable map; thus the open set $W \subset \mathbb{P}^5$ of such conics is likewise an open subset of the Kontsevich space $\overline{M}_0(\mathbb{P}^2,2)$.

But when a family $\mathcal{C} \subset \Delta \times \mathbb{P}^2$ of conics specializes to a double line $C_0 = 2L$, the limiting stable map is a finite, degree 2 map $f : C \to L$, with C either isomorphic to $\mathbb{P}^1$, or two copies of $\mathbb{P}^1$ meeting at a point. Such a map is characterized, up to automorphisms of the domain curve, by its branch divisor $B \subset L$, a divisor of degree 2. (If B consists of two distinct points, $C \cong \mathbb{P}^1$, while if $B = 2p$ for some $p \in L$, the curve C will be reducible.) Thus we have a birational morphism

$$\pi : \overline{M}_0(\mathbb{P}^2,2) \to \mathcal{H}_{2m+1}(\mathbb{P}^2) = \mathbb{P}^5$$

from the Kontsevich space to the Hilbert scheme, with 2-dimensional fibers over the locus in $\mathbb{P}^5$ corresponding to double lines.

2.2.2. Conics in space By contrast, there is not a regular map in either direction between the Hilbert scheme of conics in space and the Kontsevich space $\overline{M}_0(\mathbb{P}^3,2)$. Of course there is a common open set: its points correspond to reduced conics—that is, embedded nodal curves of degree 2. To see that this does not extend to a regular map in either direction, note first that, as before, if $\mathcal{C} \subset \Delta \times \mathbb{P}^3$ is a family of conics specializing to a double line C_0, the limiting stable map is a finite, degree 2 cover $f : C \to L$, and this cover is not determined by the flat limit C_0 of the schemes $C_t \subset \mathbb{P}^3$. But on the other hand the scheme C_0 is again a complete intersection of a plane and a quadric surface, which is to say it lies in a unique plane H; and this plane is not determined by the data of the map f.

The relationship in this case between the Hilbert scheme and the Kontsevich space is what's called in higher-dimensional birational geometry a *flip*: the Kontsevich space is obtained from the Hilbert scheme $\mathcal{H}$ by blowing up the locus of double lines, and then blowing down the exceptional divisor along another ruling. (The blow-up of $\mathcal{H}$ along the double line locus could be described as the space of pairs $(H; (C, C^*))$, where $H \subset \mathbb{P}^3$ is a plane and (C, C^*) a complete conic in $H \cong \mathbb{P}^2$.)

2.2.3. Plane cubics Here, we do have a regular map from the Kontsevich space $\overline{M}_1(\mathbb{P}^2, 3)$ to the Hilbert scheme $\mathcal{H}_{3m}(\mathbb{P}^2) \cong \mathbb{P}^9$, and it does some interesting things: it blows up the locus of triple lines, much as in the example of plane conics, and the locus of cubics consisting of a double line and a line as well. But it also blows up the locus of cubics with a cusp, and cubics consisting of a conic and a tangent line, and these are trickier: the blow-up along the locus of cuspidal cubics, for example, can be obtained either by three blow-ups with smooth centers, or one blow-up along a nonreduced scheme supported on this locus.

But what we really want to illustrate here is that the Kontsevich space $\overline{M}_1(\mathbb{P}^2, 3)$ is not irreducible—in fact, it's not even 9-dimensional! For example, maps of the form $f : C \to \mathbb{P}^2$ with C consisting of the union of an elliptic curve E and a copy of $\mathbb{P}^1$, with f mapping $\mathbb{P}^1$ to a nodal plane cubic C_0 and mapping E to a smooth point of C_0 form a 10-dimensional family of stable maps; in fact, these form an open subset of a second irreducible component of $\overline{M}_1(\mathbb{P}^2, 3)$. And there's also a third component, whose general member $f : C \to \mathbb{P}^2$ has domain C an elliptic curve, with two $\mathbb{P}^1$s attached, with the map f contracting the elliptic curve and sending the two rational tails to a line and a conic in $\mathbb{P}^2$.

2.2.4. Twisted cubics Here the shoe is on the other foot. The Hilbert scheme $\mathcal{H} = \mathcal{H}_{3m+1}$ has, as we saw, a second irreducible component besides the closure $\mathcal{H}_0$ of the locus of actual twisted cubics, and the presence of this component makes it difficult to work with. For example, it takes quite a bit of analysis to see that $\mathcal{H}_0$ is smooth, since we have no simple way of describing its tangent space; see [12] for details. By contrast, the Kontsevich space is irreducible, and has only relatively mild (finite quotient) singularities.

2.2.5. Report Card for the Kontsevich Space As with the Hilbert scheme, there are difficulties in using the Kontsevich space:

(1) **It has extraneous components**. These arise in a completely different way from the extraneous components of the Hilbert scheme, but they're there. A typical example of an extraneous component of the Kontsevich space $\overline{M}_g(\mathbb{P}^r, d)$ would consist of maps $f : C \to \mathbb{P}^r$ in which C was the union of a rational curve $C_0 \cong \mathbb{P}^1$, mapping to a rational curve of degree d in $\mathbb{P}^r$, and C_1 an arbitrary curve of genus g meeting C_0 in one point and on which f was constant; if the curve C_1 does not itself admit a nondegenerate map of degree d to $\mathbb{P}^r$, this map can't be smoothed.

So, using the Konsevitch space rather than the Hilbert scheme doesn't solve this problem, but it does provide a frequently useful alternative: there are situations where the Kontsevich space has extraneous components and the Hilbert scheme not—like the case of plane cubics described above—and also situations where the reverse is true, such as the case of twisted cubics.

(2) **No one knows what's in the closure of the locus of smooth curves**. This, unfortunately, remains an issue with the Kontsevich space. Even in the case of the space $\overline{M}_g(\mathbb{P}^2, d)$ parametrizing plane curves, where it might be hoped that the Kontsevich space would provide a better compactification of the Severi variety than simply taking its closure in the space $\mathbb{P}^N$ of all plane curves of degree d, the fact that we don't know which stable maps are smoothable represents a real obstacle to its use.

(3) **It has points corresponding to highly singular schemes, and these tend to be in turn highly singular points of $\overline{M}_g(\mathbb{P}^r, d)$**. Still true; but in this respect, at least, it might be said that the Kontsevich space represents an improvement over the Hilbert scheme: even when the image $f(C)$ of a stable map $f : C \to \mathbb{P}^r$ is highly singular, the fact that the domain of the map is at worst nodal makes the deformation theory of the map relatively tractable.

2.3. Caveat: should we restrict attention to smooth curves, or reduced ones?

Before we leave the topic of the choice of parameter space, we should mention one other choice. We have declared that we are primarily interested in the space $\mathcal{H}^{\circ}_{g,r,d}$ parametrizing smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ of degree d and genus g. But there is at least one plausible alternative: we could replace the condition of smoothness with the weaker condition of being simply reduced.

Now, it may seem at first that. as long as we're concerned only with coarse invariants of the space $\mathcal{H}^{\circ}_{g,r,d}$—enumerating its irreducible components, and their dimensions—it shouldn't matter whether we work with $\mathcal{H}^{\circ}_{g,r,d}$ or the larger open subset $\mathcal{H}'_{g,d,r}$ of reduced, irreducible and nondegenerate curves, and for the most part this is true: the vast majority of components of $\mathcal{H}'_{g,r,d}$ have dense open subsets lying in $\mathcal{H}^{\circ}_{g,r,d}$. But not always: while you may have to work to locate them, there are irreducible components of $\mathcal{H}'_{g,r,d}$ whose general point corresponds to a reduced but singular curve. This can happen for local or global reasons: Hartshorne, in [8], shows that there exist reduced and irreducible curves $C \subset \mathbb{P}^3$ that are not smoothable; and in the following section we'll see how to construct examples of curves in higher-dimensional space $\mathbb{P}^r$ that have just one node as singularity and that are still not smoothable.

We have chosen here to work with the more restricted class of smooth curves. But this is largely for reasons of convenience: if we allow singularities, the question arises of whether to talk in terms of the arithmetic genus or the geometric genus; and while the former is clearly the more natural choice when we're talking about the Hilbert scheme of curves of higher genus (as we will in the following Section), the Brill-Noether-theoretic approach to curves of lower genus that we take in Section 4 is better suited to working with the geometric genus. The fact is, many of the statements we make, and the techniques we employ, can be extended to the larger class of reduced, irreducible and nondegenerate curves, but they're more complicated; and in the interests of an intelligible exposition we've opted to do everything in the simpler setting.

3. The existence problem

We start with the first question: when there exists a smooth, irreducible and nondegenerate curve of degree d and genus g in $\mathbb{P}^r$. This was answered for $r = 3$ by Halphen in the 19^{th} century, though there was a gap in his argument; the correct proof was given in [4]. A wonderful survey of our knowledge of curves in $\mathbb{P}^3$ is given in [7] and [9]. The answer for $r = 4$ and 5 was given in [13]; and we have a conjectured answer—or at least an algorithm for determining the answer in any given case—in general; we'll describe this conjecture below.