Donald L. Cohn

MEASURE THEORY

测度论

Birkhäuser

世界图书出版公司
www.wpcbj.com.cn

Measure Theory

Donald L. Cohn

BIRKHÄUSER
Boston · Basel · Berlin

图书在版编目(CIP)数据

测度论 = Measure Theory:英文/(美)科恩(Cohn,D. L.)
著.—影印本.—北京:世界图书出版公司北京
公司,2011.12
ISBN 978-7-5100-4058-0

Ⅰ.①测… Ⅱ.①科… Ⅲ.①测度论—英文
Ⅳ.①O174.12

中国版本图书馆 CIP 数据核字(2011)第 217083 号

书　　名:	Measure Theory
作　　者:	Donald L. Cohn
中 译 名:	测度论
责任编辑:	高蓉　刘慧
出 版 者:	世界图书出版公司北京公司
印 刷 者:	三河市国英印务有限公司
发　　行:	世界图书出版公司北京公司(北京朝内大街 137 号　100010)
联系电话:	010-64021602,010-64015659
电子信箱:	kjb@wpcbj.com.cn
开　　本:	24 开
印　　张:	16
版　　次:	2012 年 01 月
版权登记:	图字:01-2011-3253
书　　号:	978-7-5100-4058-0/O·906　　定　价: 49.00 元

To Linda, Henry, and Edward

Preface

This book is intended as a straightforward treatment of the parts of measure theory necessary for analysis and probability. The first five or six chapters form an introduction to measure and integration, while the last three chapters should provide the reader with some tools that are necessary for study and research in any of a number of directions. (For instance, one who has studied Chapters 7 and 9 should be able to go on to interesting topics in harmonic analysis, without having to pause to learn a new theory of integration and to reconcile it with the one he or she already knows.) I hope that the last three chapters will also prove to be a useful reference.

Chapters 1 through 5 deal with abstract measure and integration theory, and presuppose only the familiarity with the topology of Euclidean spaces that a student should acquire in an advanced calculus course. Lebesgue measure on **R** (and on $\mathbf{R}^d$) is constructed in Chapter 1 and is used as a basic example thereafter.

Chapter 6, on differentiation, begins with a treatment of changes of variables in $\mathbf{R}^d$, and then gives the basic results on the almost everywhere differentiation of functions on **R** (and measures on $\mathbf{R}^d$). The first section of this chapter makes use of the derivative (as a linear transformation) of a function from $\mathbf{R}^d$ to $\mathbf{R}^d$; the necessary definitions and facts are recalled, with appropriate references. The rest of the chapter has the same prerequisites as the earlier chapters.

Chapter 7 contains a rather thorough treatment of integration on locally compact Hausdorff spaces. I hope that the beginner can learn the basic facts from Sections 2 and 3 without too much trouble. These sections, together with Section 4 and the first part of Section 6, cover almost everything the typical analyst needs to know about regular measures. The technical facts needed for dealing with very large locally compact Hausdorff spaces are included in Sections 5 and 6.

In Chapter 8 I have tried to collect those parts of the theory of analytic sets that are of everyday use in analysis and probability. I hope it will serve both as an introduction and as a useful reference.

Chapter 9 is devoted to integration on locally compact groups. In addition to a construction and discussion of Haar measure, I have included a brief introduction to convolution on $L^1(G)$ and on the space of finite signed or complex regular Borel measures on G. The details are provided for arbitrary locally compact groups, but in such a way that a reader who is interested only in second countable groups should find it easy to make the appropriate omissions.

Chapters 7, 8, and 9 presuppose a little background in general topology. The necessary facts are reviewed, and so some facility with arguments involving topological spaces and metric spaces is actually all that is required. The reader who can work through Sections 7.1 and 8.1 should have no trouble.

In addition to the main body of the text, there are five appendices. The first four explain the notation used and contain some elementary facts from set theory, calculus, and topology; they should remind the reader of a few things he or she may have forgotten, and should thereby make the book quite self-contained. The fifth appendix contains an introduction to the Bochner integral.

Each section ends with some exercises. They are, for the most part, intended to give the reader practice with the concepts presented in the text. Some contain examples, additional results, or alternative proofs, and should provide a bit of perspective. Only a few of the exercises are used later in the text itself; these few are provided with hints, as needed, that should make their solution routine.

I believe that no result in this book is new. Hence the lack of a bibliographic citation should never be taken as a claim of originality. The notes at the ends of chapters occasionally tell where a theorem or proof first appeared; most often, however, they point the reader to alternative presentations or to sources of further information.

The system used for cross-references within the book should be almost self-explanatory. For example, Proposition 1.3.5 and Exercise 1.3.7 are to be found in Section 3 of Chapter 1, while C.1 and Theorem C.8 are to be found in Appendix C.

There are a number of people to whom I am indebted, and whom I would like to thank. First there are those from whom I learned integration theory, whether through courses, books, papers, or conversations; I won't try to name them, but I thank them all. I would like to thank R. M. Dudley and W. J. Buckingham, who read the original manuscript, and J. P. Hajj, who helped me with the proofreading. These three read the book with much care and thought, and provided many useful suggestions. (I must, of course, accept responsibility for ignoring a few of their suggestions and for whatever mistakes remain.) Finally, I thank my wife, Linda, for typing and providing editorial advice on the manuscript, for helping with the proofreading, and especially for her encouragement and patience during the years it took to write this book.

Donald L. Cohn

Donald L. Cohn
Department of Mathematics
Suffolk University
Boston, MA 02114

Library of Congress Cataloging In-Publication Data

Cohn, Donald L 1942-
Measure theory.

Bibliography: p.
Includes index.
1. Measure theory. I. Title.
QA312.C56 515.4'2 80-14768
ISBN 3-7643-3003-1

CIP-Kurztitelaufnahme der Deutschen Bibliothek

Cohn, Donald L.:
Measure theory / Donald L. Cohn. - Boston, Basel, Stuttgart : Birkhäuser, 1980.
ISBN 3-7643-3003-1

Reprinted 1993, 1996, 1997

Birkhäuser

ISBN 0-8176-3003-1
ISBN 3-7643-3003-1

Contents

1. Measures
1. Algebras and sigma-algebras 1
2. Measures 8
3. Outer measures 14
4. Lebesgue measure 26
5. Completeness and regularity 35
6. Dynkin classes 44

2. Functions and Integrals
1. Measurable functions 48
2. Properties that hold almost everywhere 58
3. The integral 61
4. Limit theorems 70
5. The Riemann integral 75
6. Measurable functions again, complex-valued functions, and image measures 79

3. Convergence
1. Modes of convergence 85
2. Normed spaces 90
3. Definition of $\mathscr{L}^p$ and L^p 98
4. Properties of $\mathscr{L}^p$ and L^p 106
5. Dual spaces 113

4. Signed and Complex Measures
1. Signed and complex measures 121
2. Absolute continuity 131
3. Singularity 140
4. Functions of bounded variation 143
5. The duals of the L^p spaces 149

5. Product Measures
1. Constructions 154
2. Fubini's theorem 158
3. Applications 162

6. Differentiation
1. Change of variable in $\mathbf{R}^d$ 167
2. Differentiation of measures 177
3. Differentiation of functions 184

7. Measures on Locally Compact Spaces
1. Locally compact spaces 196
2. The Riesz representation theorem 205
3. Signed and complex measures; duality 217
4. Additional properties of regular measures 226
5. The μ^*-measurable sets and the dual of L^1 232
6. Products of locally compact spaces 240

8. Polish Spaces and Analytic Sets
1. Polish spaces 251
2. Analytic sets 261
3. The separation theorem and its consequences 272
4. The measurability of analytic sets 278
5. Cross sections 284
6. Standard, analytic, Lusin, and Souslin spaces 288

9. Haar Measure
1. Topological groups 297
2. The existence and uniqueness of Haar measure 303
3. Properties of Haar measure 312
4. The algebras $L^1(G)$ and $M(G)$ 317

Appendices
A. Notation and set theory 328
B. Algebra 334
C. Calculus and topology in $\mathbf{R}^d$ 339
D. Topological spaces and metric spaces 342
E. The Bochner integral 350

Bibliography 361
Index of notation 367
Index 369

1 | Measures

1. ALGEBRAS AND SIGMA-ALGEBRAS

Let X be an arbitrary set. A collection $\mathscr{A}$ of subsets of X is an *algebra* on X if

(a) $X \in \mathscr{A}$,

(b) for each set A that belongs to $\mathscr{A}$ the set A^c belongs to $\mathscr{A}$,

(c) for each finite sequence $A_1, \ldots, A_n$ of sets that belong to $\mathscr{A}$ the set $\bigcup_{i=1}^n A_i$ belongs to $\mathscr{A}$, and

(d) for each finite sequence $A_1, \ldots, A_n$ of sets that belong to $\mathscr{A}$ the set $\bigcap_{i=1}^n A_i$ belongs to $\mathscr{A}$.

Of course, in conditions (b), (c), and (d) we have required that $\mathscr{A}$ be *closed* under complementation, under the formation of finite unions, and under the formation of finite intersections. It is easy to check that closure under complementation and closure under the formation of finite unions together imply closure under the formation of finite intersections (use the fact that $\bigcap_{i=1}^n A_i = (\bigcup_{i=1}^n A_i^c)^c$). Thus we could have defined an algebra using only conditions (a), (b), and (c). A similar argument shows that we could have used only conditions (a), (b), and (d).

Again let X be an arbitrary set. A collection $\mathscr{A}$ of subsets of X is a *σ-algebra** on X if

(a) $X \in \mathscr{A}$,

(b) for each set A that belongs to $\mathscr{A}$ the set A^c belongs to $\mathscr{A}$,

*The terms *field* and *σ-field* are sometimes used in place of algebra and σ-algebra.

(c) for each infinite sequence $\{A_i\}$ of sets that belong to $\mathscr{A}$ the set $\bigcup_{i=1}^{\infty} A_i$ belongs to $\mathscr{A}$, and
(d) for each infinite sequence $\{A_i\}$ of sets that belong to $\mathscr{A}$ the set $\bigcap_{i=1}^{\infty} A_i$ belongs to $\mathscr{A}$.

Thus a σ-algebra on X is a family of subsets of X that contains X and is closed under complementation, under the formation of countable unions, and under the formation of countable intersections. Note that, as in the case of algebras, we could have used only conditions (a), (b), and (c), or only conditions (a), (b), and (d), in our definition.

Each σ-algebra on X is an algebra on X since, for example, the union of the finite sequence $A_1, A_2, \ldots, A_n$ is the same as the union of the infinite sequence $A_1, A_2, \ldots, A_n, A_n, A_n, \ldots$.

If X is a set and if $\mathscr{A}$ is a family of subsets of X that is closed under complementation, then X belongs to $\mathscr{A}$ if and only if $\emptyset$ belongs to $\mathscr{A}$. Thus in the definitions of algebras and σ-algebras given above we can replace condition (a) with the requirement that $\emptyset$ be a member of $\mathscr{A}$. Furthermore, if $\mathscr{A}$ is a family of subsets of X that is non-empty, closed under complementation, and closed under the formation of finite or countable unions, then $\mathscr{A}$ must contain X: if the set A belongs to $\mathscr{A}$, then X, since it is the union of A and A^c, must also belong to $\mathscr{A}$. Thus in our definitions of algebras and σ-algebras we can replace condition (a) with the requirement that $\mathscr{A}$ be non-empty.

In case $\mathscr{A}$ is a σ-algebra on the set X, it is sometimes convenient to call a subset of X *$\mathscr{A}$-measurable* if it belongs to $\mathscr{A}$.

We turn to some examples.

1. Let X be a set, and let $\mathscr{A}$ be the collection of all subsets of X. Then $\mathscr{A}$ is a σ-algebra on X.

2. Let X be a set, and let $\mathscr{A} = \{\emptyset, X\}$. Then $\mathscr{A}$ is a σ-algebra on X.

3. Let X be an infinite set, and let $\mathscr{A}$ be the collection of all finite subsets of X. Then $\mathscr{A}$ does not contain X and is not closed under complementation, and so is not an algebra (or a σ-algebra) on X.

4. Let X be an infinite set, and let $\mathscr{A}$ be the collection of all subsets A of X such that either A or A^c is finite. Then $\mathscr{A}$ is an algebra on X (check this), but is not closed under the formation of countable unions, and so is not a σ-algebra.

5. Let X be an uncountable set, and let $\mathscr{A}$ be the collection of all countable (i.e., finite or countably infinite) subsets of X. Then $\mathscr{A}$ does not contain X and is not closed under complementation, and so is not an algebra.

6. Let X be a set, and let $\mathscr{A}$ be the collection of all subsets A of X such that either A or A^c is countable. Then $\mathscr{A}$ is a σ-algebra.

7. Let $\mathscr{A}$ be the collection of all subsets of $\mathbf{R}$ that are unions of finitely many intervals of the form $(a,b]$, $(a,+\infty)$, or $(-\infty,b]$. It is easy to check that each set that belongs to $\mathscr{A}$ is the union of a finite disjoint collection of intervals of the types listed above, and then to check that $\mathscr{A}$ is an algebra on $\mathbf{R}$ (the empty set belongs to $\mathscr{A}$, since it is the union of the empty, and hence finite, collection of intervals). The algebra $\mathscr{A}$ is not a σ-algebra; for example, the bounded open subintervals of $\mathbf{R}$ are unions of sequences of sets in $\mathscr{A}$, but do not themselves belong to $\mathscr{A}$.

Next we consider ways of constructing σ-algebras.

Proposition 1.1.1.

Let X be a set. Then the intersection of an arbitrary non-empty collection of σ-algebras on X is a σ-algebra on X.

Proof. Let $\mathscr{C}$ be a non-empty collection of σ-algebras on X, and let $\mathscr{A}$ be the intersection of the σ-algebras that belong to $\mathscr{C}$. It is enough to check that $\mathscr{A}$ contains X, is closed under complementation, and is closed under the formation of countable unions. The set X belongs to $\mathscr{A}$, since it belongs to each σ-algebra that belongs to $\mathscr{C}$. Now suppose that $A \in \mathscr{A}$. Each σ-algebra that belongs to $\mathscr{C}$ contains A and so contains A^c; thus A^c belongs to the intersection $\mathscr{A}$ of these σ-algebras. Finally, suppose that $\{A_i\}$ is a sequence of sets that belong to $\mathscr{A}$, and hence to each σ-algebra in $\mathscr{C}$. Then $\cup A_i$ belongs to each σ-algebra in $\mathscr{C}$, and so to $\mathscr{A}$. ■

The reader should note that the union of a family of σ-algebras can fail to be a σ-algebra (see Exercise 5).

Proposition 1.1.1 implies the following result, which is a basic tool for the construction of σ-algebras.

Corollary 1.1.2.

Let X be a set, and let $\mathscr{F}$ be a family of subsets of X. Then there is a smallest σ-algebra on X that includes $\mathscr{F}$.

Of course, to say that $\mathscr{A}$ is the smallest σ-algebra on X that includes $\mathscr{F}$ is to say that $\mathscr{A}$ is a σ-algebra on X that includes $\mathscr{F}$, and that every σ-algebra on X that includes $\mathscr{F}$ also includes $\mathscr{A}$. This smallest σ-algebra on X that includes $\mathscr{F}$ is clearly unique; it is called the σ-algebra *generated* by $\mathscr{F}$, and is often denoted by $\sigma(\mathscr{F})$.

Proof. Let $\mathscr{C}$ be the collection of all σ-algebras on X that include $\mathscr{F}$. Then $\mathscr{C}$ is non-empty, since it contains the σ-algebra that consists of all subsets of X. The intersection of the σ-algebras that belong to $\mathscr{C}$ is, according to Proposition 1.1.1, a σ-algebra; it includes $\mathscr{F}$ and is included in every σ-algebra on X that includes $\mathscr{F}$. ■

We now use the preceding corollary to define an important family of σ-algebras. The *Borel σ-algebra* on $\mathbf{R}^d$ is the σ-algebra on $\mathbf{R}^d$ generated by the collection of open subsets of $\mathbf{R}^d$, and is denoted by $\mathscr{B}(\mathbf{R}^d)$. The *Borel subsets* of $\mathbf{R}^d$ are those that belong to $\mathscr{B}(\mathbf{R}^d)$. In case $d = 1$, one generally writes $\mathscr{B}(\mathbf{R})$ in place of $\mathscr{B}(\mathbf{R}^1)$.

Proposition 1.1.3.
The σ-algebra $\mathscr{B}(\mathbf{R})$ of Borel subsets of $\mathbf{R}$ is generated by each of the following collections of sets:

(a) the collection of all closed subsets of $\mathbf{R}$;
(b) the collection of all subintervals of $\mathbf{R}$ of the form $(-\infty, b]$;
(c) the collection of all subintervals of $\mathbf{R}$ of the form $(a, b]$.

Proof. Let $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$ be the σ-algebras generated by the collections of sets in parts (a), (b), and (c) of the proposition. We shall show that $\mathscr{B}(\mathbf{R}) \supset \mathscr{B}_1 \supset \mathscr{B}_2 \supset \mathscr{B}_3$, and then that $\mathscr{B}_3 \supset \mathscr{B}(\mathbf{R})$; this will establish the proposition. Since $\mathscr{B}(\mathbf{R})$ includes the family of open subsets of $\mathbf{R}$ and is closed under complementation, it includes the family of closed subsets of $\mathbf{R}$; thus it includes the σ-algebra generated by the closed subsets of $\mathbf{R}$, namely $\mathscr{B}_1$. The sets of the form $(-\infty, b]$ are closed and so belong to $\mathscr{B}_1$; consequently $\mathscr{B}_2 \subset \mathscr{B}_1$. Since

$$(a, b] = (-\infty, b] \cap (-\infty, a]^c,$$

each set of the form $(a, b]$ belongs to $\mathscr{B}_2$; thus $\mathscr{B}_3 \subset \mathscr{B}_2$. Finally, note that each open subinterval of $\mathbf{R}$ is the union of a sequence of sets of the form $(a, b]$, and that each open subset of $\mathbf{R}$ is the union of a sequence of open intervals (see Proposition C.4). Thus each open subset of $\mathbf{R}$ belongs to $\mathscr{B}_3$, and so $\mathscr{B}(\mathbf{R}) \subset \mathscr{B}_3$. ■

As we proceed, the reader should note the following properties of the σ-algebra $\mathscr{B}(\mathbf{R})$:

1. It contains virtually* every subset of **R** that is of interest in analysis.
2. It is small enough that it can be dealt with in a fairly constructive manner.

It is largely these properties that explain the importance of $\mathscr{B}(\mathbf{R})$.

Proposition 1.1.4.

The σ-algebra $\mathscr{B}(\mathbf{R}^d)$ of Borel subsets of $\mathbf{R}^d$ is generated by each of the following collections of sets:

(a) the collection of all closed subsets of $\mathbf{R}^d$;
(b) the collection of all closed half-spaces in $\mathbf{R}^d$ that have the form $\{(x_1, \dots, x_d): x_i \leq b\}$ for some index i and some b in $\mathbf{R}$;
(c) the collection of all rectangles in $\mathbf{R}^d$ that have the form $\{(x_1, \dots, x_d): a_i < x_i \leq b_i \text{ for } i = 1, \dots, d\}$.

Proof. This proposition can be proved with essentially the argument that was used for Proposition 1.1.3, and so most of the proof is omitted. To see that the σ-algebra generated by the rectangles of part (c) is included in the σ-algebra generated by the half-spaces of part (b), note that each strip that has the form

$$\{(x_1, \dots, x_d): a < x_i \leq b\}$$

for some i is the difference of two of the half-spaces in part (b), and that each of the rectangles in part (c) is the intersection of d such strips.■

Let us look in more detail at some of the sets in $\mathscr{B}(\mathbf{R}^d)$. Let $\mathscr{G}$ be the family of all open subsets of $\mathbf{R}^d$, and let $\mathscr{F}$ be the family of all closed subsets of $\mathbf{R}^d$. (Of course $\mathscr{G}$ and $\mathscr{F}$ depend on the dimension d, and it would have been more precise to write $\mathscr{G}(\mathbf{R}^d)$ and $\mathscr{F}(\mathbf{R}^d)$.) Let $\mathscr{G}_\delta$ be the collection of all intersections of sequences of sets in $\mathscr{G}$, and let $\mathscr{F}_\sigma$ be the collection of all unions of sequences of sets in $\mathscr{F}$. Sets in $\mathscr{G}_\delta$ are often called G_δ's, and sets in $\mathscr{F}_\sigma$, F_σ's. The letters G and F presumably stand for the German word Gebiet and the French word fermé, and the letters σ and δ for the German words Summe and Durchschnitt.

*See Chapter 8 for some interesting and useful sets that are not Borel sets.

Proposition 1.1.5.

Each closed subset of $\mathbf{R}^d$ is a G_δ, and each open subset of $\mathbf{R}^d$ is an F_σ.

Proof. Suppose that F is a closed subset of $\mathbf{R}^d$. We need to construct a sequence $\{U_n\}$ of open subsets of $\mathbf{R}^d$ such that $F = \bigcap_n U_n$. For this define U_n by

$$U_n = \{x \in \mathbf{R}^d : \|x-y\| < 1/n \quad \text{for some } y \text{ in } F\}.$$

(Note that U_n is empty if F is empty.) It is clear that each U_n is open and that $F \subset \bigcap_n U_n$. The reverse inclusion follows from the fact that F is closed (note that each point in $\bigcap_n U_n$ is the limit of a sequence of points in F). Hence each closed subset of $\mathbf{R}^d$ is a G_δ.

If U is open, then U^c is closed, and so is a G_δ. Thus there is a sequence $\{U_n\}$ of open sets such that $U^c = \bigcap_n U_n$. The sets U_n^c are then closed, and $U = \bigcup_n U_n^c$; hence U is an F_σ. ■

For an arbitrary family $\mathscr{S}$ of sets let $\mathscr{S}_\sigma$ be the collection of all unions of sequences of sets in $\mathscr{S}$, and let $\mathscr{S}_\delta$ be the collection of all intersections of sequences of sets in $\mathscr{S}$. We can iterate the operations represented by σ and δ, obtaining from the class $\mathscr{G}$ the classes $\mathscr{G}_\delta$, $\mathscr{G}_{\delta\sigma}$, $\mathscr{G}_{\delta\sigma\delta}$, ..., and from the class $\mathscr{F}$ the classes $\mathscr{F}_\sigma$, $\mathscr{F}_{\sigma\delta}$, $\mathscr{F}_{\sigma\delta\sigma}$, ... (Note that $\mathscr{G} = \mathscr{G}_\sigma$ and $\mathscr{F} = \mathscr{F}_\delta$. Note also that $\mathscr{G}_{\delta\delta} = \mathscr{G}_\delta$, that $\mathscr{F}_{\sigma\sigma} = \mathscr{F}_\sigma$, and so on.) It now follows (see Proposition 1.1.5) that all the inclusions indicated in the following diagram are valid.

$$\begin{array}{ccccccccc}
\mathscr{G} & \subset & \mathscr{G}_\delta & \subset & \mathscr{G}_{\delta\sigma} & \subset & \mathscr{G}_{\delta\sigma\delta} & \subset & \cdots \\
 & \times & & \times & & \times & & \times & \\
\mathscr{F} & \subset & \mathscr{F}_\sigma & \subset & \mathscr{F}_{\sigma\delta} & \subset & \mathscr{F}_{\sigma\delta\sigma} & \subset & \cdots
\end{array}$$

It turns out that no two of these classes of sets are equal, and that there are Borel sets that belong to none of them (see Exercises 7 and 9 in Section 8.2).

A sequence $\{A_i\}$ of sets is called *increasing* if $A_i \subset A_{i+1}$ holds for each i, and *decreasing* if $A_i \supset A_{i+1}$ holds for each i.

Proposition 1.1.6.

Let X be a set, and let $\mathscr{A}$ be an algebra on X. Then $\mathscr{A}$ is a σ-algebra if either

(a) $\mathscr{A}$ is closed under the formation of unions of increasing sequences of sets, or

(b) $\mathscr{A}$ is closed under the formation of intersections of decreasing sequences of sets.

Proof. First suppose that condition (a) holds. Since $\mathscr{A}$ is an algebra, we can check that it is a σ-algebra by verifying that it is closed under the formation of countable unions. Suppose that $\{A_i\}$ is a sequence of sets that belong to $\mathscr{A}$. For each n let $B_n = \cup_{i=1}^{n} A_i$. The sequence $\{B_n\}$ is increasing, and, since $\mathscr{A}$ is an algebra, each B_n belongs to $\mathscr{A}$; thus assumption (a) implies that $\cup_n B_n$ belongs to $\mathscr{A}$. However, $\cup_i A_i$ is equal to $\cup_n B_n$, and so belongs to $\mathscr{A}$. Thus $\mathscr{A}$ is closed under the formation of countable unions, and so is a σ-algebra.

Now suppose that condition (b) holds. It is enough to check that condition (a) holds. If $\{A_i\}$ is an increasing sequence of sets that belong to $\mathscr{A}$, then $\{A_i^c\}$ is a decreasing sequence of sets that belong to $\mathscr{A}$, and so condition (b) implies that $\cap_i A_i^c$ belongs to $\mathscr{A}$. Since $\cup_i A_i = (\cap_i A_i^c)^c$, it follows that $\cup_i A_i$ belongs to $\mathscr{A}$. Thus condition (a) follows from condition (b), and the proof is complete. ■

EXERCISES

1. Find the σ-algebra on **R** that is generated by the collection of all one-point subsets of **R**.
2. Show that $\mathscr{B}(\mathbf{R})$ is generated by the collection of intervals $(-\infty, t]$ for which the end-point t is a rational number.
3. Show that $\mathscr{B}(\mathbf{R})$ is generated by the collection of all compact subsets of **R**.
4. Show that if $\mathscr{A}$ is an algebra of sets, and if $\cup_n A_n$ belongs to $\mathscr{A}$ whenever $\{A_n\}$ is a sequence of disjoint sets in $\mathscr{A}$, then $\mathscr{A}$ is a σ-algebra.
5. Show by example that the union of a collection of σ-algebras on a set X can fail to be a σ-algebra on X. (Hint: There are examples in which X is a small finite set.)
6. Find an infinite collection of subsets of **R** that contains **R**, is closed under the formation of countable unions, and is closed under the formation of countable intersections, but is not a σ-algebra.
7. Let $\mathscr{S}$ be a collection of subsets of the set X. Show that for each A in $\sigma(\mathscr{S})$ there is a countable subfamily $\mathscr{C}_0$ of $\mathscr{S}$ such that $A \in \sigma(\mathscr{C}_0)$. (Hint: Let $\mathscr{A}$ be the union of the σ-algebras $\sigma(\mathscr{C})$, where $\mathscr{C}$ ranges over the countable subfamilies of $\mathscr{S}$, and show that $\mathscr{A}$ is a σ-algebra that satisfies $\mathscr{S} \subset \mathscr{A} \subset \sigma(\mathscr{S})$.)
8. Find all σ-algebras on **N**.
9. (a) Show that **Q** is an F_σ, but not a G_δ, in **R**. (Hint: Use the Baire category theorem, Theorem D.37.)

   (b) Find a subset of **R** that is neither an F_σ nor a G_δ.